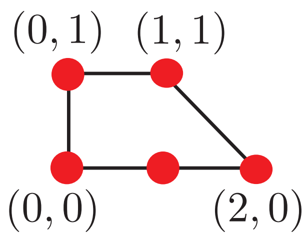
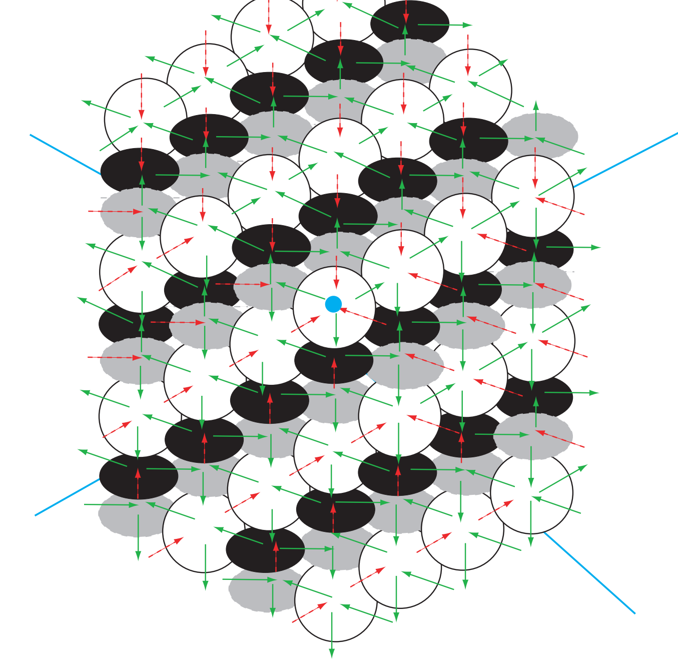


$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,, \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,, \\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,, \\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,, \\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,, \\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,,
\end{aligned}
$$



Crystal Meltings Revisited

Masahito Yamazaki

KAVLI IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

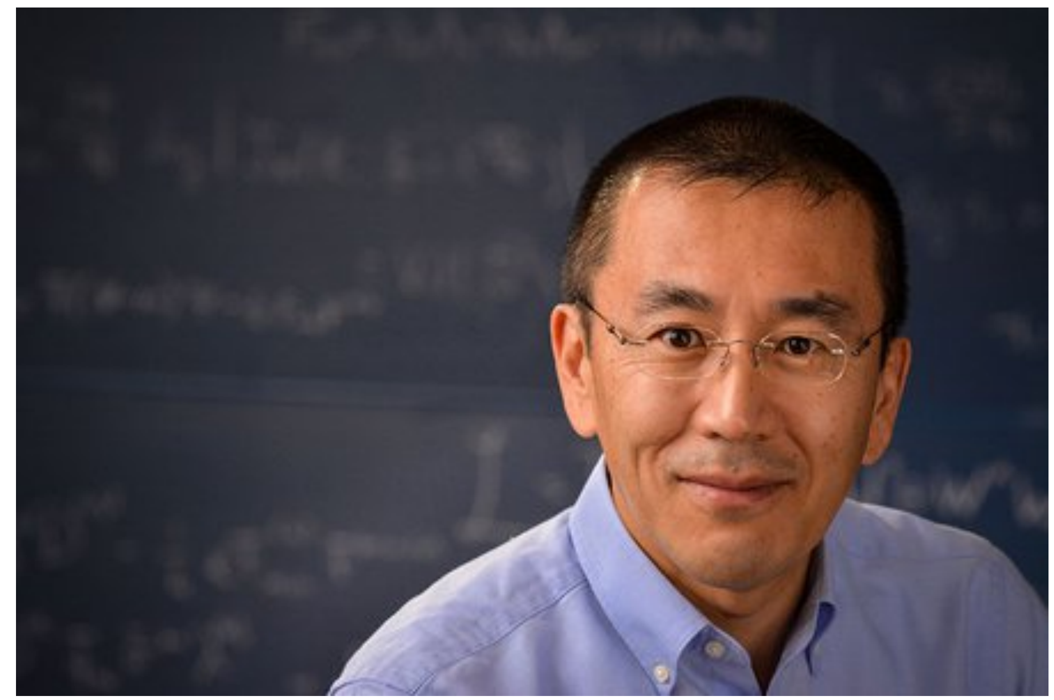
Hirosifest @ Kavli IPMU

October 20, 2022

Turning Point

Through a chain of unexpected events,
I became Hirosi's student

IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE



This is one of the best things
which happened in my life

and has opened up a whole new world to me…


MathSciNet
fedora

I coauthored 4 papers with Hirosi which became the basis of my Ph.D. thesis

Communications in Mathematical Physics

0811.2801

Crystal Melting and Toric Calabi-Yau Manifolds

Hirosi Ooguri[1,2], **Masahito Yamazaki**[1,2,3]

[1] California Institute of Technology, 452-48, Pasadena, CA 91125, USA
[2] Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8586, Japan

PRL **102,** 161601 (2009) PHYSICAL REVIEW LETTERS week ending 24 APRIL 2009

0902.3996

Emergent Calabi-Yau Geometry

Hirosi Ooguri[1,2] and Masahito Yamazaki[1,2,3]

[1]*California Institute of Technology, Pasadena, California 91125, USA*
[2]*Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8586, Japan*
[3]*Department of Physics, University of Tokyo, Hongo 7-3-1, Tokyo 113-0033, Japan*
(Received 27 February 2009; published 21 April 2009)

Crystal Melting

Generalization of [Okounov-Reshetikhin-Vafa]

We studied BPS state counting problem

Type IIA on toric CY_3

+ on D0/D2/D4/D6-branes on 0/2/4/6-cycles

$\underbrace{\text{D0/D2/D4/D6-branes}}_{\text{charge } \gamma \in H^{\text{even}}}$

$$Z_{BPS} \equiv \sum_{\gamma} \underbrace{\Omega_{\gamma}}\, \underbrace{g^{\gamma}}$$

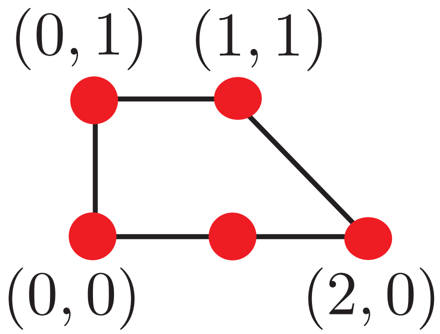
formal counting parameter

BPS degeneracy

$\shortparallel$

Donaldson-Thomas inv.

