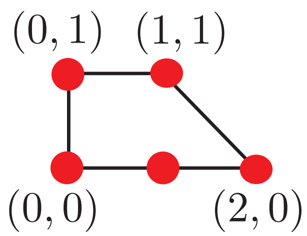
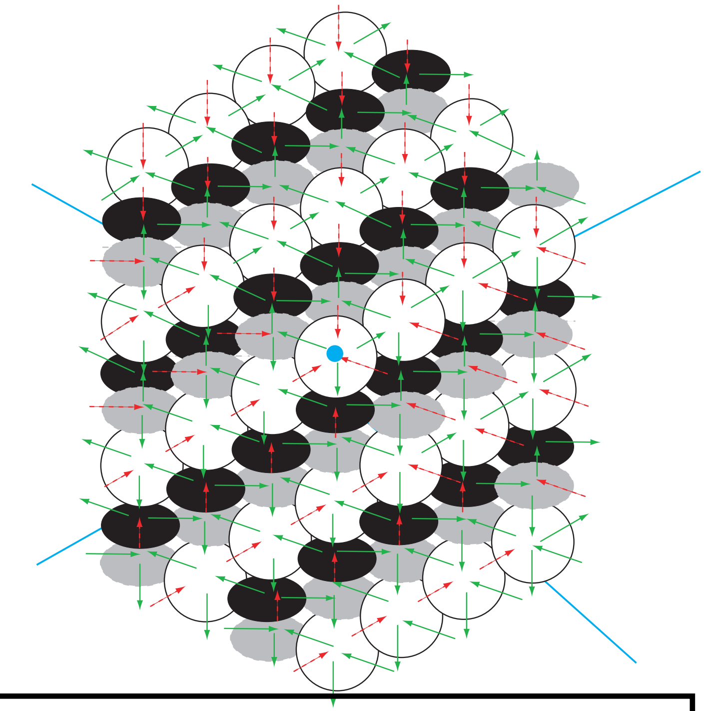


$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,\\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,,\\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,,
\end{aligned}
$$



Quiver Yangians and Donaldson-Thomas Invariants

Masahito Yamazaki

IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

86th Geometry Symposium

Aug 31, 2021

Based on

Wei Li + MY

(2003.08909 [hep-th])

Dmitry Galakhov + MY

(2008.07006 [hep-th])

Dimitry Galakhov+Wei Li + MY

(2106.01230 [hep-th])

(2108.10286 [hep-th])

⋯ and many works in the literature

e.g. Rapcak-Soibelman-Yang-Zhao ('18, '20)

Also earlier works, e.g.

Hirosi Ooguri + MY (0811.2810 [hep-th])

MY (Ph.D. thesis, 1002.1709 [hep-th])

MY (Master thesis, 0803.4474 [hep-th])

Overview

Donaldson-Thomas invariants

(toric CY_3)

$\mathcal{M}\left(\begin{matrix}\text{stable}\\ \text{sheaf}\end{matrix}\right)$

Donaldson-Thomas invariants
(toric CY_3)

$\mathcal{M}\left(\substack{\text{stable}\\ \text{sheaf}}\right)$

$\xrightarrow[\text{localization } T]{\text{equivariant}}$

crystal melting
(combinatorics fixed pt set)

$\left(\mathcal{M}\left(\substack{\text{stable}\\ \text{sheaf}}\right)\right)^T$

Donaldson-Thomas invariants

(toric CY_3)

$\mathcal{M}$(stable sheaf)

$\xrightarrow{\text{equivariant localization}}$

T

crystal melting

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$(\mathcal{M}(\text{stable sheaf}))^T$

rep.

Quiver Yangian

(∞-dim. algebra)

Donaldson-Thomas invariants
(toric CY_3)

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equivariant localization

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crystal melting
(combinatorics fixed pt set)

$\left(\mathcal{M}\left(\begin{matrix}\text{stable}\\ \text{sheaf}\end{matrix}\right)\right)^T$

$\boxed{\text{Quiver w/ potential}}$

rep.

character of rep.

Quiver Yangian
(∞-dim. algebra)

Donaldson-Thomas invariants

(toric CY_3)

$\mathcal{M}$ (stable sheaf)

equivariant localization

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crystal melting

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$(\mathcal{M}(\text{stable sheaf}))^T$

Quiver w/ potential

rep.

Quiver Yangian

(∞-dim. algebra)

character of rep.

string theory

- Donaldson - Thomas Invariants
- Quiver w/ Potential
- Quiver Yangian ← Algebra
- Representations of Quiver Yangian

Representation

(generalized)

Donaldson-Thomas invariants

- X : Calabi-Yau 3-fold "10d theory on X" (Type IIA)

$\mathcal{D} := D^b\mathrm{coh}\, X$: "category of B-branes"

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- $\gamma \in H^*(X, \mathbb{Q}) \xleftarrow{\mathrm{ch}} K(\mathcal{D})$: "D-brane charge"

- $\theta \in \mathrm{Stab}(X)$: stability condition for $\mathcal{D} = D^b \mathrm{coh}\, X$

[Douglas, Bridgeland, ...]

$\theta = (Z, \mathcal{A})$

$$\left(\begin{array}{l} * \; Z : H^*(X, \mathbb{Q}) \to \mathbb{C} \\ * \; \mathcal{A} : \text{heart of bounded t-structure} \\ + \; (\text{Harder-Narasimhan property}) \end{array} \right)$$

$X, \gamma, \theta \rightsquigarrow$ Donaldson-Thomas inv.

$$\Omega^\theta(\gamma) := \int_{[\mathcal{M}^\theta_s(\gamma)]_{vir}} 1 \in \mathbb{Q}$$

$X, Y, \theta \rightsquigarrow$ Donaldson-Thomas inv.

$$\Omega^\theta(\gamma) := \int_{[\mathcal{M}_s^\theta(\gamma)]_{vir}} 1 \in \mathbb{Q}$$

$\rightsquigarrow$ Generating function / partition function

$$Z^\theta(q) := \sum_{\gamma \in H^*(X, \mathbb{Q})} \pm\, \Omega^\theta(\gamma)\, q^\gamma$$

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$\rightsquigarrow$ Generating function / partition function

$$Z^\theta(q) := \sum_{\gamma \in H^*(X,\mathbb{Q})} \pm\, \Omega^\theta(\gamma)\, q^\gamma$$

* $Z^\theta(q)$: piecewise constant in $\theta \in Stab(X)$

"wall crossing phenomenon" [Kontsevich-Soibelman, Joyce-Song ....]

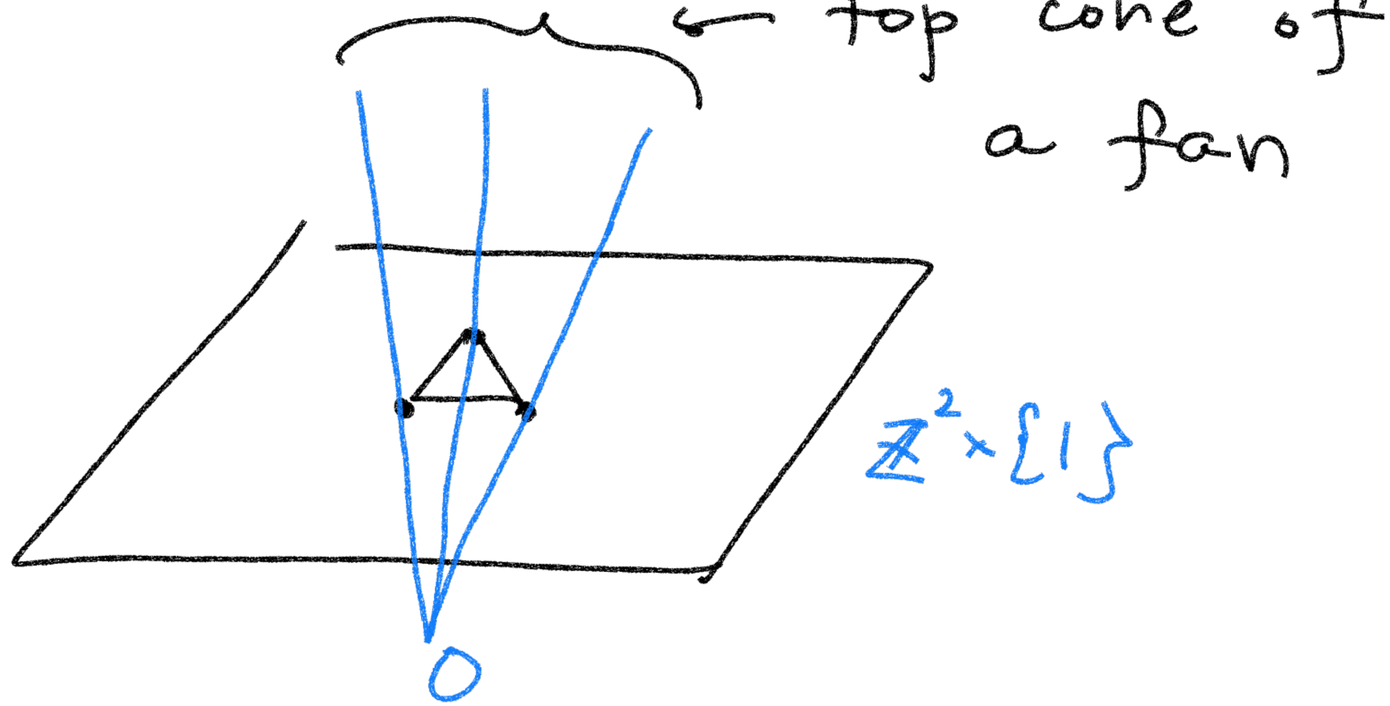
* $Z_{DT}^{\theta_*}(q) = Z_{GW}(g_s)$: MNOP conjecture

$q = e^{ig_s}$

[Maulik-Nekrasov-Okounkov-Pandharipande]

X : toric CY_3

$\Delta \subset \mathbb{Z}^2$



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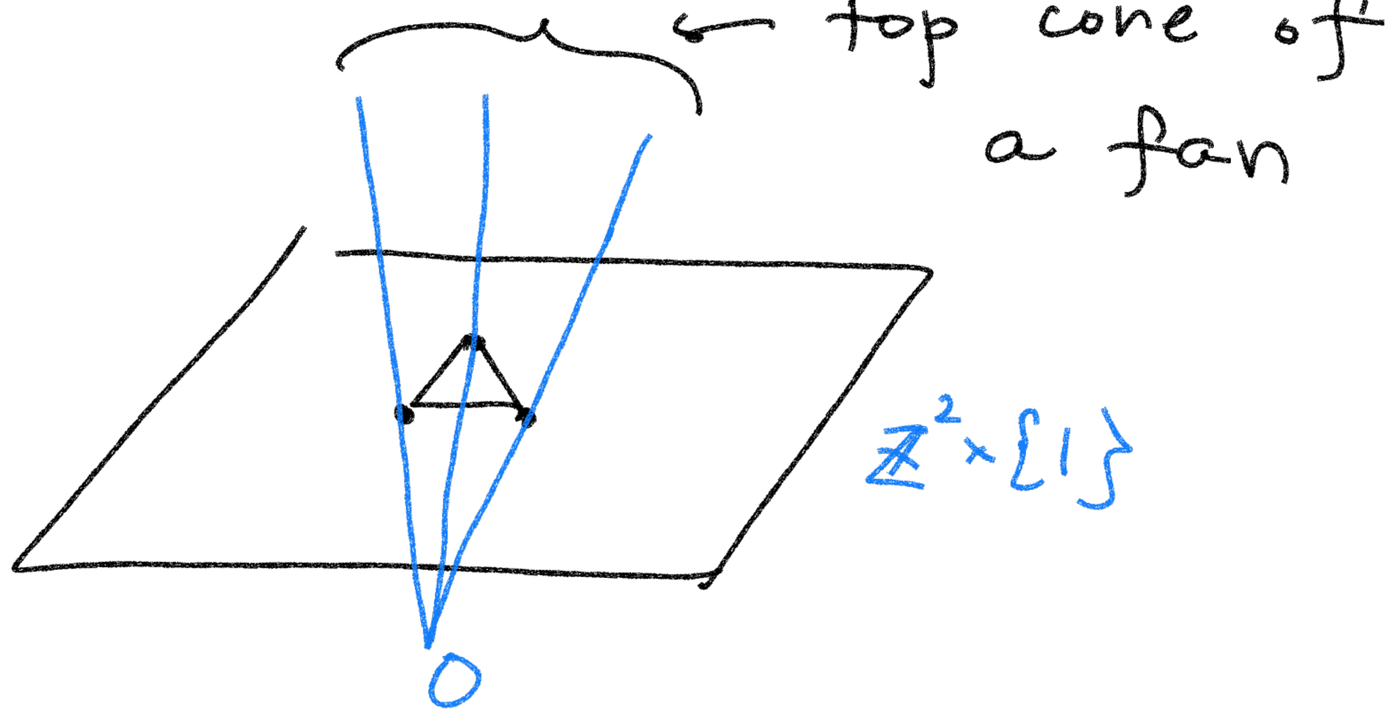