toric diagram $(xy = zw^2)$



$\Delta \subset \mathbb{Z}^2$

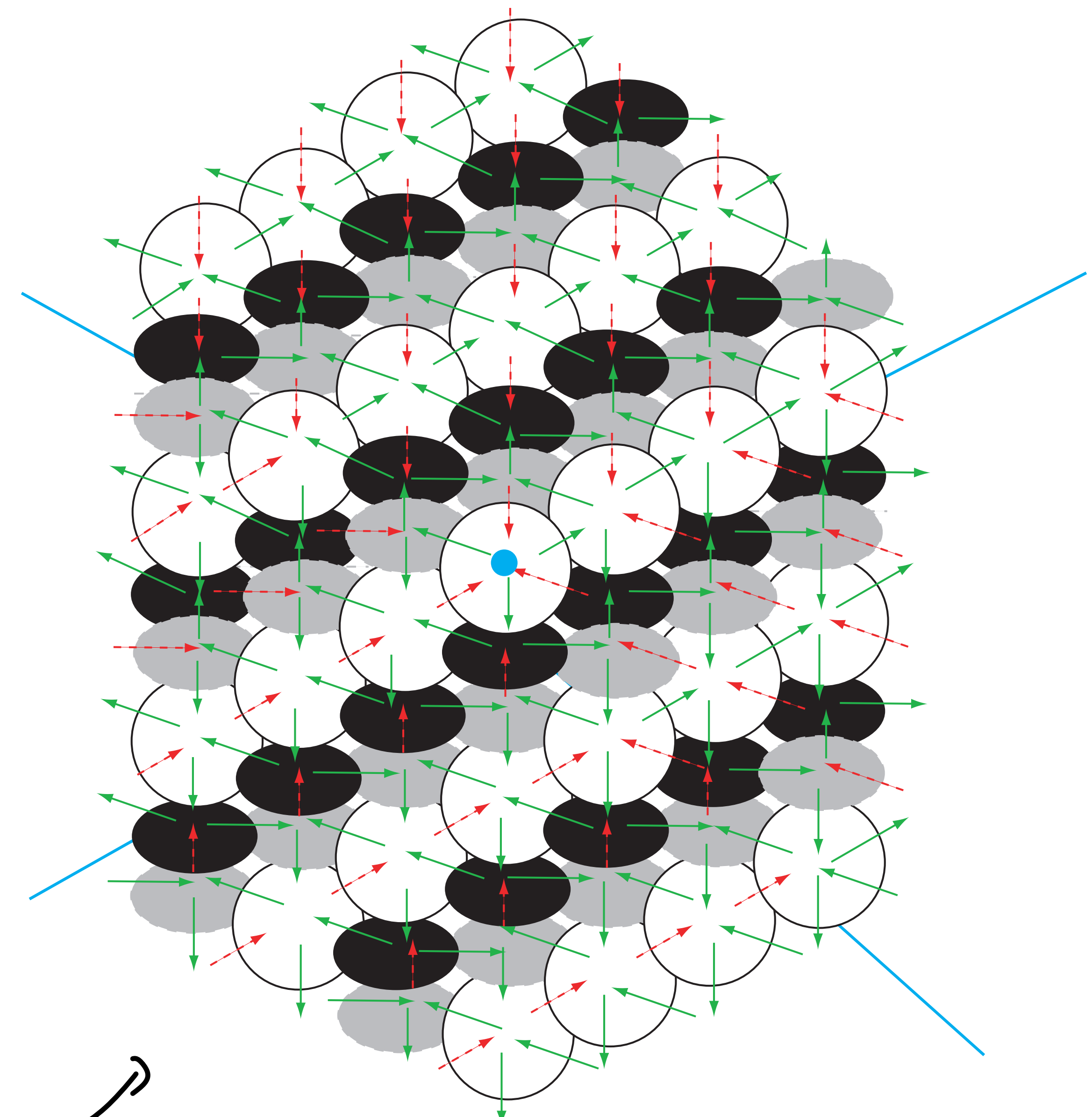
[Ooguri-MY '08]

Type IIA on toric CY_3

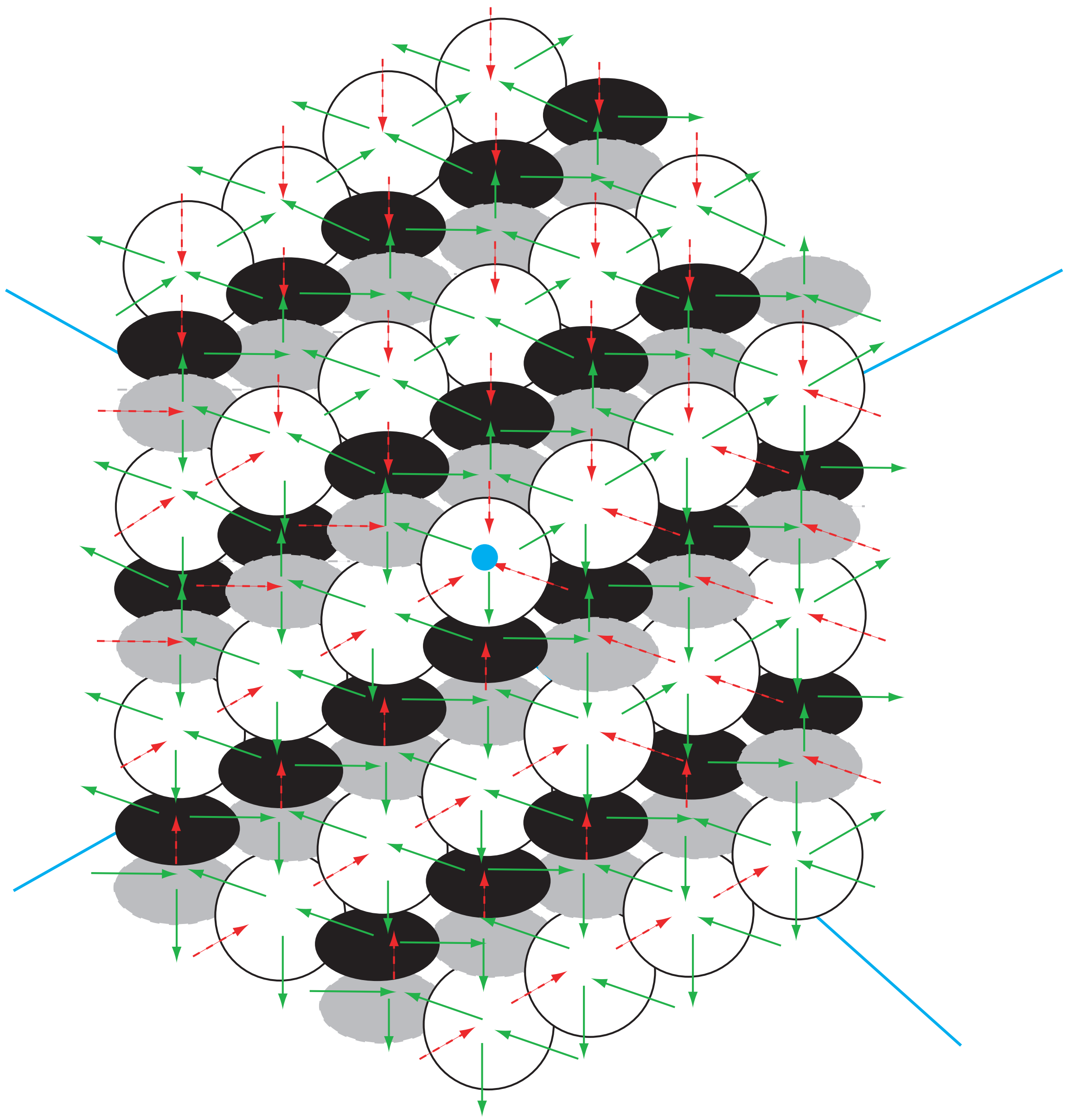
+ D6/D4/D2/D0

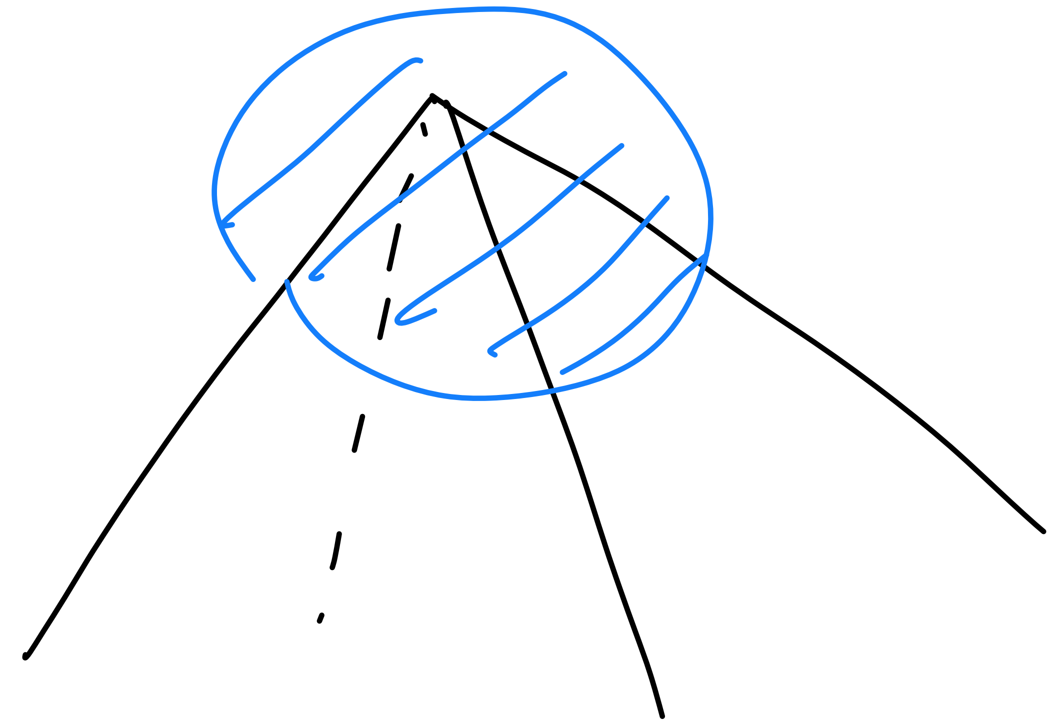
$\mathcal{N}=4$ SQM on D-brane

$\chi(\mathcal{M}_{\text{vacuum}})$

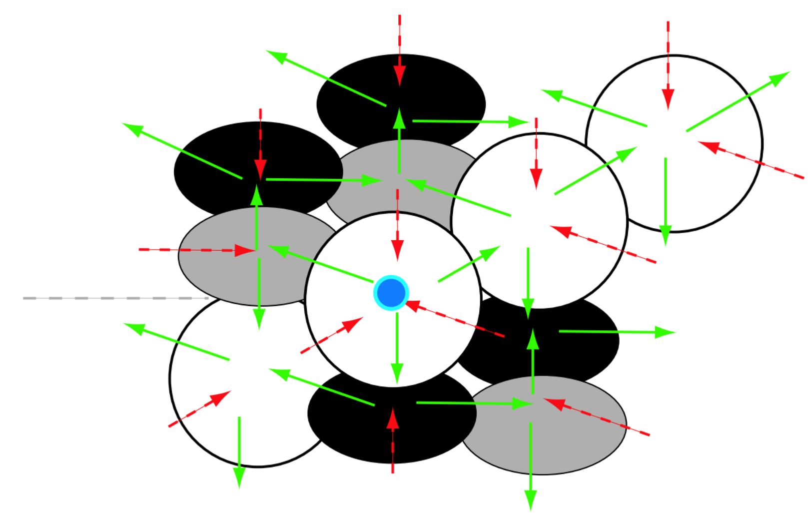


torus fixed point





"Take from top"



$$Z_{BPS} = Z_{crystal} = \sum_{\Lambda} q^{\Lambda}$$

For toric CY3 without compact 4-cycles, Z has an infinite-product form

[Szendroi, Young, Nagao,⋯]

conifold : $xy = zw$: $Z \sim \prod_{n \geq 0} (1 - Q q^n)^n$

SPP : $xy = zw^2$: $Z \sim \prod_{n \geq 0} \frac{(1 - Q_1 q^n)^n (1 - Q_1 Q_2 q^n)^n}{(1 - Q_2 q^n)^n}$

Why?

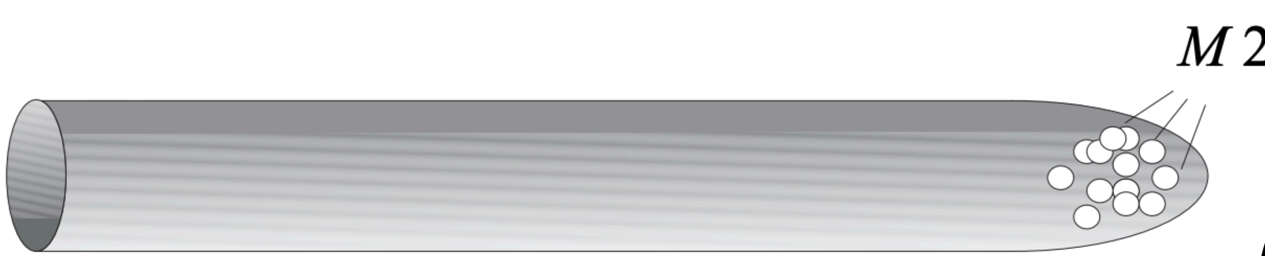
M-theory explanation of the infinite product

[Aganagic, Ooguri, Vafa, Y]



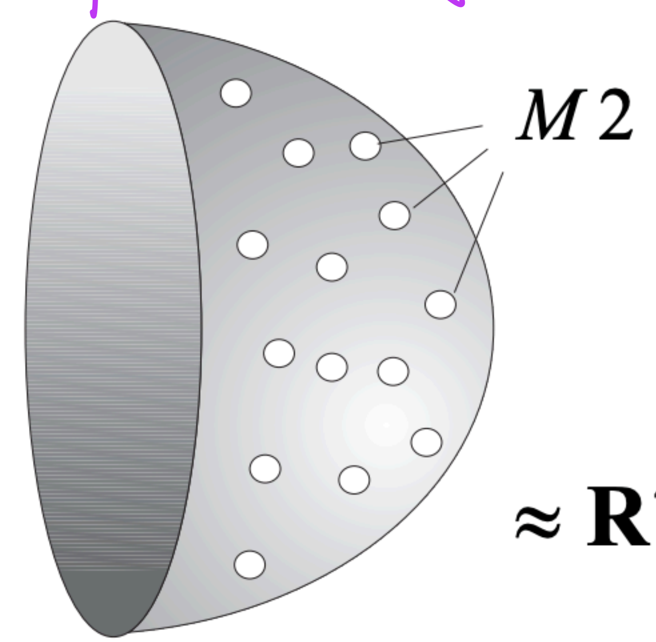

IIA on D0/D2$_1$/D6 (CY$_3$ has no 4-cycle)

M on TN$_1$ + M2 + KK



$\approx S^1 \times \mathbf{R}^3$ Taub-NUT

Spinning M2-brane particles



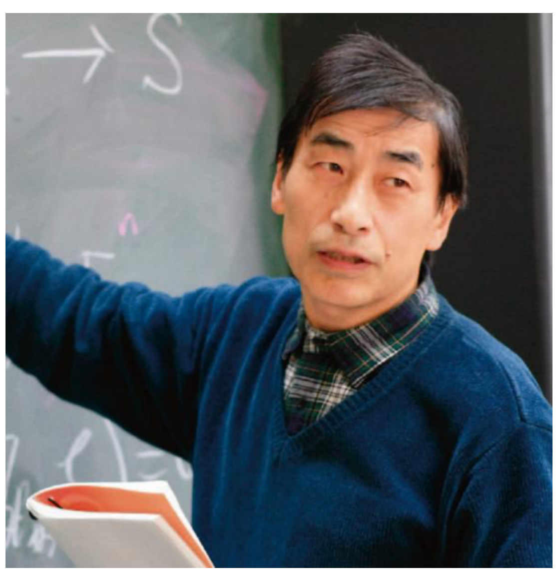
[Gaiotto, Strominger, Yin; Dijkraaf, Verlinde, Vafa]

Underlying Algebra?

$$Z \sim \prod_{n\geq 0} (1-Q_1 q^n)^n (1-Q_1 Q_2 q^n)^n (1-Q_2 q^n)^{-n}$$

$$\Big(\underset{\text{odd}}{n\delta+\alpha_1}, \ \underset{\text{even}}{n\delta+\alpha_2}, \ \underset{\text{odd}}{n\delta+\alpha_1+\alpha_2} \Big)?$$

[Nagao-MY] discussed chamber structures in terms of affine Weyl groups



Kyoji Saito

Elliptic !!

Michio Jimbo

Quantum toroidal !!

??

Later important developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by [B. Feigin, E. Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka, ⋯], also in higher spin algebras [Gaberdiel, Gopakumar; Li, Peng,⋯]

I now have the answer to my original question:

Quiver Yangian

Quiver Yangians

Based on

Wei Li + MY

(2003.08909 [hep-th])