$X \circlearrowright T = U(1)^3$

$\rightsquigarrow$

$\mathcal{M}^{\theta}_{s}(H) \circlearrowright T$ torus action

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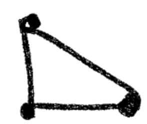

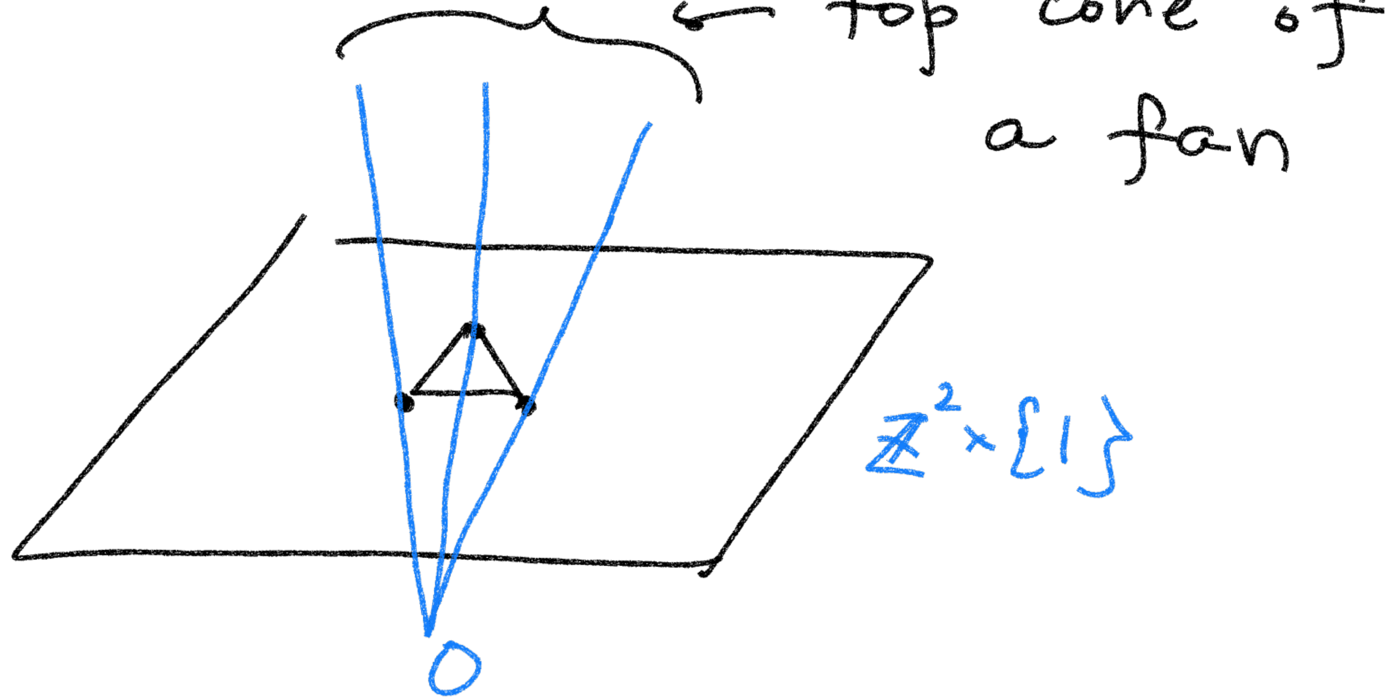

$X \circlearrowleft T = U(1)^3$

$\rightsquigarrow$

$\mathcal{M}^{\theta}_{s}(\mathcal{H}) \circlearrowleft T$ torus action

$\rightsquigarrow$

$[\mathcal{M}^{\theta}_{s}(\mathcal{H})]^T$: fixed point set $\overset{1:1}{\Longleftrightarrow}$ molten crystal $\swarrow$ defined later

(generalization of (plane) partitions)

Examples (for cyclic Θ & some framing)

* $\mathbb{C}^3$ $\quad Z = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} =: M(q)$ [MacMahon]

* $\mathbb{C}^3/\mathbb{Z}_2$ $\quad Z = M(q)^2 \prod_{n=1}^{\infty} \frac{1}{(1-Qq^n)^n (1-Q^{-1}q^n)^n}$

[Young ('08)]

* $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$ $\quad Z = M(q)^2 \prod_{n=1}^{\infty} (1-Qq^n)^n (1-Q^{-1}q^n)^n$

[Szendrői ('07), Young ('07)]

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$\downarrow$

$\mathbb{P}^1$

[Szendrői ('07), Young ('07)]

– string theory explanation of infinite product

[Aganagic – Ooguri – Vafa – MY ('09)]

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* $\mathbb{C}^3$ $\quad Z = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} =: M(q)$ [MacMahon]

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$\downarrow$

$\mathbb{P}^1$

[Szendroi ('07), Young ('07)]

- string theory explanation of infinite product

[Aganagic - Ooguri - Vafa - MY ('09)]

- mathematically, character of some algebra?

[Lie superalgebra? cf. Nagao, Nagao-MY ('09)]

Quiver w/ Potential

- Quiver $\underset{\substack{\uparrow\\ \text{vertex}}}{}$ $Q = (Q_0, Q_1)$: an oriented graph

Q_0: vertex, Q_1: arrow

- (super) potential W : formal sum (over $\mathbb{C}$) of closed cyclic paths

independent data

- quiver path algebra : $\underbrace{\mathbb{C}Q}_{\text{the set of paths w/ product = concatenation}} \Big/ \underbrace{(\partial W)}_{\text{Jacobian ideal}}$

the set of paths

w/ product = concatenation

Jacobian ideal

$$\left[\text{e.g. } \frac{\partial}{\partial a_2}(a_1 a_2 a_3) = a_3 a_1 \right]$$

Claim

For given toric CY3 X

$\exists$ quiver w/ relations $\overbrace{(Q, W)}^{\text{not unique}}$ s.t.

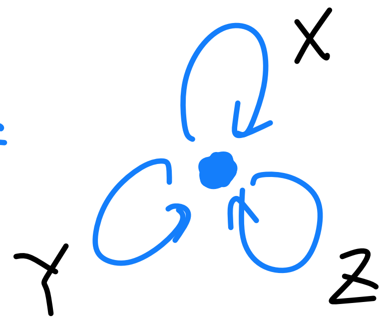
$$D^b\mathrm{coh}(X) \simeq D^b \mathrm{mod}\, \mathbb{C}Q/(\partial W)$$

[.... , Ueda - MY, Ueda - Ishii, ...]

From now on

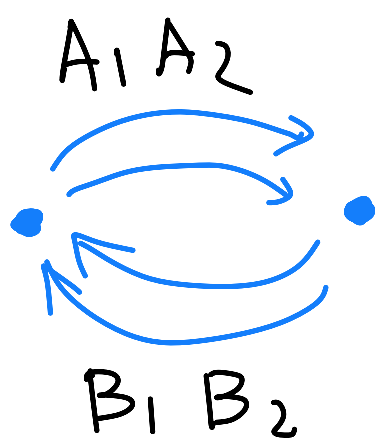
$$\begin{array}{ccc} D^b \mathrm{coh}\, X & \simeq & D^b \mathrm{mod}\, \mathbb{C}Q/(\partial W) \\ \Big\downarrow {\scriptstyle \theta\text{-stable objects}} & & \Big\downarrow {\scriptstyle \theta\text{-stable objects}} \\ \text{DT inv.} & = & \text{DT inv.} \end{array}$$

Quiver Q & Potential W ⟵ toric CY_3

* $Q =$ 

$W = (XYZ) - (XZY) \qquad (CY_3 = \mathbb{C}^3)$

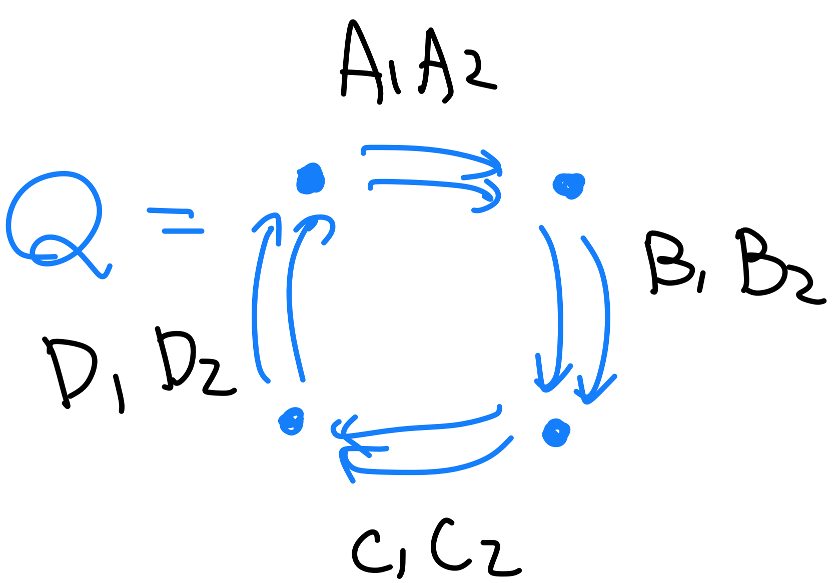
$\rightsquigarrow XY = YX$ etc.

* $Q =$ 

$W = (A_1 B_1 A_2 B_2) - (A_1 B_2 A_2 B_1)$

$(CY_3 = \text{conifold})$

[von den Bergh]

* $Q =$ 

$W = (A_1 B_1 C_1 D_1) - (A_1 B_2 C_1 D_2) - (A_2 B_1 C_2 D_1) + (A_2 B_2 C_2 D_2)$

$(CY_3 = K_{\mathbb{P}^1 \times \mathbb{P}^1})$

Claim

quiver w/ potential $\rightsquigarrow$ quiver on $\mathbb{T}^2$

(periodic quiver)

$Q = (Q_0, Q_1)$ & W

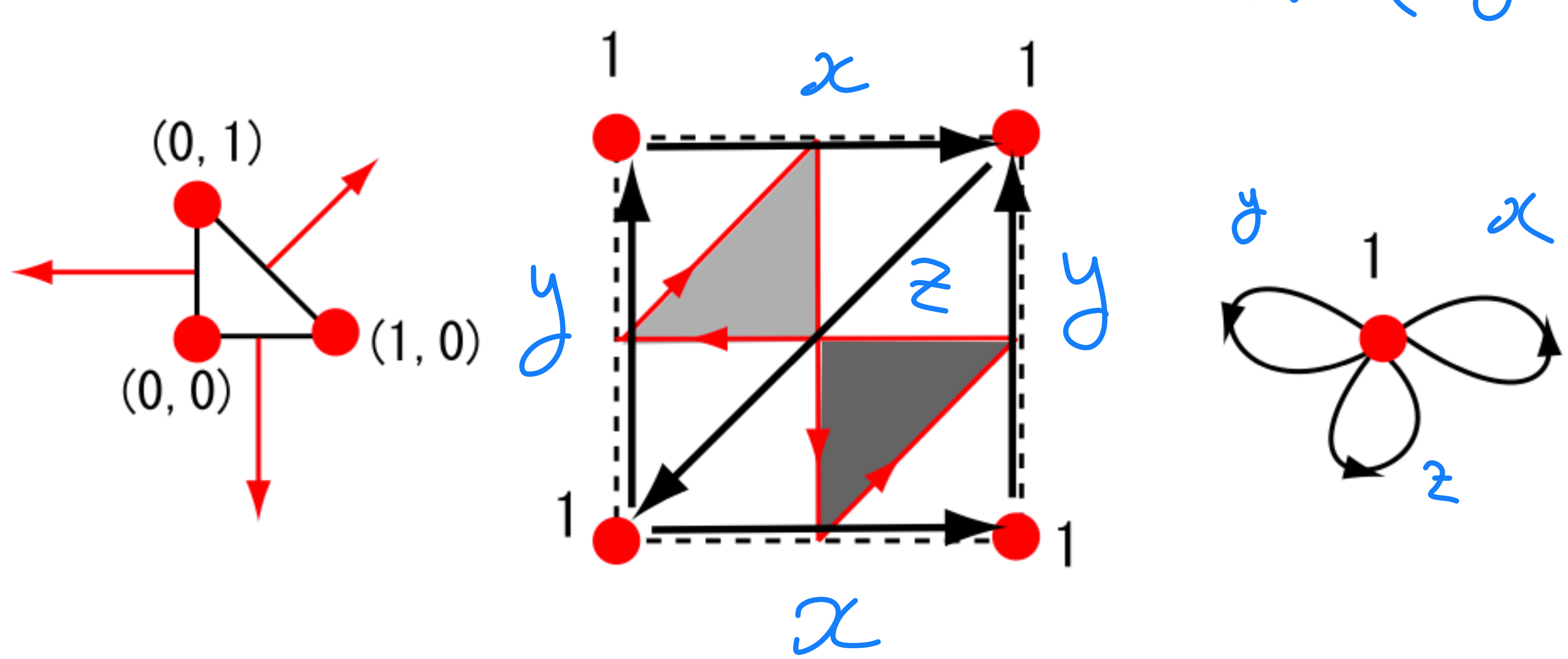
$\tilde{Q} = (Q_0, Q_1, Q_2)$, $Q_2 = W$