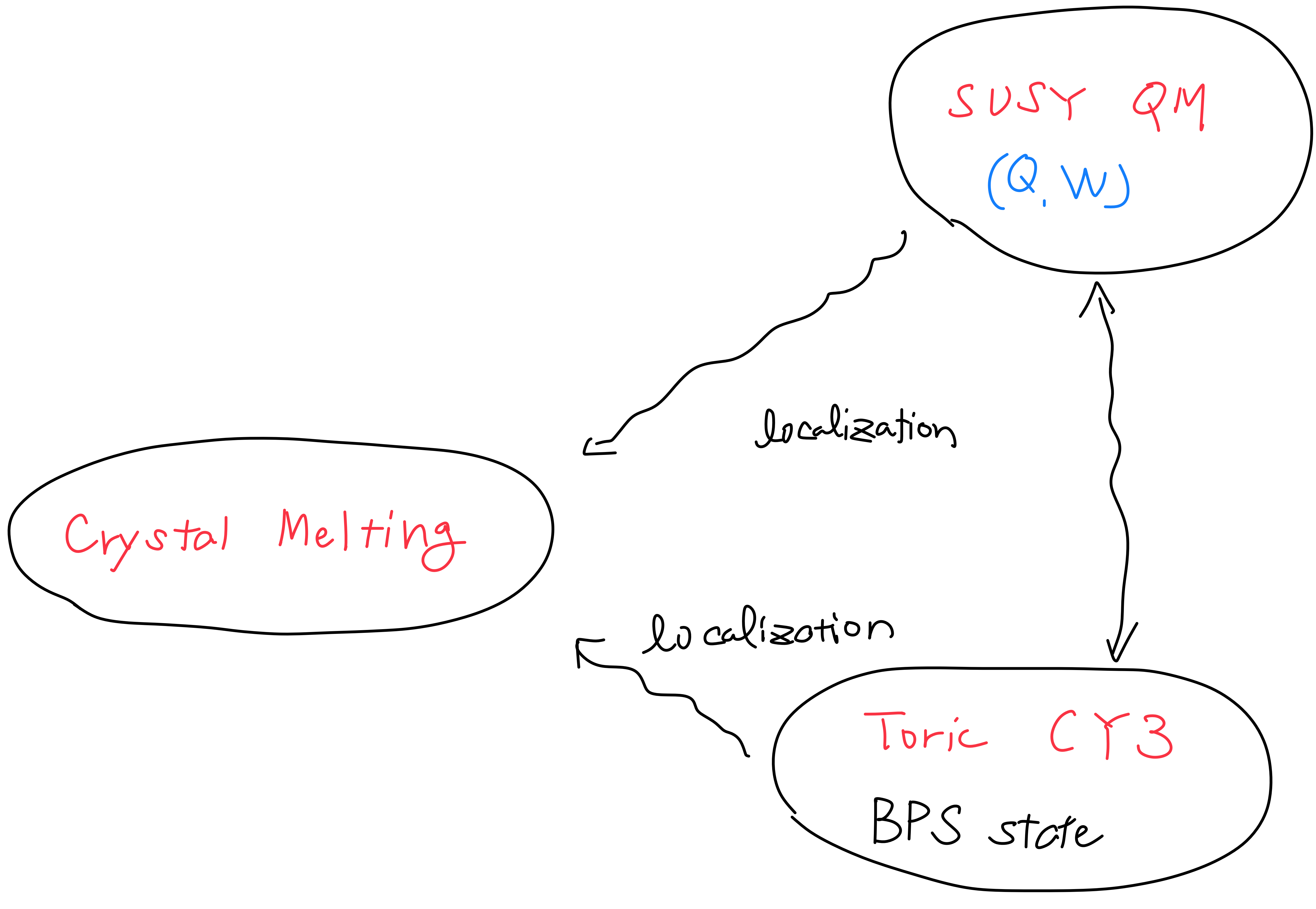
Dmitry Galakhov + MY

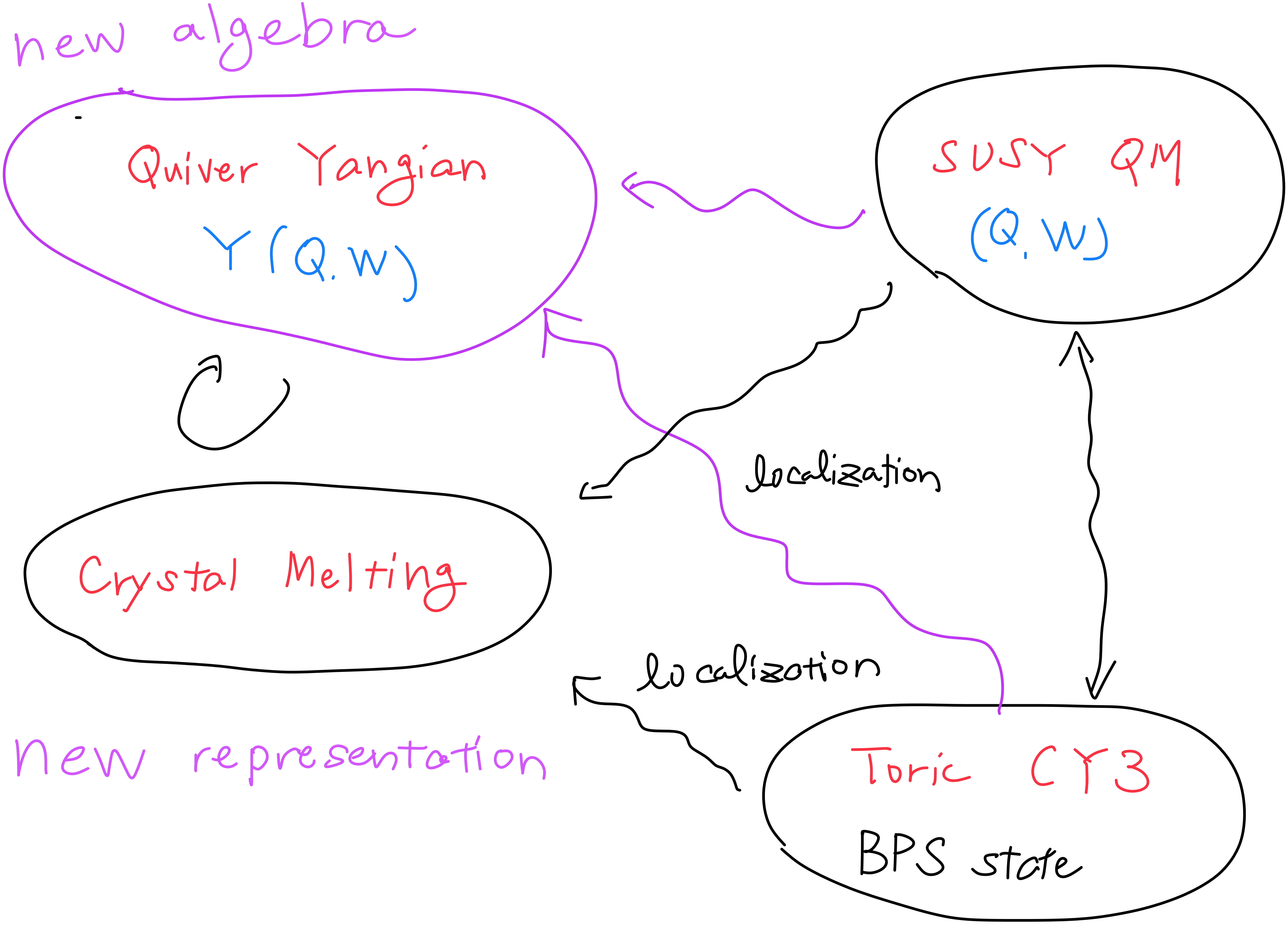
(2008.07006 [hep-th])

Dimitry Galakhov+Wei Li + MY

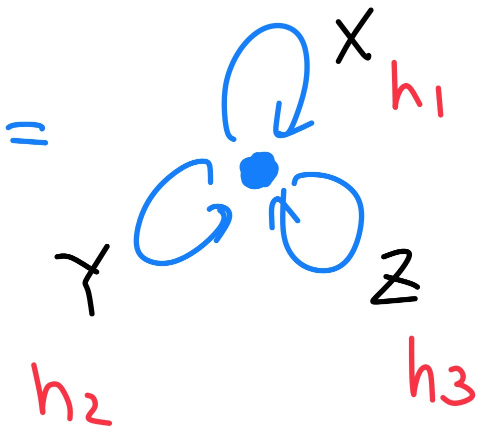
(2106.01230 [hep-th])

See Y (2206.13340 [hep-th] for review)


SUSY QM
(Q, W)
localization
Crystal Melting
localizotion
Toric CY3
BPS state


new algebra
Quiver Yangian
Y(Q,W)
SUSY QM
(Q,W)
localization
Crystal Melting
localization
new representation
Toric CY3
BPS state

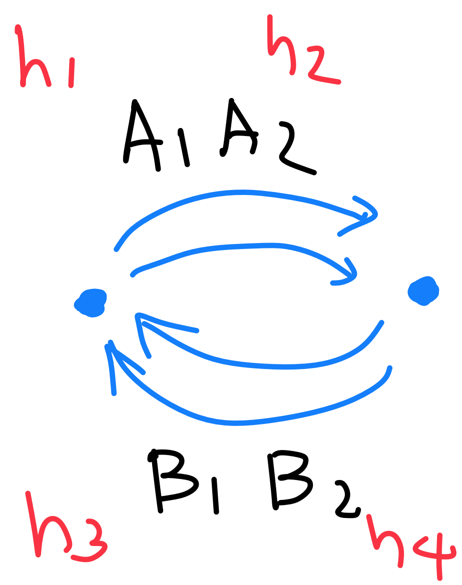
Quiver Q & Superpotential W ⟵ toric CY_3

* $Q =$ 

$W = \mathrm{Tr}(XYZ - XZY)$

$h_1 + h_2 + h_3 = 0$

$CY_3 = \mathbb{C}^3$

* $Q =$ 

$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

$h_1 + h_2 + h_3 + h_4 = 0$

$CY_3 =$ conifold

(h_I : flavor charges / equiv. param.)

Generators

(z : spectral parameter)

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}}, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}}, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}},$$

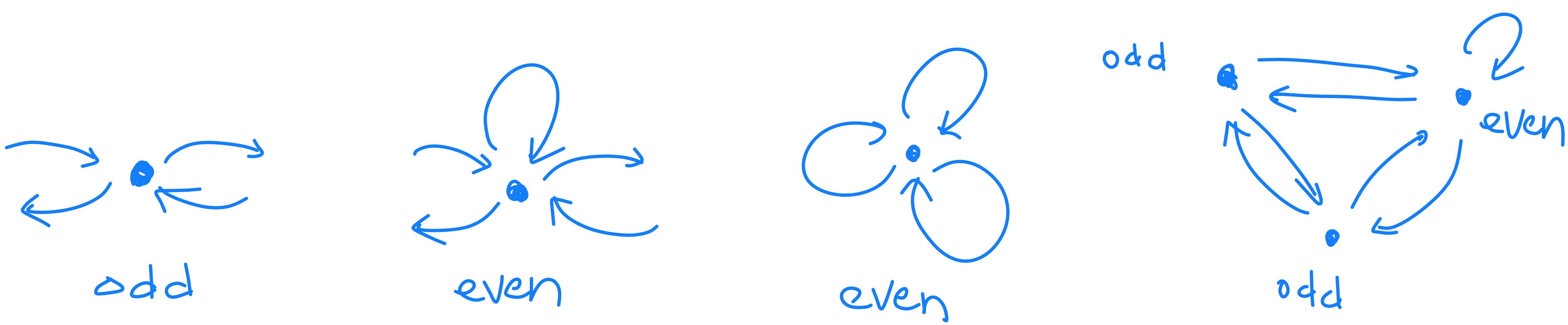
$n = -k$

"k- shifted Quiver Yangian"

a : quiver vertex

$\mathbb{Z}_2$- grading

$$|a| = \begin{cases} 0 & (\exists \text{ edge } I \text{ s.t. } I \text{ starts and ends at } a) \\ 1 & (\text{otherwise}) \end{cases}$$



Relations

$Y(Q,W)$

$$\psi^{(a)}(z)\,\psi^{(b)}(w) = \psi^{(b)}(w)\,\psi^{(a)}(z)\;,$$
$$\psi^{(a)}(z)\,e^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\;,$$
$$e^{(a)}(z)\,e^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\;,$$
$$\psi^{(a)}(z)\,f^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\;,$$
$$f^{(a)}(z)\,f^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\;,$$
$$\left[e^{(a)}(z), f^{(b)}(w)\right\} \sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\;,$$

"$\simeq$" means equality up to $z^n w^{m\geq 0}$ terms

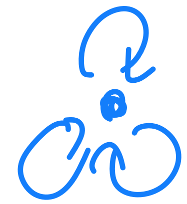
"$\sim$" means equality up to $z^{n\geq 0} w^m$ and $z^n w^{m\geq 0}$ terms

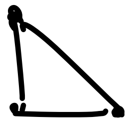
bonding factor

equivariant weight

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

edge

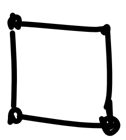
* $\mathbb{C}^3 \rightsquigarrow Q=$  $\rightsquigarrow Y(\widehat{\mathfrak{gl}}_1)$



$W=\mathrm{Tr}(XYZ-XZY)$

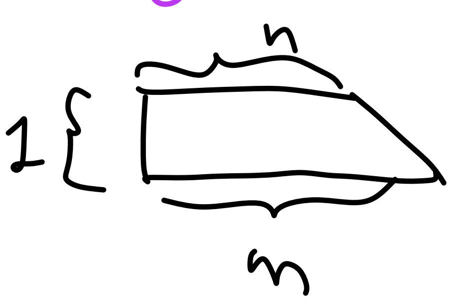
[Miki; Ding-Iohara;⋯
Tsymbaulik; Prochazka;
Gaberdiel, Gopakumar, Li, Peng,⋯]

* conifold $\rightsquigarrow Q=$  $\rightsquigarrow Y(\widehat{\mathfrak{gl}}_{1|1})$



$W=\mathrm{Tr}(A_1B_1A_2B_2-A_1B_2A_2B_1)$

* $xy=z^n w^m \rightsquigarrow Y(\widehat{\mathfrak{gl}}_{m|n})$ [Rapcak; Bezerra-Mukhin]

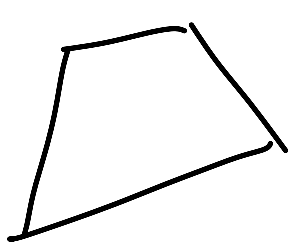


infinite product form

↑ Yes

* general toric $CY_3 \rightsquigarrow Y(Q,W)$

↓ No ?



$\Delta \subset \mathbb{Z}^2$

has no "$\mathfrak{g}$"

We can <u>derive</u> quiver Yangian representations

by "bootstrapping" from crystal

[Li-MY '20]

by equivariant localization in SUSY QM

[Galakhov-MY '20]

Representations from Crystal Melting

cf. earlier developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by [Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka; Rapcak; Gaberdiel, Gopakumar; Li, Peng,⋯]

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

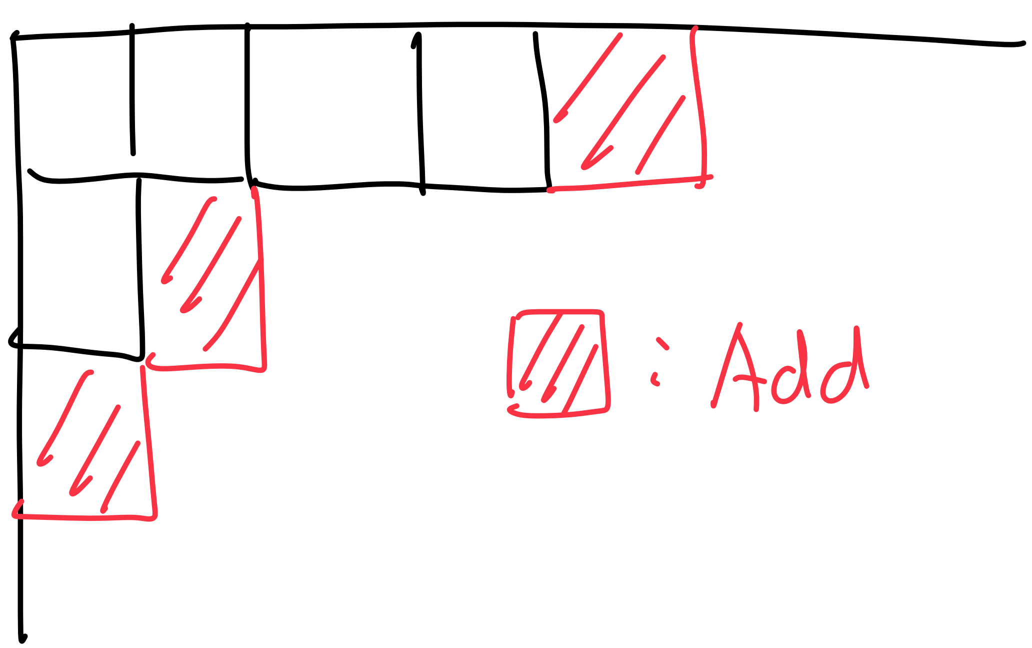
crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle ,$$

add/remove an atom



(2D crystal case)

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \ ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \ ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \ ,$$

poles for atom $\boxed{a}$

add/remove an atom

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \ ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \ ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \ ,$$

poles for atom $\boxed{a}$

$$h(\boxed{a}) \equiv \sum_{I\in \mathrm{path}[\mathfrak{o}\to\boxed{a}]} h_I \ .$$

$\Psi_K^{(a)}$:
$$\Psi^{(a)}_{\mathrm{K}}(u) = \psi^{(a)}_0(z) \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{b\Rightarrow a}(u-h(\boxed{b})) \ ,$$