[Hanany, Kennaway, Franco, Vegh, Wecht ('05-)

[Ueda, MY, Ishii, ... ('06-)]

e.g. $\mathbb{C}^3$

$W = (xyz) - (xzy)$



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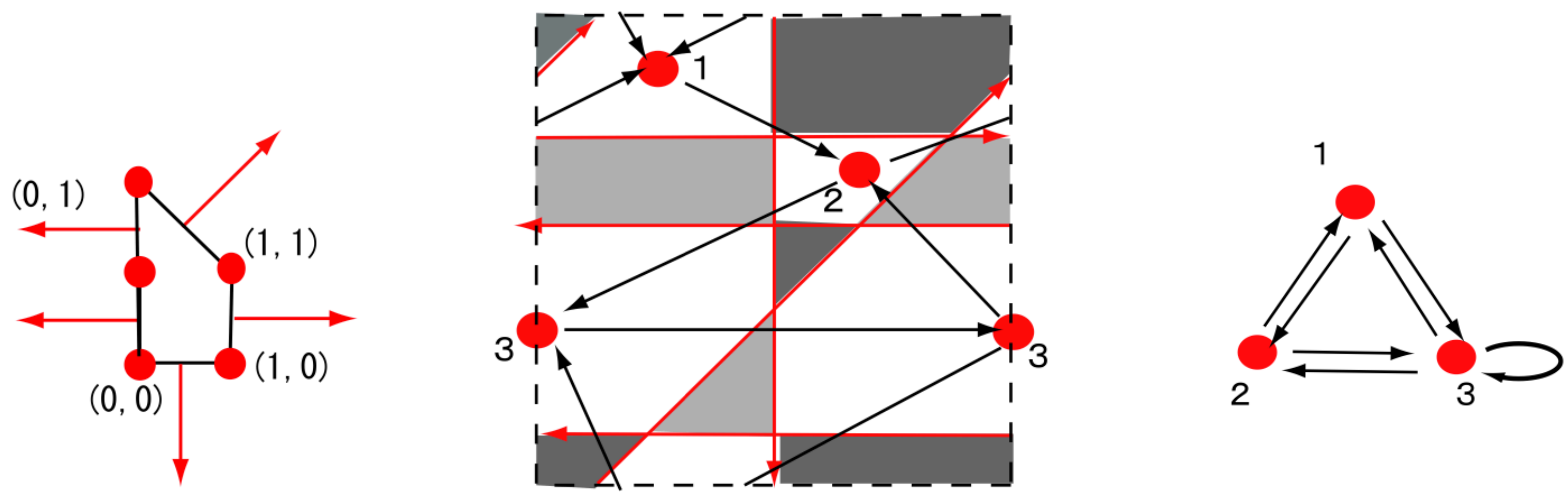
[Hanany, Kennaway, Franco, Vegh, Wecht ('05–)]
[Ueda, MY, Ishii, ... ('06)–]

e.g. $xy = zw^2$



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[Hanany, Kennaway, Franco, Vegh, Wecht, ... ('05–)

[Ueda, MY, Ishii, ... ('06–)]

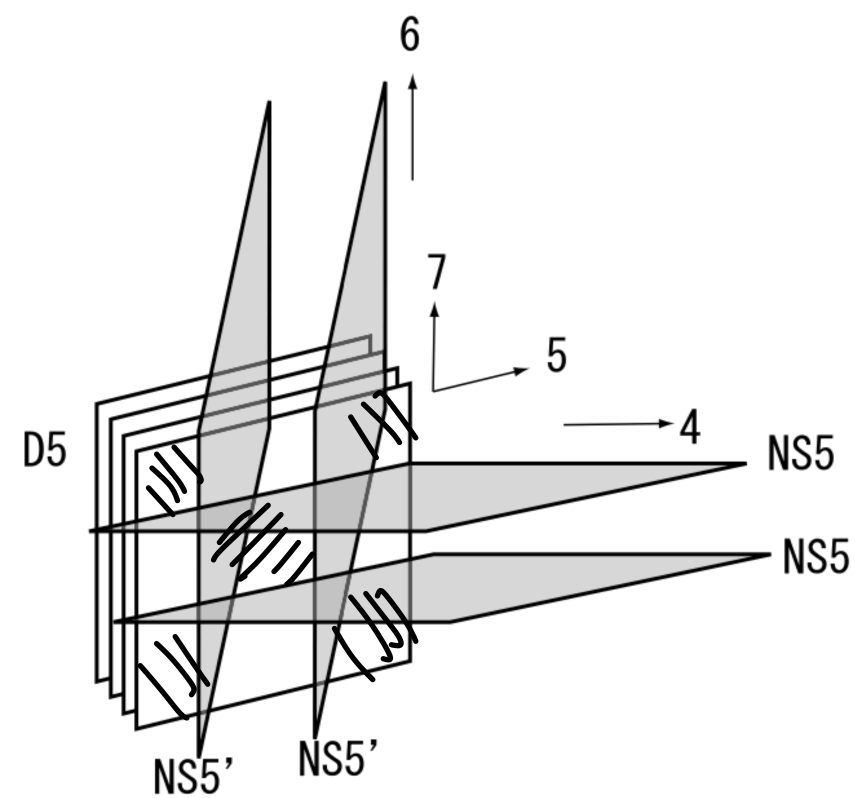
※ geometrical reasoning :

degeneration of Lagrangian in

mirror geometry!

[MY thesis ('08)

Shende-Treumann-Williams-Zaslow ('15)]



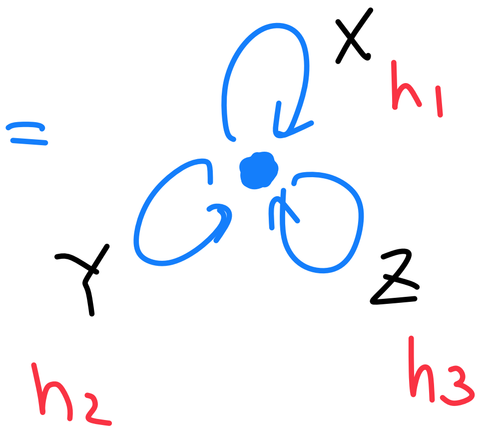
Shifted

Quiver Yangian

Wei Li + MY ('20)

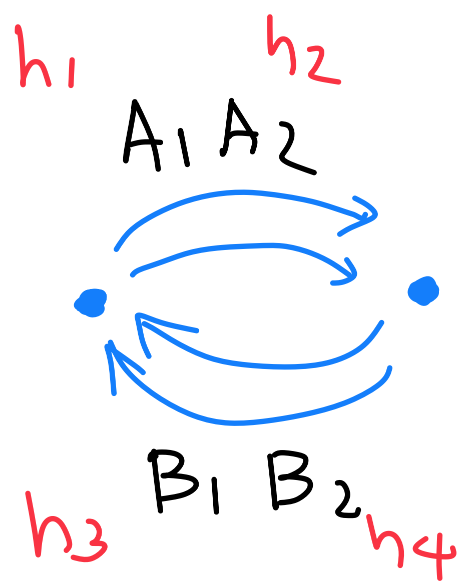
Dimitry Galakhov + Wei Li + MY ('21)

Quiver Q & Potential W ← toric CY_3

* $Q =$ 

$W = \mathrm{Tr}(XYZ - XZY)$ $\quad (CY_3 = \mathbb{C}^3)$

$h_1 + h_2 + h_3 = 0$

* $Q =$ 

$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

$(CY_3 = \text{conifold})$

$h_1 + h_2 + h_3 + h_4 = 0$

* Assign equivariant parameters h_I consistent w/ W

edge

Generators

(z: spectral parameter)

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}}\,, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}}\,, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}}\,,$$

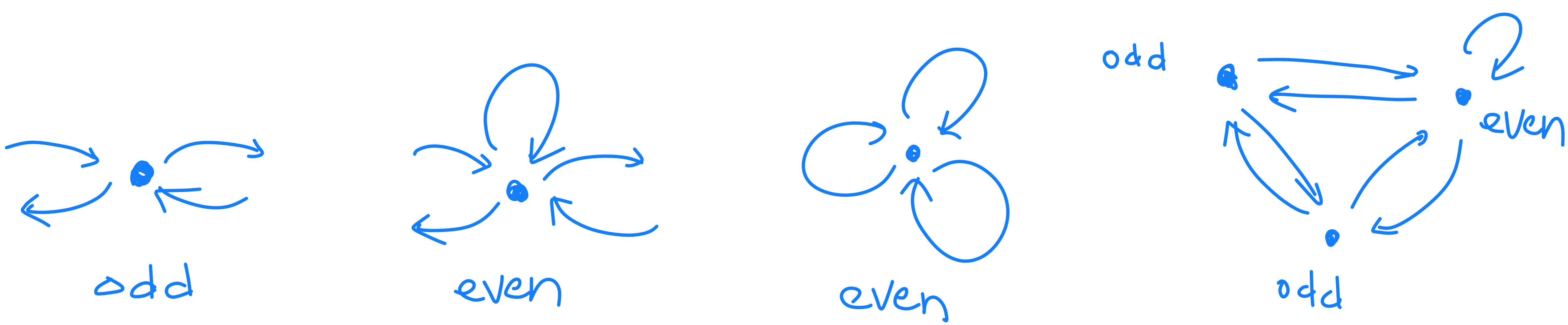
$n=-k$

"k-shifted Quiver Yangian"

a : quiver vertex

$\mathbb{Z}_2$-grading

$$|a| = \begin{cases} 0 & (\exists \text{ edge } I \text{ s.t. } I \text{ starts and ends at } a) \\ 1 & (\text{otherwise}) \end{cases}$$



Relations

Y(Q, W)

$$\psi^{(a)}(z)\,\psi^{(b)}(w) = \psi^{(b)}(w)\,\psi^{(a)}(z)\;,$$
$$\psi^{(a)}(z)\,e^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\;,$$
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$$\left[e^{(a)}(z), f^{(b)}(w)\right\} \sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\;,$$

"$\simeq$" means equality up to $z^n w^{m\geq 0}$ terms

"$\sim$" means equality up to $z^{n\geq 0} w^m$ and $z^n w^{m\geq 0}$ terms

bonding factor

equivariant weight

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

edge

Relations

$Y(Q,W)$

$$\psi^{(a)}(z)\,\psi^{(b)}(w)=\psi^{(b)}(w)\,\psi^{(a)}(z)\,,$$

$$\psi^{(a)}(z)\,e^{(b)}(w)\simeq\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,$$

$$\boxed{e^{(a)}(z)\,e^{(b)}(w)\sim(-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,,}$$