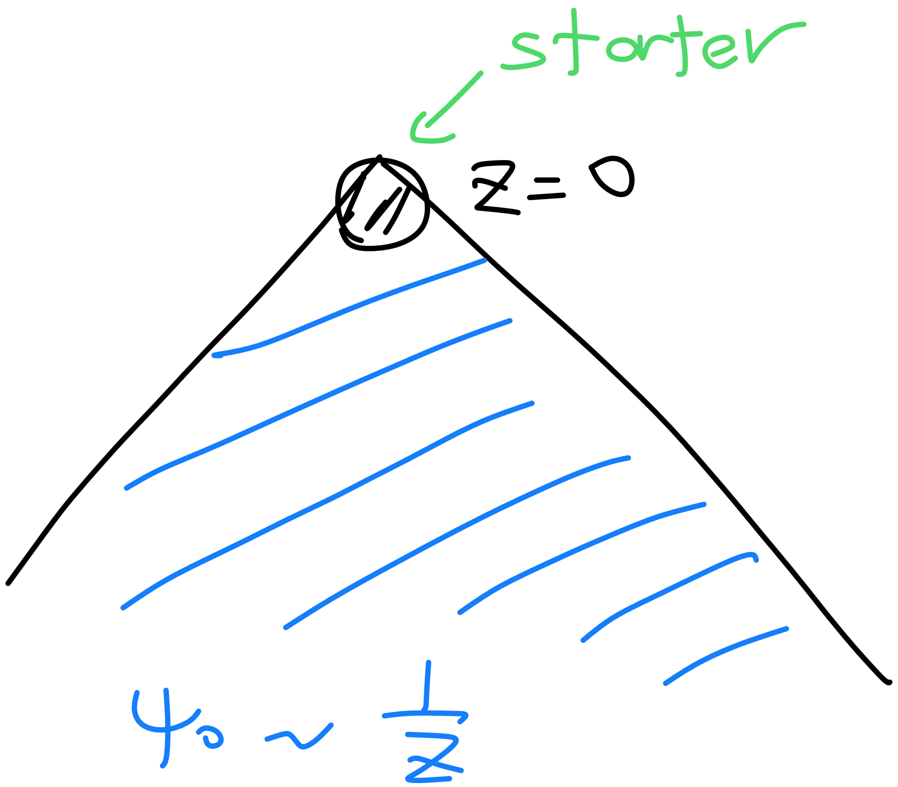
$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

$E^{(a)}/F^{(a)}$:
$$E^{(a)}/F^{(a)} = \sqrt{\pm \operatorname*{Res}_{u=h(\boxed{a})} \Psi_K^{(a)}(u)}$$

$$\psi^{(a)}(z)|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}|\emptyset\rangle$$

[Galakhov-Li-MY '21]

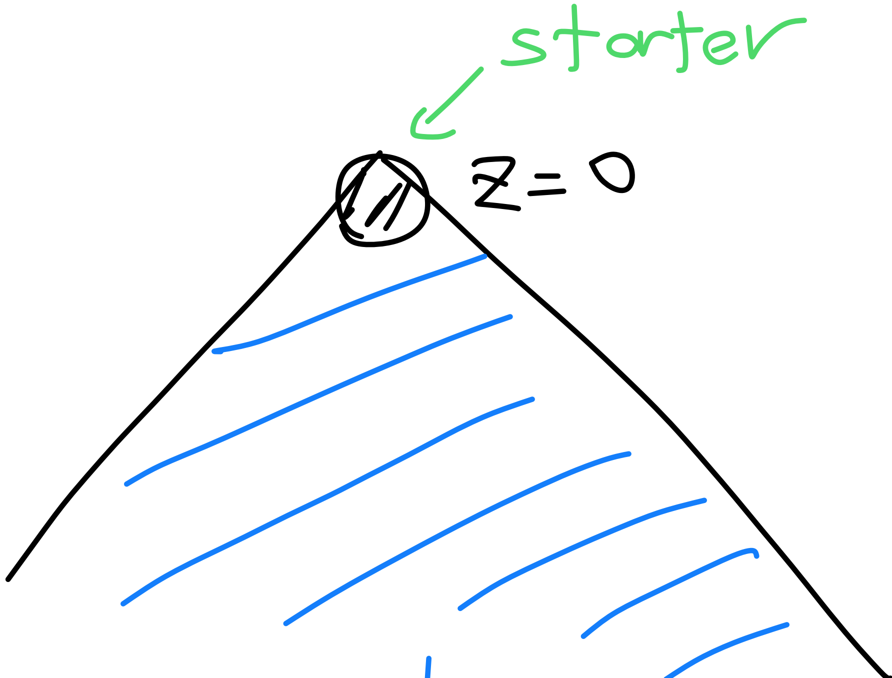
vacuum charge function $\longleftrightarrow$ representation



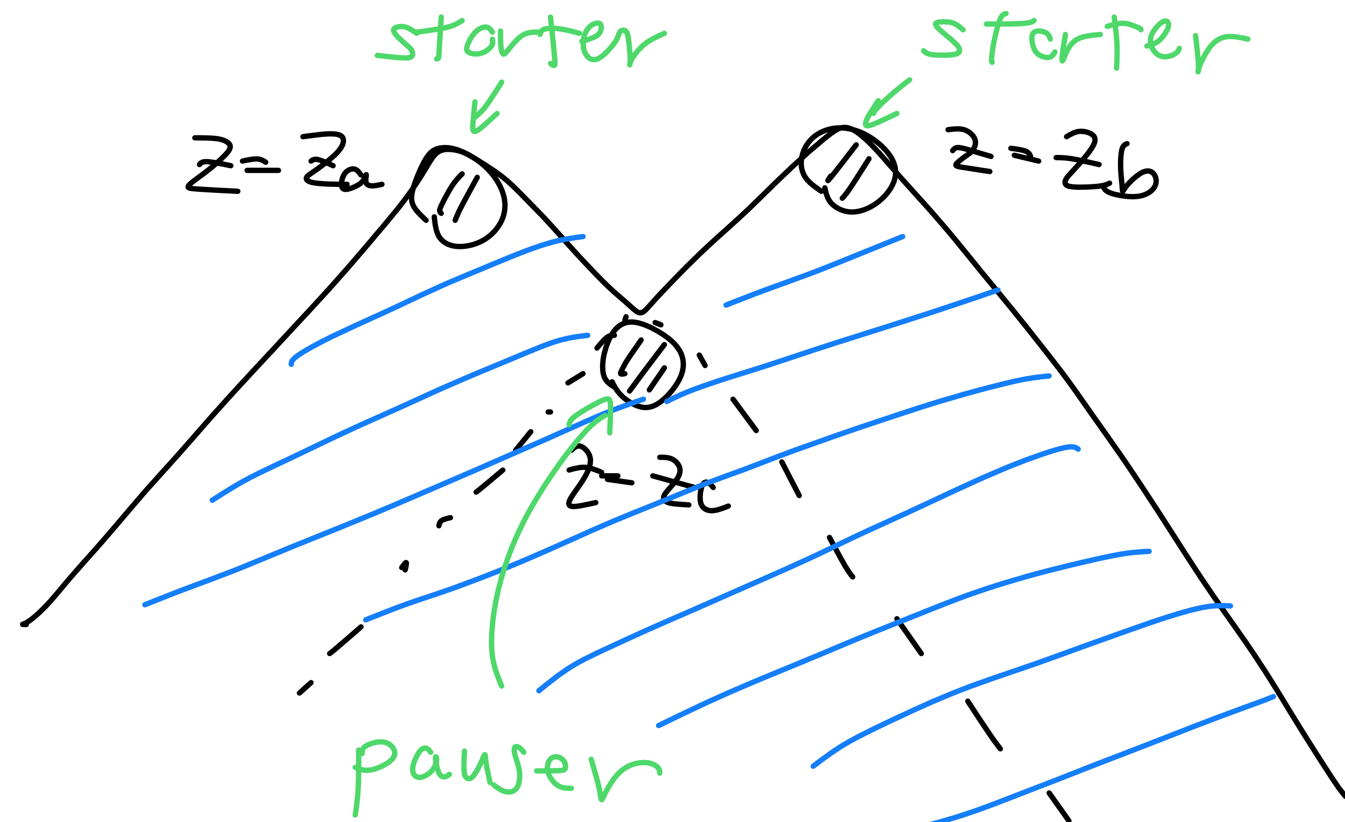
[Galakhov-Li-MY '21]

$$\psi^{(a)}(z)|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}|\emptyset\rangle$$

vacuum charge function $\longleftrightarrow$ representation

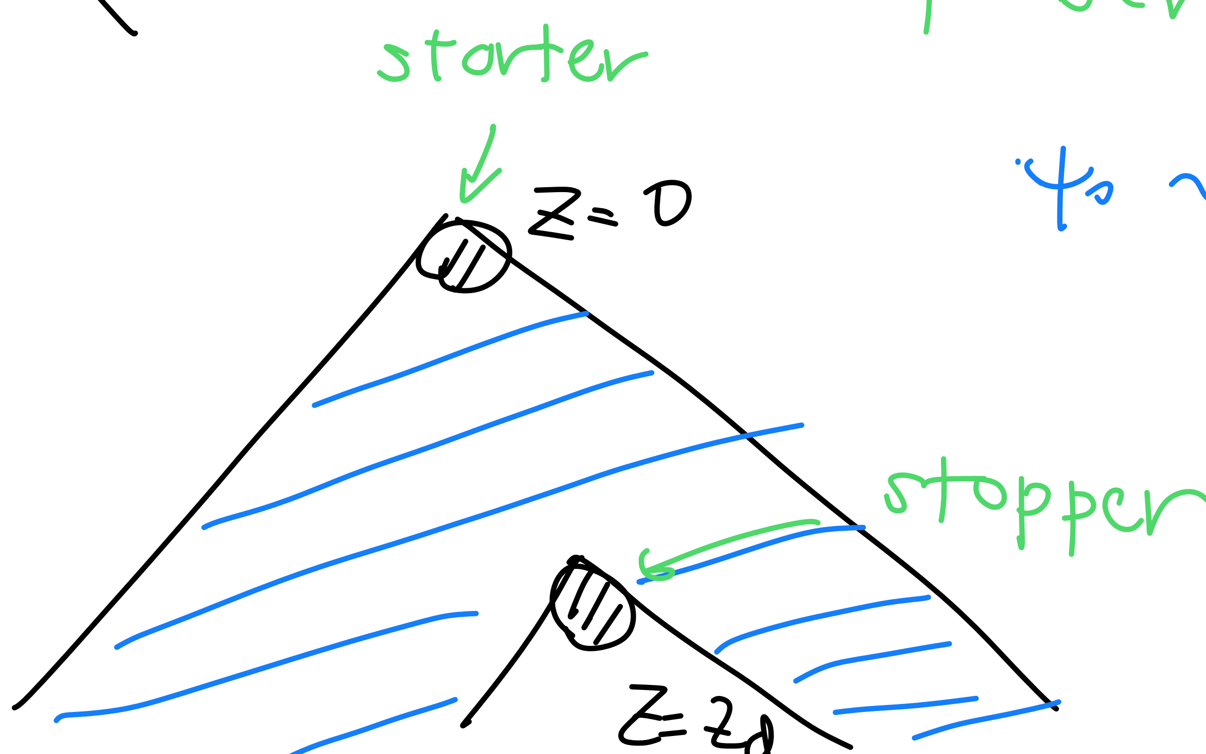


$$\psi_0 \sim \frac{1}{z}$$



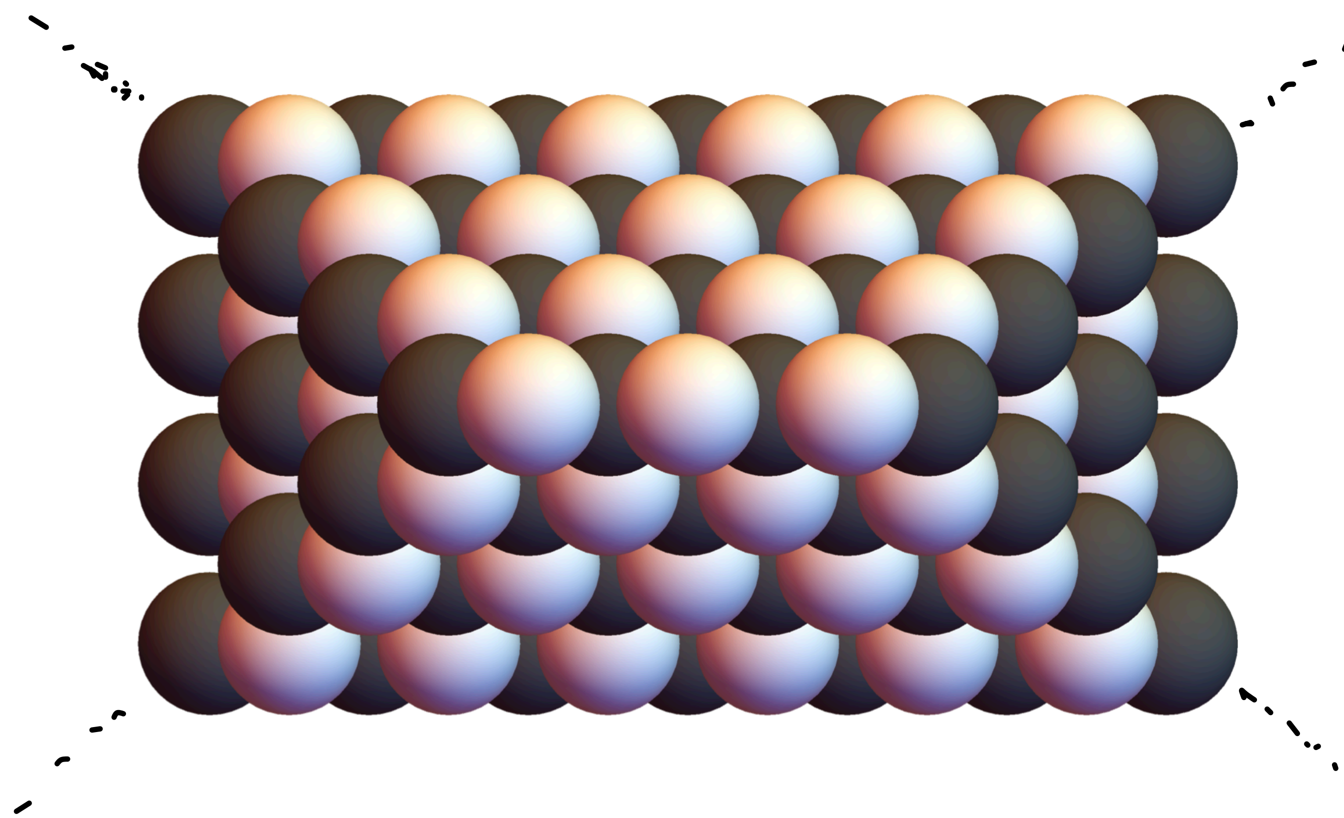
$$\psi_0 \sim \frac{z - z_c}{(z - z_a)(z - z_b)}$$

$$\psi_0 \sim \frac{z - z_d}{z}$$

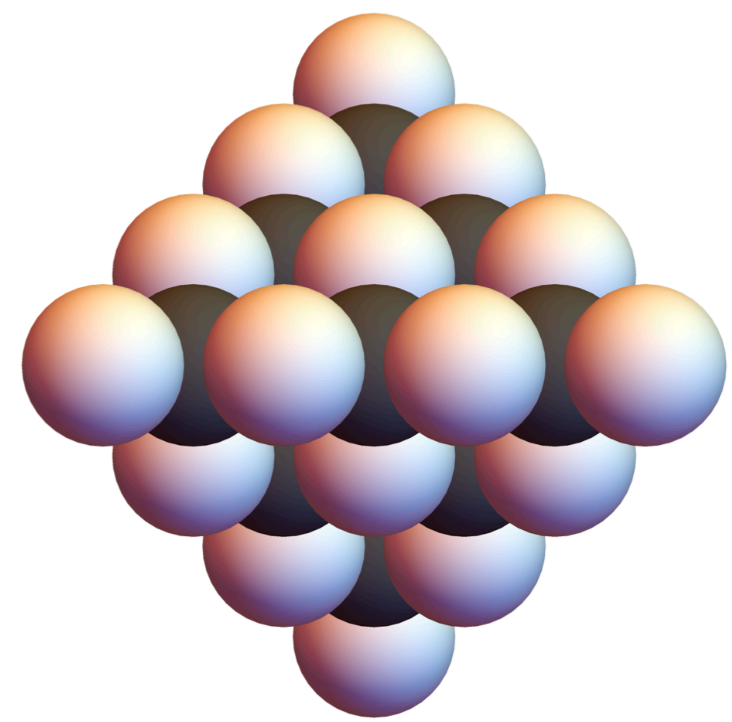


We can obtain rather general reps by using starter / pauser / stoppers

e.g. open / closed BPS state counting and their wall crossings



conifold : ∞-chamber



conifold : finite chamber

[Nagao-Nakajima; Jafferis-Moore; Chuang-Jafferis,⋯ '08]

Revisiting Gauge/Bethe

[Galakhov-Li-Y ('22)]

Puzzle of Gauge/Bethe

2d $\mathcal{N}=(2,2)$

Q, W

$\rightsquigarrow$

Vacuum equation

$$\exp\left(\frac{\partial W}{\partial \sigma}\right) = 1$$

[Nekrasov-Shatashvili ('08)]

$\parallel$

Gauge/Bethe BAE

$\Uparrow$

integrable model

∞-dim. algebra $Y_{Q,W}$

quiver Yangian ?