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$$\left[e^{(a)}(z),f^{(b)}(w)\right\}\sim-\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,,$$

cohomological Hall algebra

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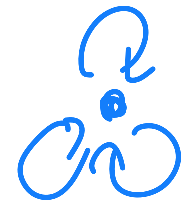
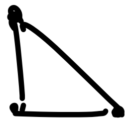
$$\varphi^{a\Rightarrow b}(u)\equiv\frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

edge

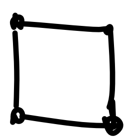
* $\mathbb{C}^3 \rightsquigarrow Q =$  $\rightsquigarrow Y(\widehat{\mathfrak{gl}}_1)$



$W = \mathrm{Tr}(XYZ - XZY)$

[Miki; Ding-Iohara;⋯
Feigin, Tsymbaulik; Prochazka;
Gaberdiel, Gopakumar, Li, Peng,⋯]

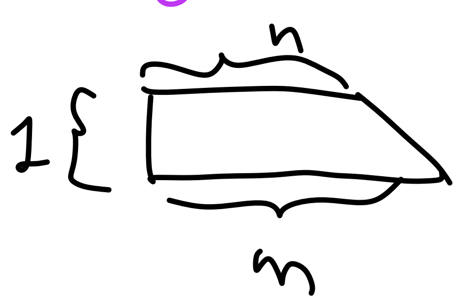
* conifold $\rightsquigarrow Q =$  $\rightsquigarrow Y(\widehat{\mathfrak{gl}}_{1|1})$



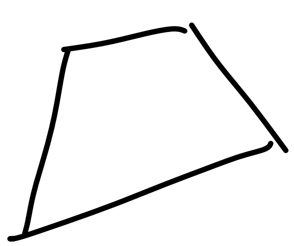
$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

* $xy = z^n w^m \rightsquigarrow Y(\widehat{\mathfrak{gl}}_{m|n})$

[Rapcak; Bezerra-Mukhin]
cf. [Nagao-MY '10]



* general toric $CY_3 \rightsquigarrow Y(Q, W)$

 $\Delta \subset \mathbb{Z}^2$

has no "$\mathfrak{g}$"

※ We can define trigonometric / elliptic versions (toroidal)

[Galakhov-Li-MY '2108
also Noshita-Watanabe]

$$Y(Q, W) \;-\; T_\beta(Q, W) \;-\; E_\tau(Q, W)$$

quiver Yangian — toroidal — elliptic

rational — trigonometric

(affine Yangian) — (quantum toroidal)

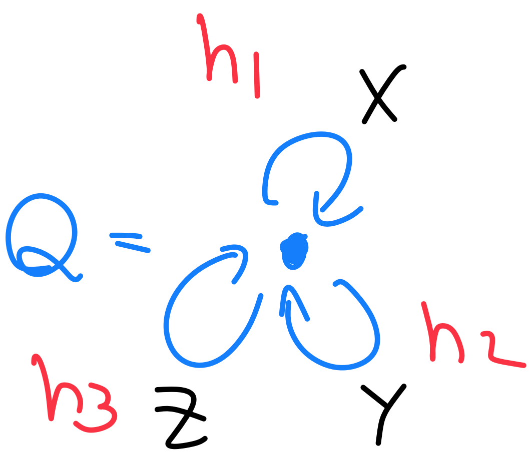
※ more generally, from formal group law & gen. cohomology

[Galakhov-Li-MY, also Yang-Zhao]

Representations from Crystal Melting

cf. earlier developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by [Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka; Rapcak; Gaberdiel, Gopakumar; Li, Peng,⋯]

example: $\mathbb{C}^3$ [Okounkov-Reshetikhin-Vafa; Iqbal, Nekrasov,...]

* $Q =$ 

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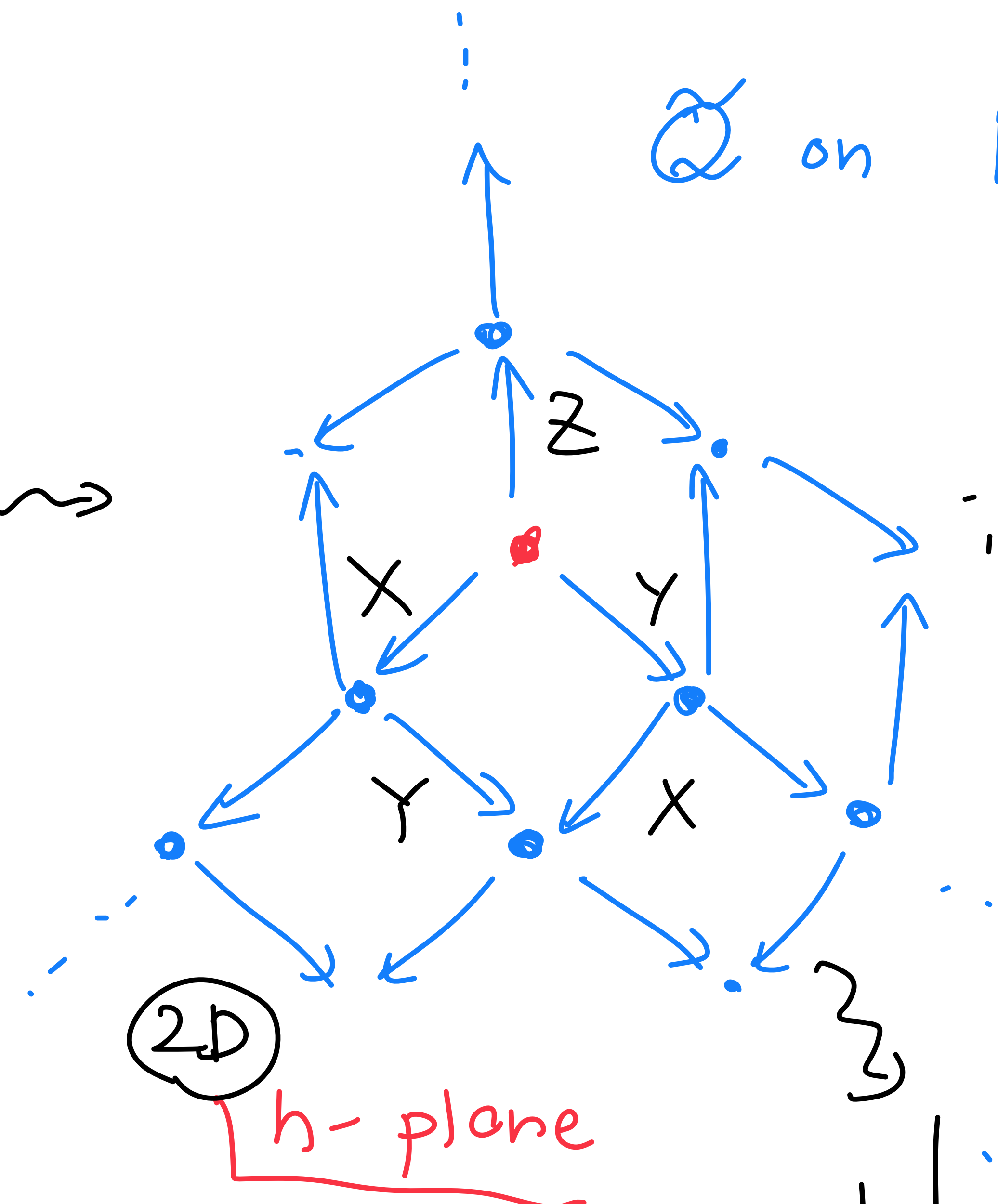
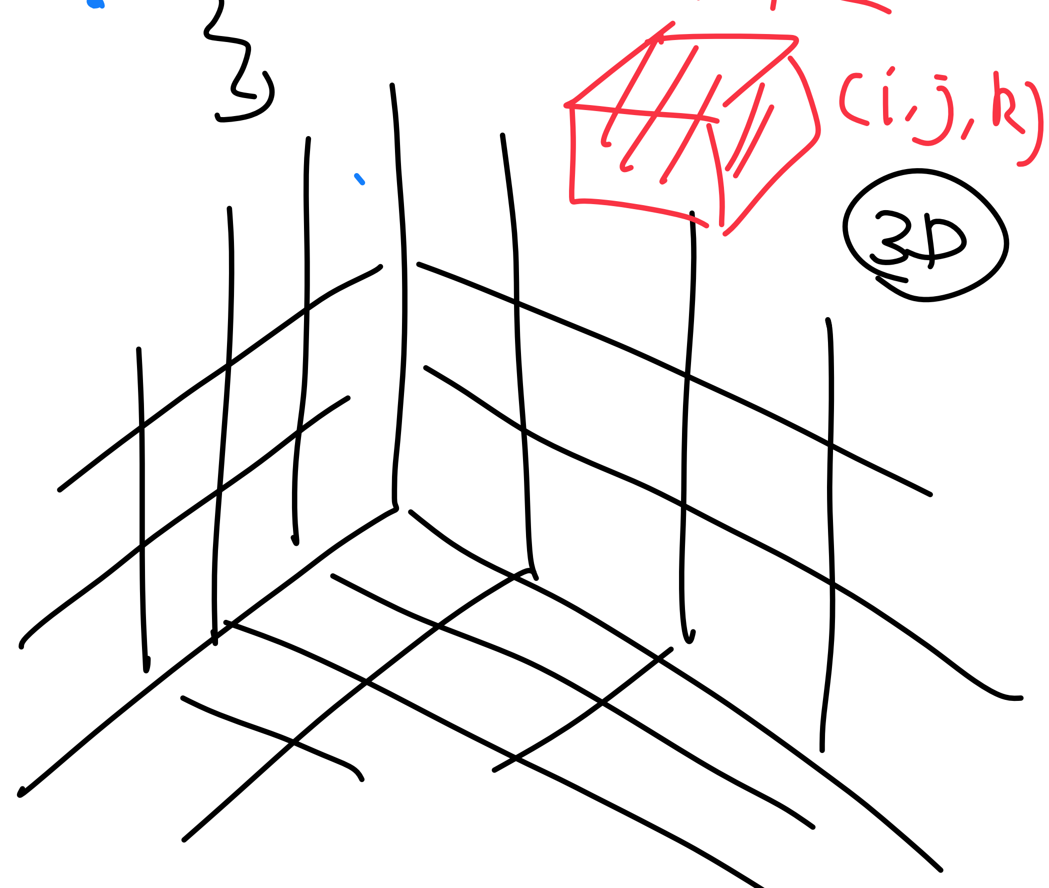
$\rightsquigarrow$ 

$\tilde{Q}$ on $\mathbb{R}^2$: (paths from •) / (relations)