We can make "crystal chains" by bringing together crystals in Spectral-parameter plane

[Galakhov-Y, Galakhov-Li-Y ('21)]

$$|\mathrm{K}_1, {}^\sharp\mathcal{C}_1\rangle_{u_1} \otimes |\mathrm{K}_2, {}^\sharp\mathcal{C}_2\rangle_{u_2} \otimes \ldots \otimes |\mathrm{K}_n, {}^\sharp\mathcal{C}_n\rangle_{u_n} .$$

- "coproduct"

$\Delta_0 : \mathrm{Rep}\, Y \to \mathrm{Rep}(Y \otimes Y)$

$$\mathbf{\Delta}_0 e = e \otimes 1 + \psi \overset{\rightarrow}{\otimes} e\,,$$
$$\mathbf{\Delta}_0 f = 1 \otimes f + f \overset{\leftarrow}{\otimes} \psi\,,$$
$$\mathbf{\Delta}_0 \psi = \psi \otimes \psi\,.$$

- However, Δ_0 does NOT reproduce

(BAE) = (vacuum equation)

We need to search "correct" Δ:

$$\Delta = \mathcal{U}^{-1} \Delta_0 \mathcal{U}$$

upper triangular

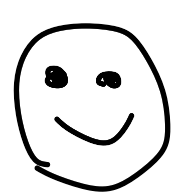
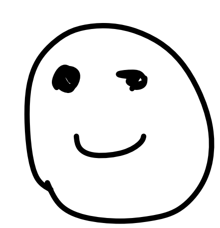
cf. stable envelope of [Maulik-Okounov]

"Yes-Go"

[Galakhov-Li-Y ('22)]
See also [Feigin-Jimbo-Miwa-Mukhin ('15)]
[Litvinov-Vilkovisky ('20)] [Chistyakova-Litvinov-Orlov ('21)]
[Kolyaskin, A. Litvinov, and A. Zhukov ('22)] [Bao ('22)]

* For 2d-crystal repr. of $Y(\hat{\mathfrak{g}})$ w/ $\mathfrak{g} = \mathfrak{gl}_{m|n},\ D(2,1;\alpha)$

Z has product form

BAE  Gauge/Bethe 

"No - Go"

[Galakhov-Li-Y ('22)]

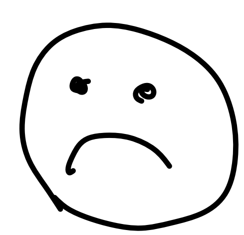
* For $Y(Q, W)$ without underlying $\mathfrak{g}$

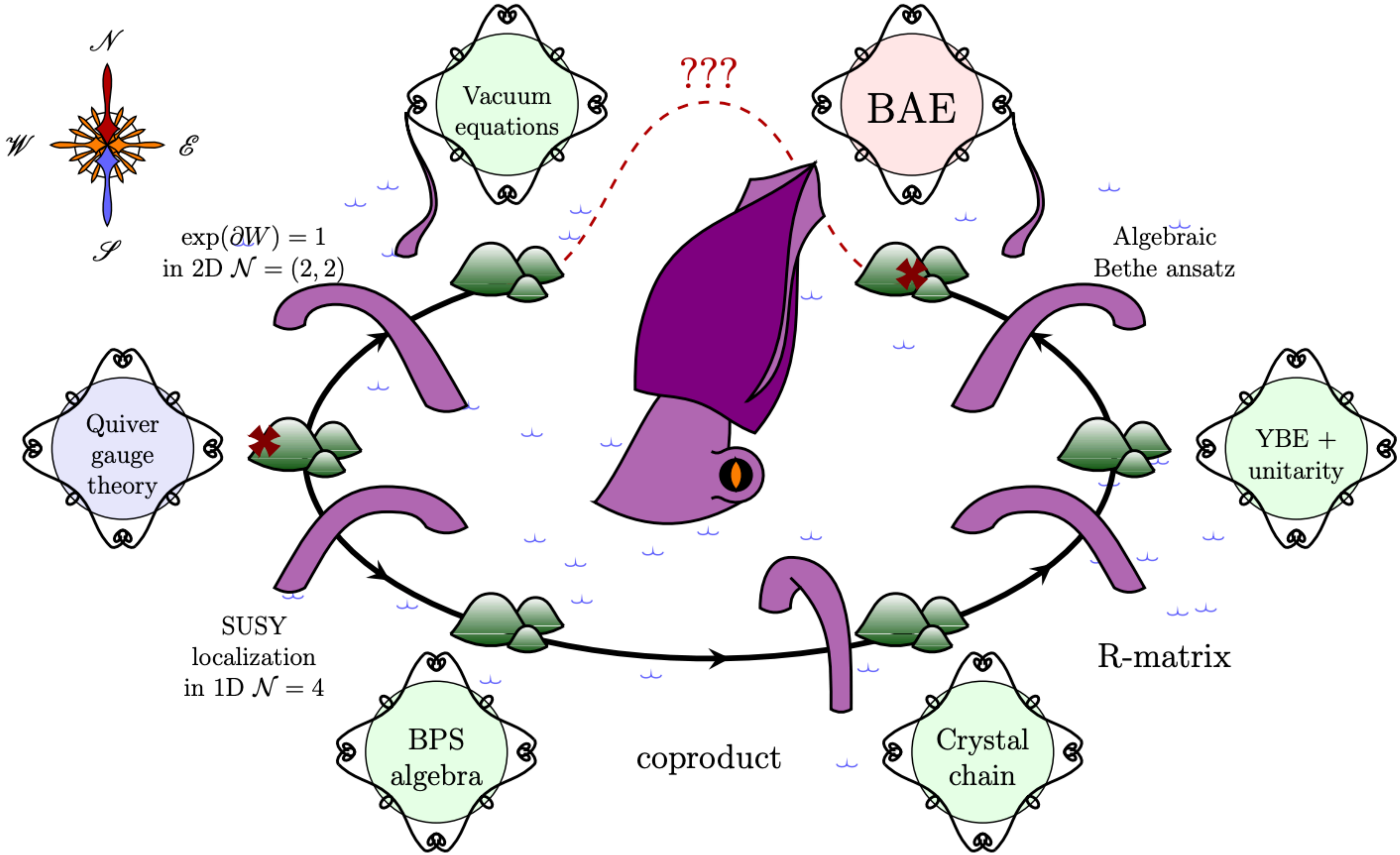
  [chiral quiver / toric CY_3 with 4-cycle]

We have obstructions (under some assumptions)

to finding consistent Δ / R

whose BAE matches vacuum eqn.





$\mathscr{N}$
$\mathscr{W}$
$\mathscr{E}$
$\mathscr{S}$
Vacuum equations
???
BAE
$\exp(\partial W) = 1$ in 2D $\mathcal{N} = (2,2)$
Algebraic Bethe ansatz
Quiver gauge theory
YBE + unitarity
SUSY localization in 1D $\mathcal{N} = 4$
R-matrix
BPS algebra
coproduct
Crystal chain

Summary

- We now have

Quiver Yangian Y = BPS algebra

underlying BPS state counting

$$Z_{BPS} = Z_{crystal} = \text{character of } \mathsf{Y}$$

$$\mathcal{M}_{BPS} \circlearrowleft \mathsf{Y}$$

$$\text{crystal} \circlearrowleft$$

Big Thank You

and

Happy birthday, Hirosi !!!