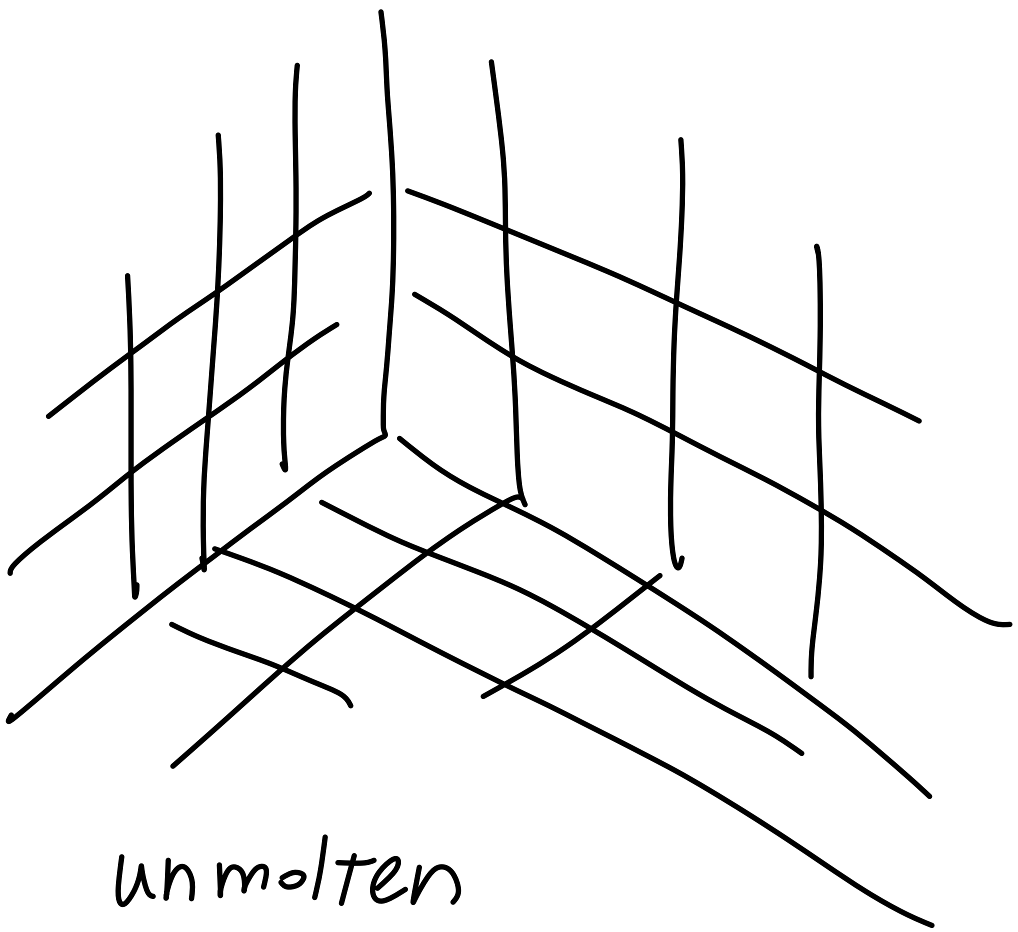
$X^i Y^j Z^k$



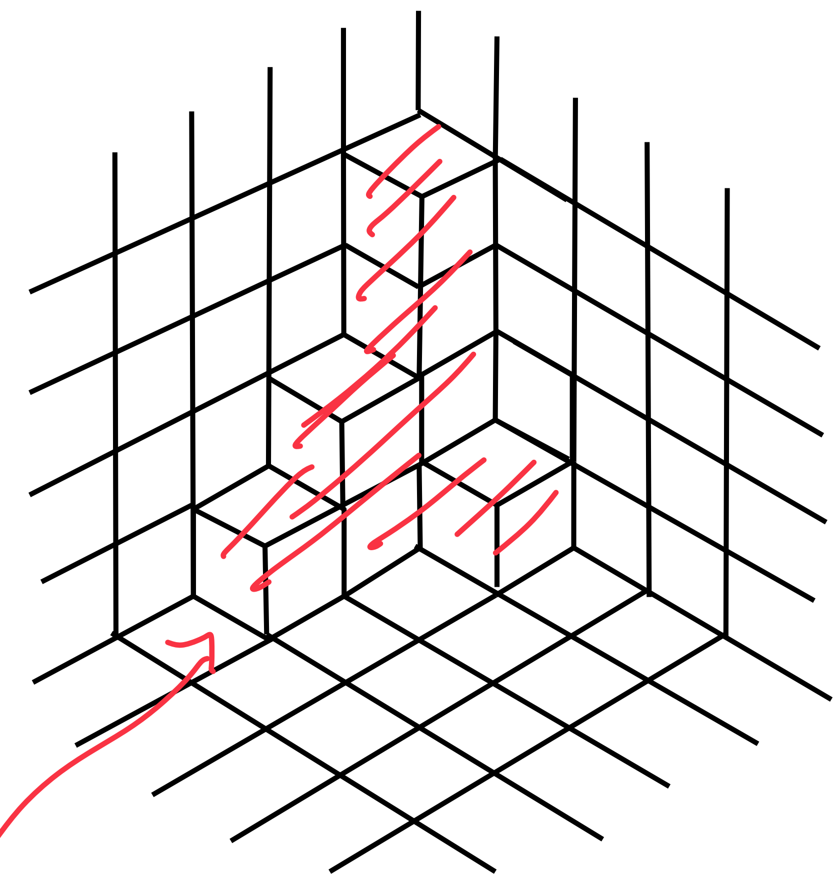
unmolten crystal

example: $\mathbb{C}^3$ [Okounkov-Reshetikhin-Vafa; Iqbal, Nekrasov,...]

plane partition



unmolten



molten

$I \subset \mathrm{Spec}[x,y,z]$

(Λ: finite set s.t. complement is an ideal I of path alg.)

The story generalizes to an arbitrary toric CY3

[Ooguri-MY '08'09]

See also [Szendroi; Mozgovoy, Reineke; Nagao, Nakajima; Ooguri, MY; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Vafa; ⋯]

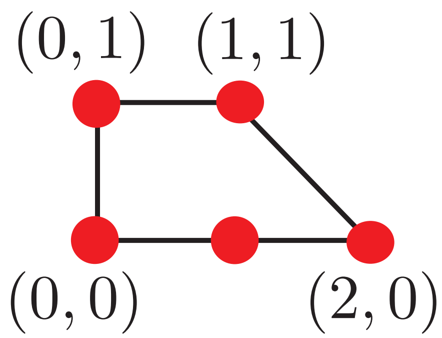
$$(Q, W) \rightsquigarrow \text{crystal } \mathcal{C} = (\text{path at a vertex}) / (\partial W) \subset \mathbb{C}Q/(\partial W)$$

open

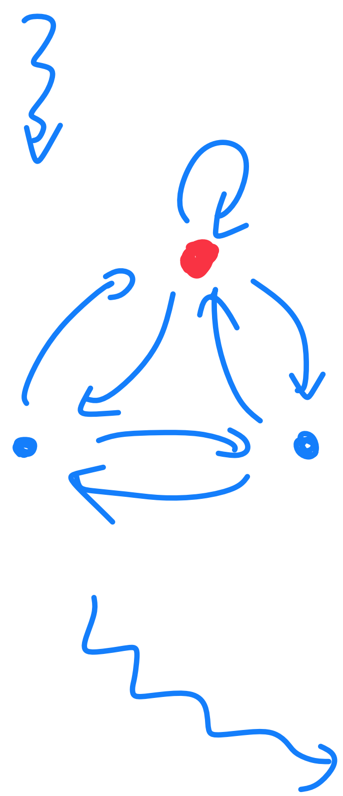
$$\rightsquigarrow \text{molten crystal} = \text{finite subset } \Lambda$$

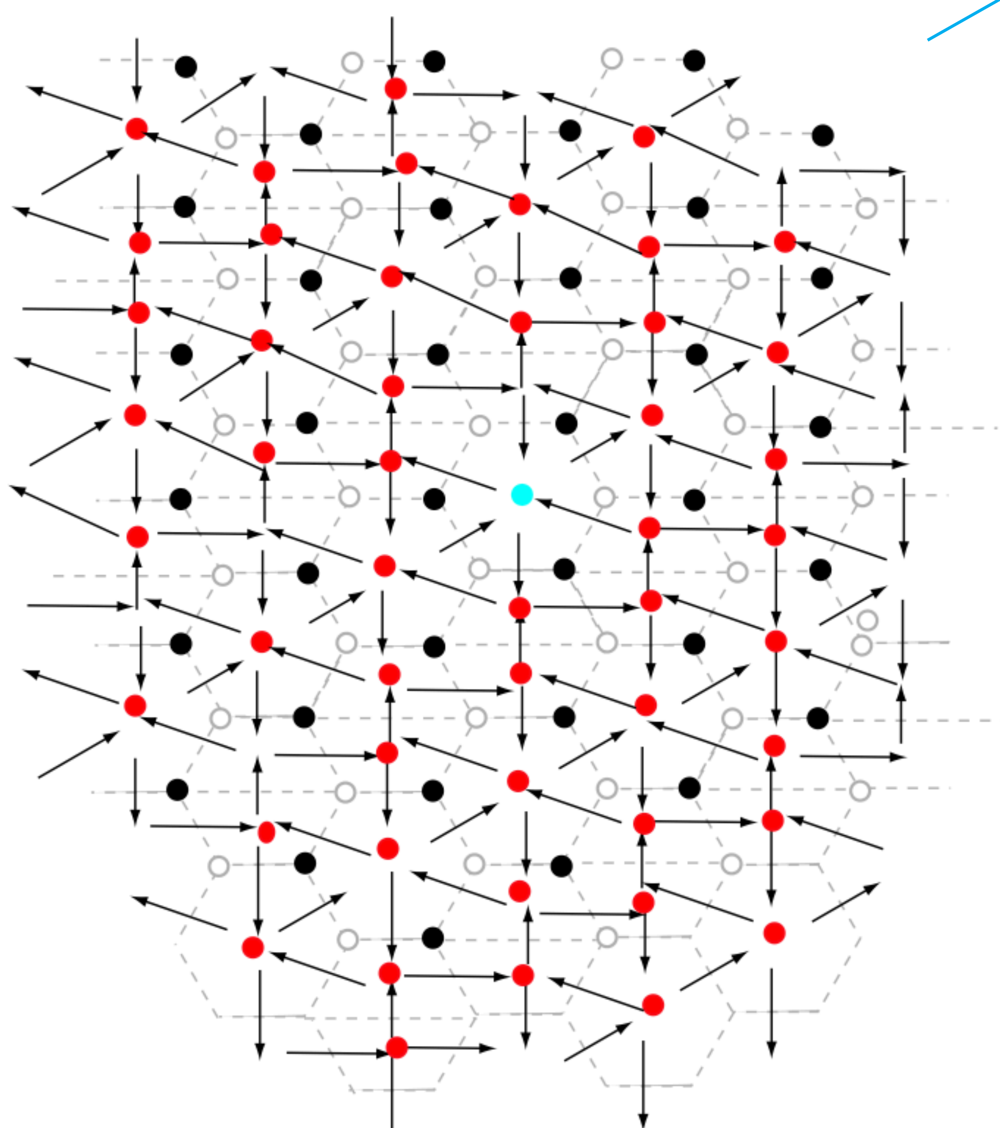
s.t. $\mathcal{C} \setminus \Lambda$: ideal of $\mathbb{C}Q/(\partial W)$

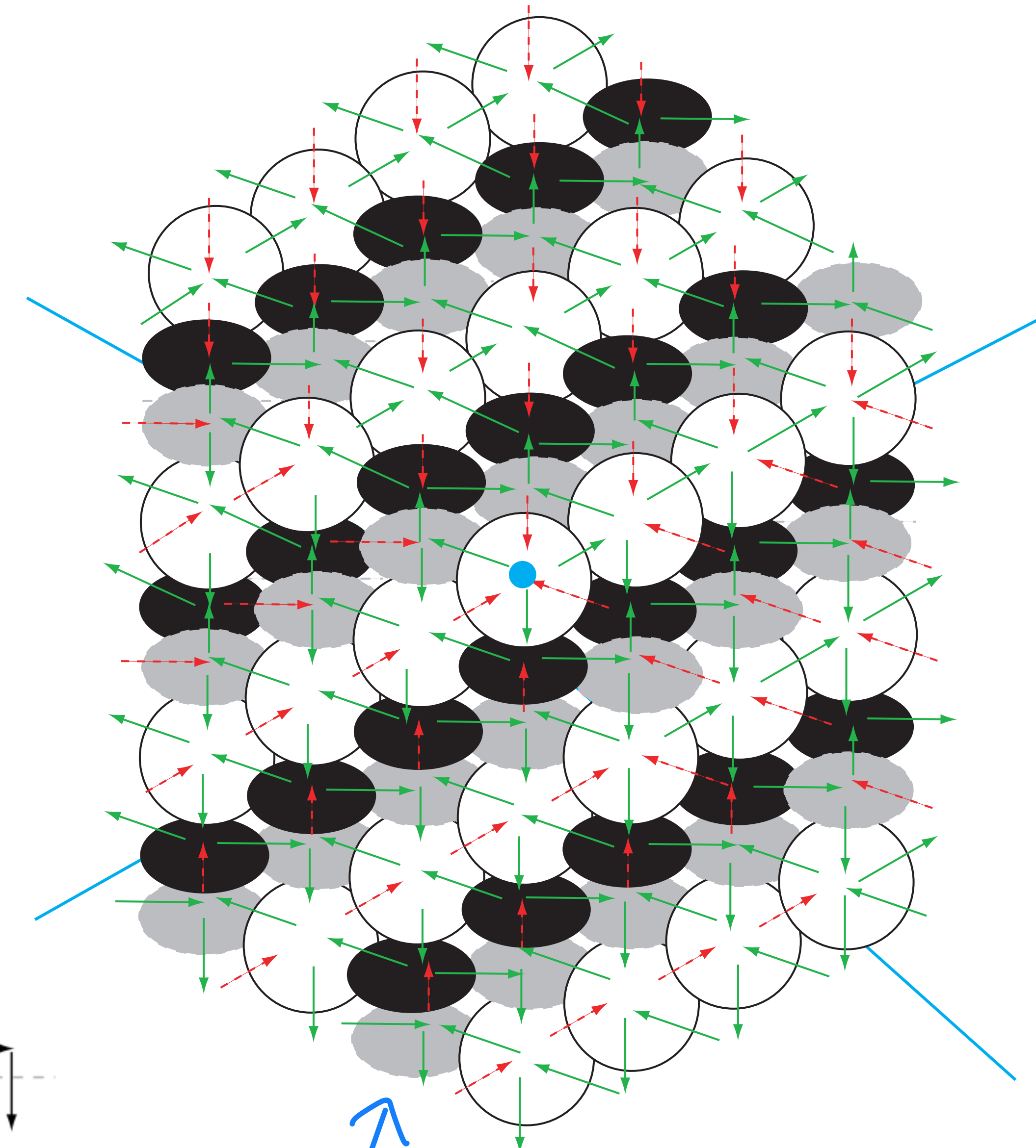
$CY_3 \qquad xy = zw^2$



$Q =$







Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \ ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a} \in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K} \to \mathrm{K} + \boxed{a})}{z - h(\boxed{a})} |\mathrm{K} + \boxed{a}\rangle \ ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a} \in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K} \to \mathrm{K} - \boxed{a})}{z - h(\boxed{a})} |\mathrm{K} - \boxed{a}\rangle \ ,$$

add/remove an atom

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi_{\mathrm{K}}^{(a)}(z)|\mathrm{K}\rangle \ ,$$

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$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\rightarrow \mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \ ,$$

poles for atom $\boxed{a}$

add/remove an atom

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poles for atom $\boxed{a}$

$$h(\boxed{a}) \equiv \sum_{I\in \mathrm{path}[\mathfrak{o}\to \boxed{a}]} h_I \; .$$

$\Psi_{\mathrm{K}}^{(a)}$:
$$\Psi_{\mathrm{K}}^{(a)}(u) = \psi_0^{(a)}(z) \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{b\Rightarrow a}(u-h(\boxed{b})) \; ,$$

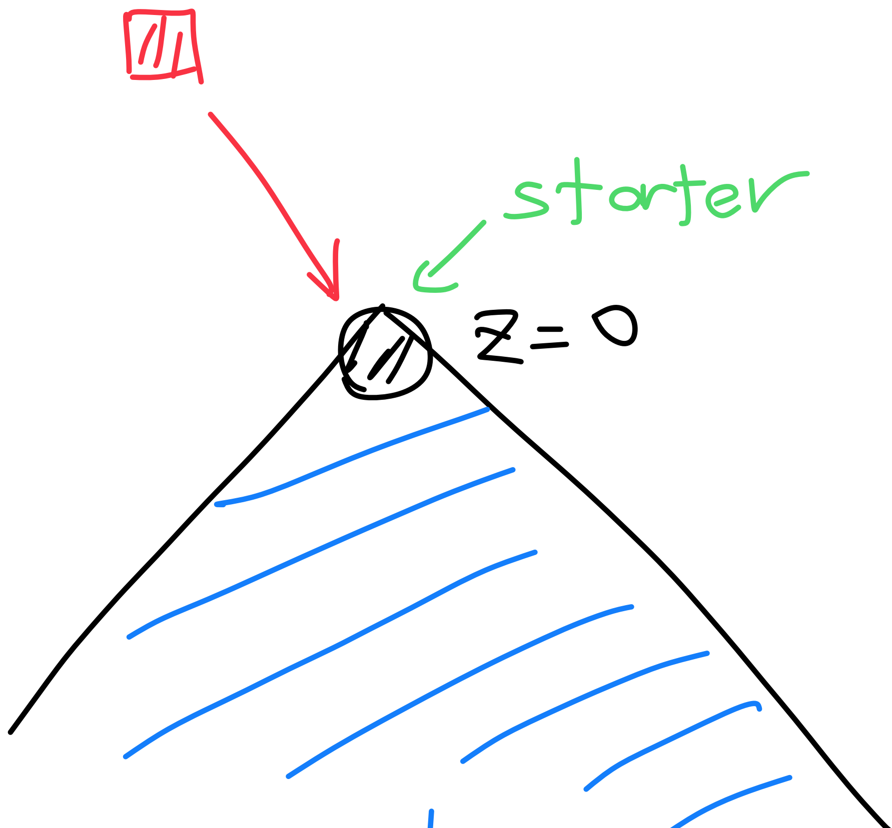
$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

$E^{(a)}/F^{(a)}$:
$$E^{(a)}/F^{(a)} = \sqrt{\pm \operatorname*{Res}_{u=h(\boxed{a})} \Psi_{\mathrm{K}}^{(a)}(u)}$$

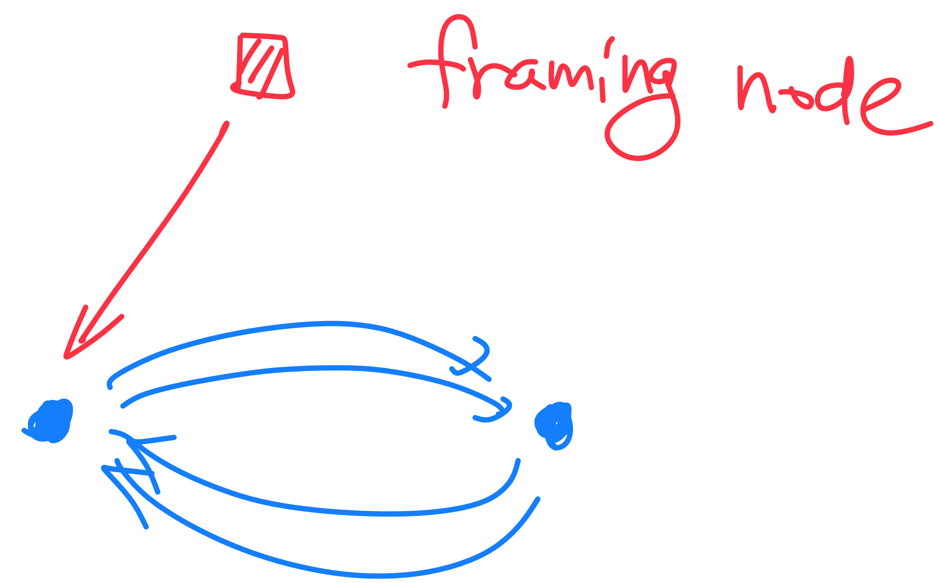
$$\psi^{(a)}(z)\,|\emptyset\rangle = \psi_0^{(a)}(z)\,|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation



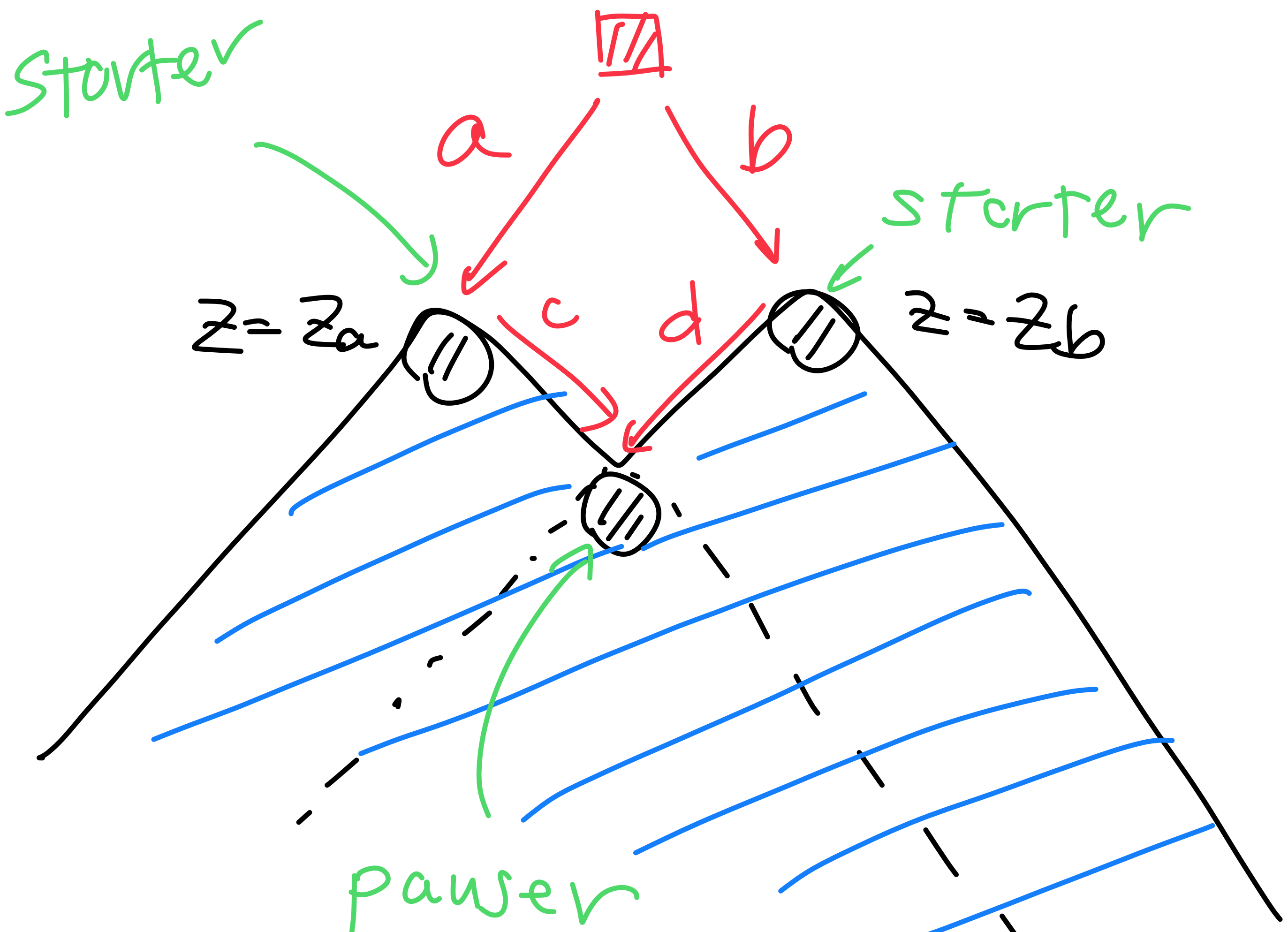
$\psi_0 \sim \frac{1}{z}$



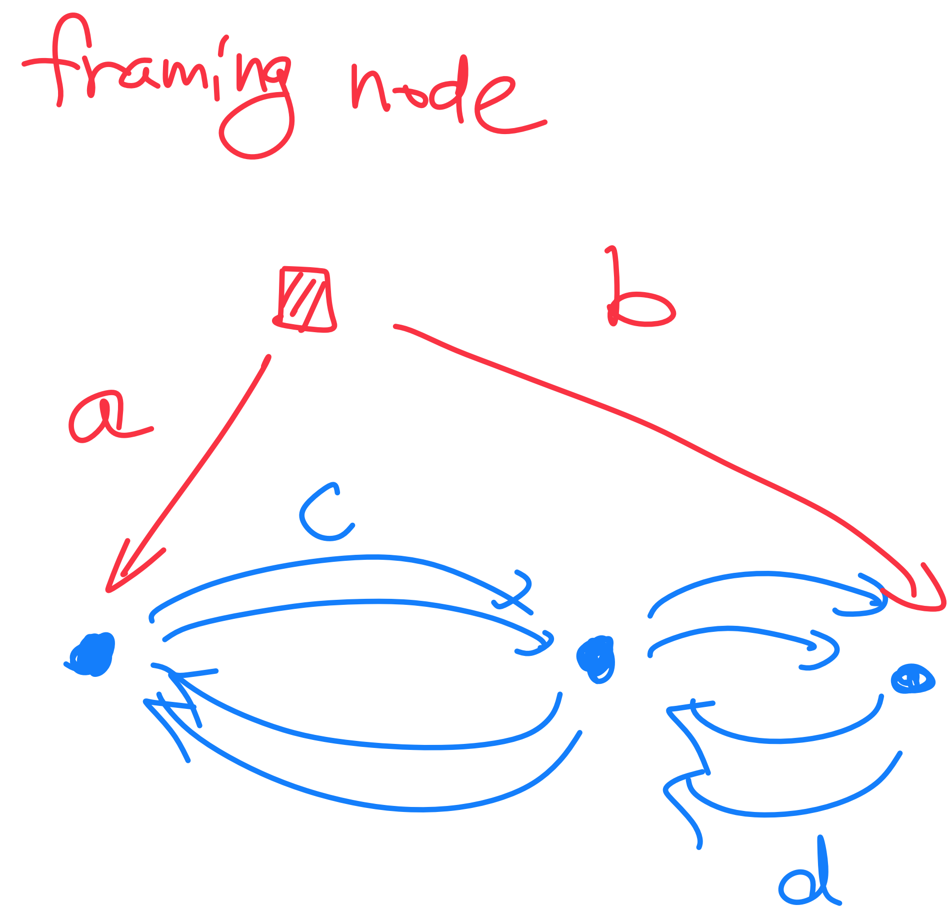
$$\psi^{(a)}(z)|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation



$$\psi_0 \sim \frac{z-z_c}{(z-z_a)(z-z_b)}$$



rel.

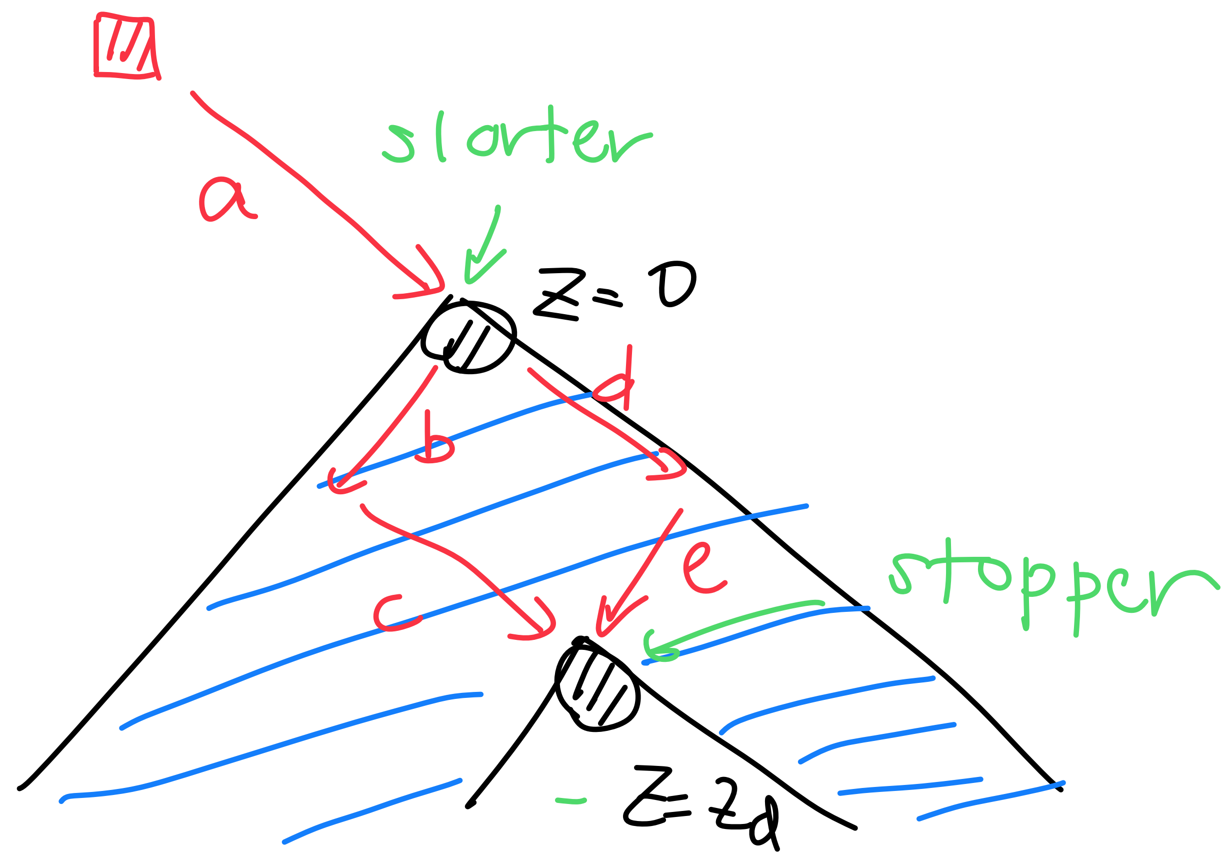
$$ca = db$$

$$\psi^{(a)}(z)|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}_{\text{vacuum charge function}}|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation

$$\psi_0 \sim \frac{z-z_d}{z}$$



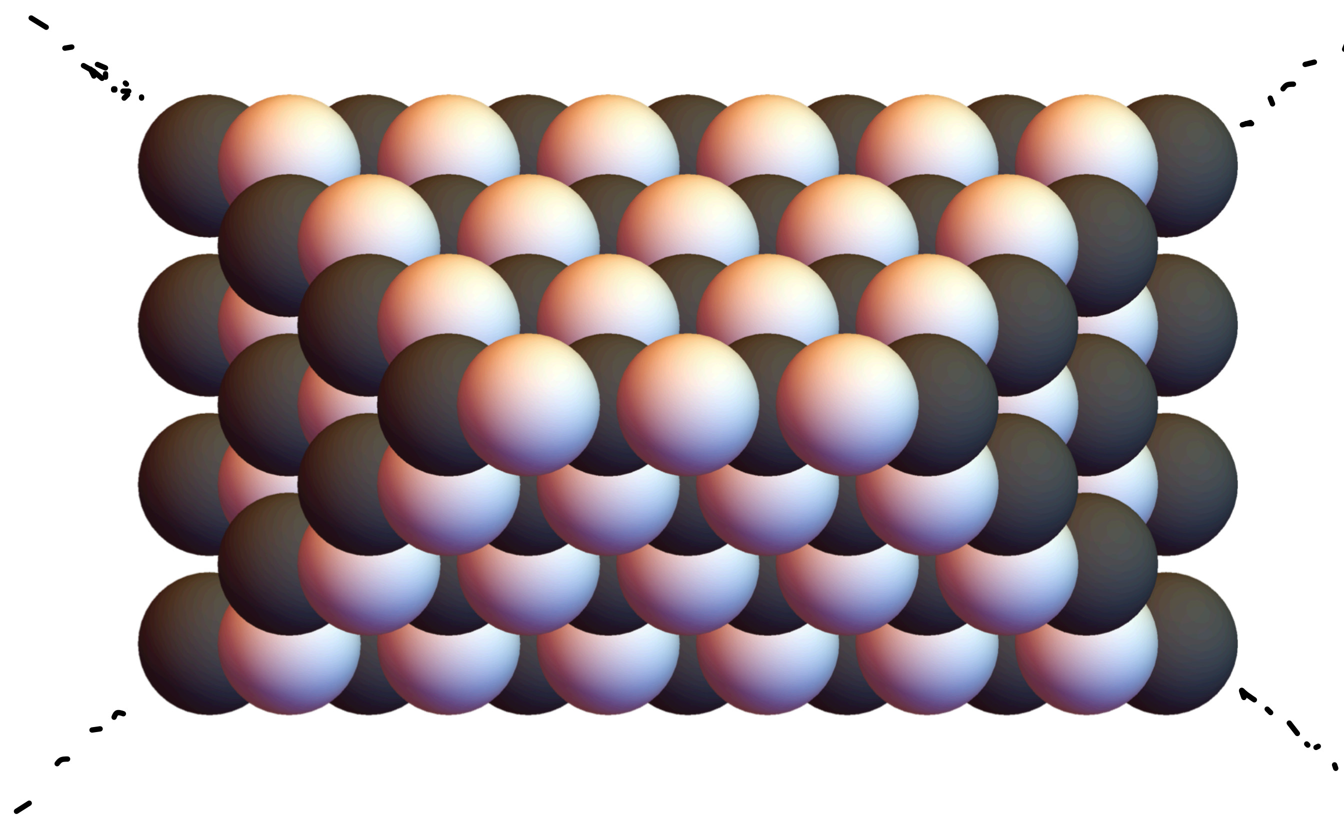
extra relation: $cba = eda = 0$

$$W \ni \phi\, cba$$

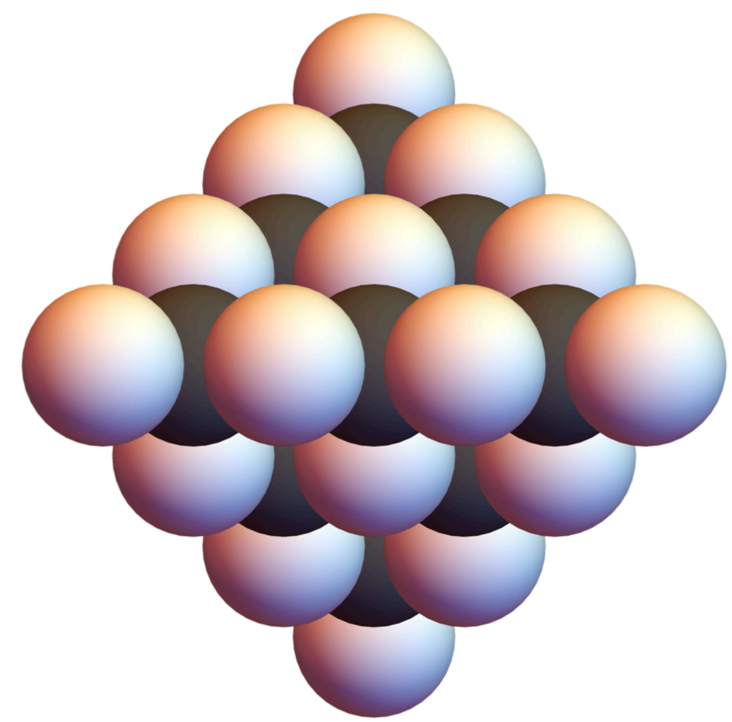
"Lagrange multiplier"

We can obtain rather general reps by using starter / pauser / stoppers

e.g. open / closed BPS state counting and their wall crossings



conifold : ∞-chamber



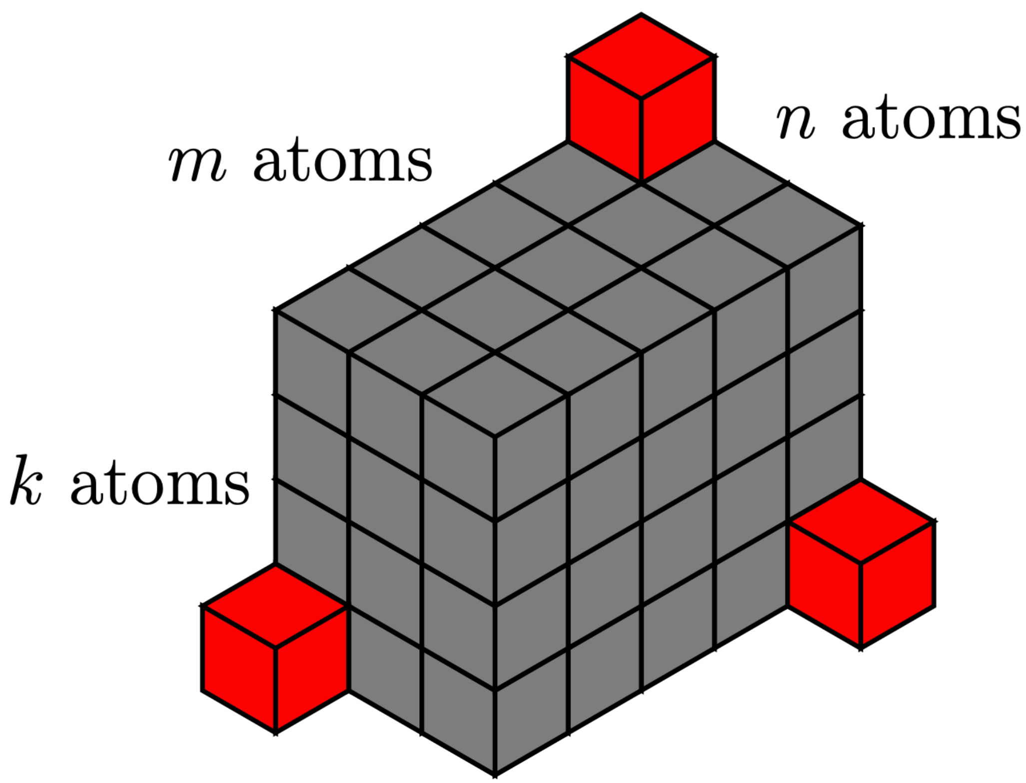
conifold : finite chamber

[Nagao-Nakajima; Jafferis-Moore; Chuang-Jafferis,⋯ '08]

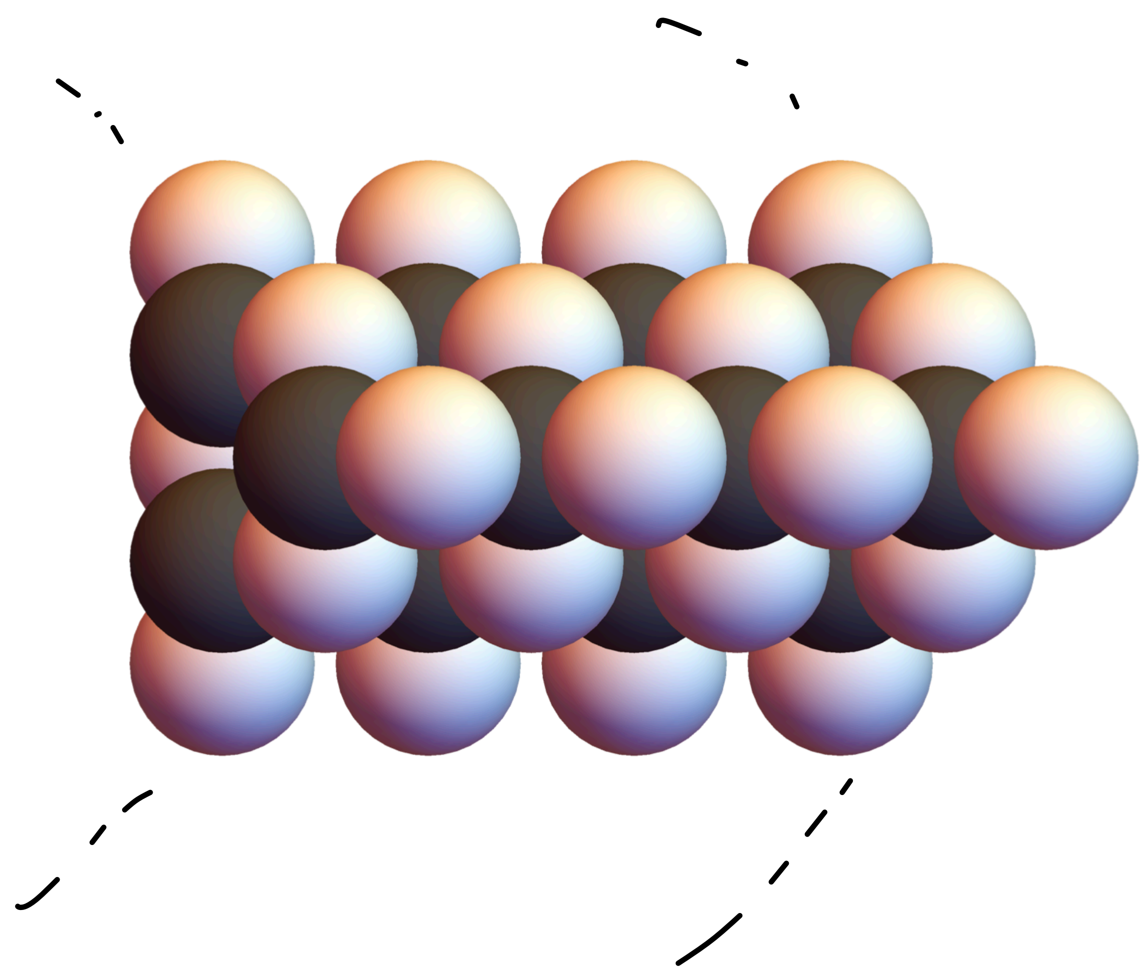
Some representations have no known

CY_3/geometry counterparts

$Y(\hat{gl}_1)$ $\mathbb{C}^3$-like



$Y(\hat{gl}_{1|1})$ conifold-like



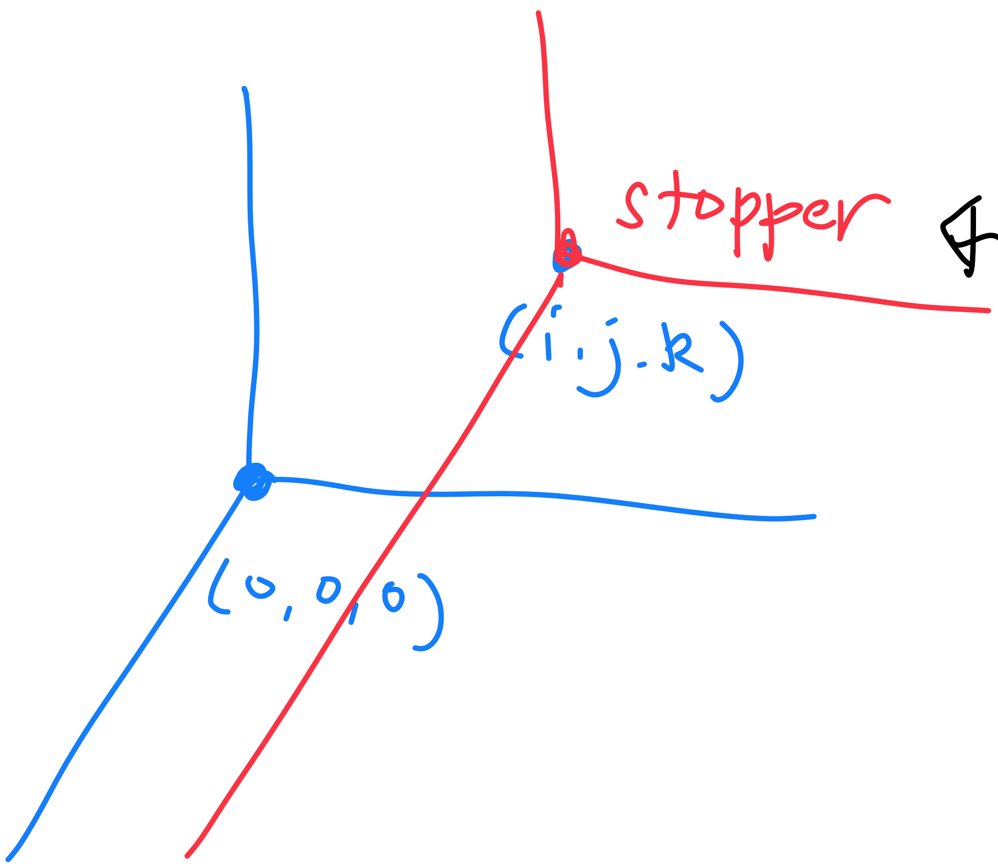
semi-infinite

[Galakhov-Li-MY '21]

Special truncations happen at non-generic equivariant parameters

e.g. $\mathbb{C}^3$ $\quad \psi_0(z) = \dfrac{z-C}{z}$ ← stopper at $z=-C$

starter at $z=0$



$C \in L h_1 + N h_2 + M(-h_1-h_2)$

$$Y(\widehat{\mathfrak{gl}}_1) \longrightarrow Y_{L,M,N}$$

$$\uparrow$$

$$\frac{\mathcal{W}_{N-M,1\cdots1|1\cdots1}[U(M|L)_\Psi]}{\mathcal{W}(M|L)_{\Psi-1}}$$

"VOA at the corner"

[Gaiotto-Rapcak, ... Genra, Nakatsuka, ...]

We can derive quiver Yangian representation

by equivariant localization in SUSY QM

[Galakhov-MY '20]

$(Q, W) \rightsquigarrow$ SUSY QM

We can <u>derive</u> quiver Yangian representation by equivariant localization in SUSY QM

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$(Q, W) \rightsquigarrow$ SUSY QM $\rightsquigarrow$ $\mathcal{U}_{\text{VAC}}$, Q supercharge $(\sim d)$

We can derive quiver Yangian representation

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[Galakhov-MY '20]

(Q, W) $\rightsquigarrow$ SUSY QM $\rightsquigarrow$ $\mathcal{M}_{\mathrm{VAC}}$

Q supercharge $(\sim d)$

$\Downarrow$

Ω-deformation by

equiv. param. $\{h_I\}$

fixed pts : crystal Λ

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effective wavefunction

$\Psi_\Lambda \sim \mathrm{Eul}_\Lambda$

We can derive quiver Yangian representation by equivariant localization in SUSY QM

[Galakhov-MY '20]

(Q, W) $\rightsquigarrow$ SUSY QM $\rightsquigarrow$ $\mathcal{M}_{\mathrm{VAC}}$

Q supercharge ($\sim d$)

Ω-deformation by equiv. param. $\{h_I\}$

fixed pts : crystal Λ

effective wavefunction

$\Psi_\Lambda \sim \mathrm{Eul}_\Lambda$

e/f generators: "Hecke modification"

$\langle \Psi_{\Lambda+\square} | e | \Psi_\Lambda \rangle$

FMT in $\mathcal{M}_\Lambda \times \mathcal{M}_{\Lambda+\square}$

We obtain algebras / repr. by

equiv. localization of $SQM_{(Q,W)}$

In all cases reproduce $Y_{(Q,W)}$

but no general proof

[Galakhov-MY '20; Galakhov-Li-MY '20]

Highly non-trivial cancellations!

[Galakhov-MY '20]

For example, for one of the Serre relations of $Y(\widehat{\mathfrak{gl}}_{3|1})$

$$\mathrm{Sym}_{z_1,z_2}\left[e^{(2)}(z_1),\left[e^{(3)}(w_1),\left[e^{(2)}(z_2),e^{(1)}(w_2)\right\}\right\}\right\}$$

$$A_2 := \operatorname*{Res}_{z_1,z_2,w_1,w_2}\langle\Lambda|A_1|\Lambda_0\rangle =$$

$$= [1,2,4,3]+[1,3,4,2]-[2,1,3,4]+[2,1,4,3]-[2,3,1,4]+[2,4,1,3]+$$
$$+[2,4,3,1]-[3,1,2,4]+[3,1,4,2]-[3,2,1,4]+[3,4,1,2]+[3,4,2,1]-$$
$$-[4,1,2,3]-[4,1,3,2]-[4,2,1,3]-[4,3,1,2] = 0\,!$$

$$[2,4,1,3]=-\frac{1}{48}\,,\quad [4,2,1,3]=-\frac{1}{96}\,,\quad [2,1,4,3]=-\frac{1}{48}\,,\quad [1,2,4,3]=\frac{1}{32}\,,$$
$$[4,1,2,3]=\frac{1}{64}\,,\quad [1,4,2,3]=\frac{1}{64}\,,\quad [4,1,3,2]=-\frac{1}{64}\,,\quad [1,4,3,2]=-\frac{1}{64}\,,$$
$$[2,4,3,1]=\frac{2\hbar_1+\hbar_2}{24\,(4\hbar_1+\hbar_2)}\,,\quad [4,2,3,1]=\frac{2\hbar_1+\hbar_2}{48\,(4\hbar_1+\hbar_2)}\,,$$
$$[2,3,4,1]=\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,\quad [3,2,4,1]=-\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[4,3,2,1]=-\frac{2\hbar_1+\hbar_2}{48\,(4\hbar_1+\hbar_2)}\,,\quad [3,4,2,1]=-\frac{(2\hbar_1+\hbar_2)^2}{24\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[2,1,3,4]=-\frac{2\hbar_1+\hbar_2}{24\,(4\hbar_1+3\hbar_2)}\,,\quad [1,2,3,4]=\frac{2\hbar_1+\hbar_2}{16\,(4\hbar_1+3\hbar_2)}\,,$$
$$[2,3,1,4]=\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,\quad [3,2,1,4]=-\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[1,3,2,4]=-\frac{2\hbar_1+\hbar_2}{16\,(4\hbar_1+3\hbar_2)}\,,\quad [3,1,2,4]=\frac{(2\hbar_1+\hbar_2)^2}{8\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[4,3,1,2]=\frac{2\hbar_1+\hbar_2}{32\,(4\hbar_1+\hbar_2)}\,,\quad [3,4,1,2]=\frac{(2\hbar_1+\hbar_2)^2}{16\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[1,3,4,2]=-\frac{2\hbar_1+\hbar_2}{32\,(4\hbar_1+3\hbar_2)}\,,\quad [3,1,4,2]=\frac{(2\hbar_1+\hbar_2)^2}{16\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}.$$

Summary

String theory

toric CY_3 : DT inv.

Quiver Yangian

$Y(Q,W)$

new algebras

quiver w/ rel.

(Q,W)

repr. in crystal melting

new repr.