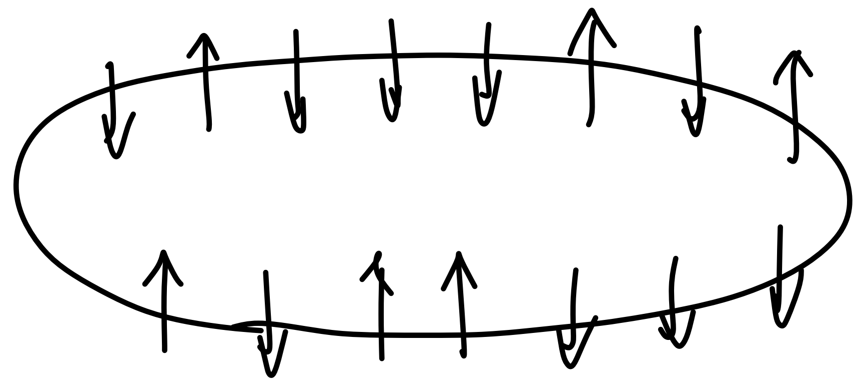
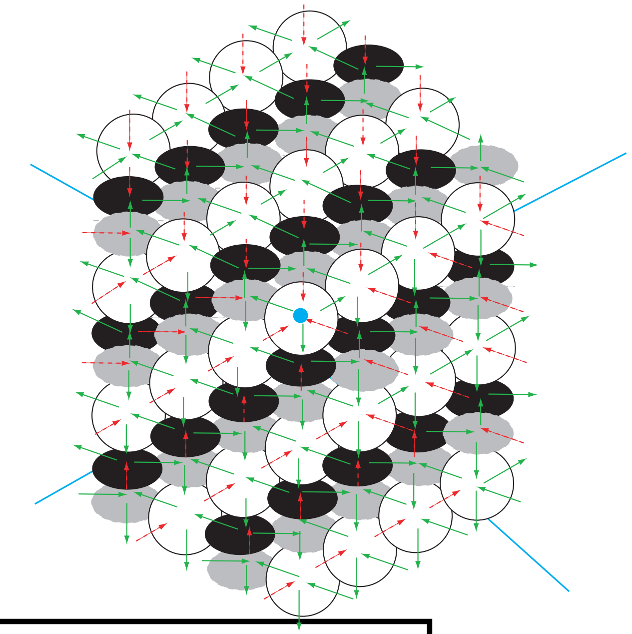




Revisiting the Gauge/Bethe Correspondence

Masahito Yamazaki

KIAS String Theory Seminar

May 22, 2023

Supersymmetric vacua and Bethe ansatz

#7

Nikita A. Nekrasov (IHES, Bures-sur-Yvette), Samson L. Shatashvili (IHES, Bures-sur-Yvette and Hamilton Math. Inst., Dublin and Trinity Coll., Dublin) (Jan, 2009)

Published in: *Nucl.Phys.B Proc.Suppl.* 192-193 (2009) 91-112 • Contribution to: ESF School in High Energy Physics and Astrophysics: Theory and Particle Physics: The LHC Perspective and Beyond • e-Print: 0901.4744 [hep-th]

pdf DOI cite 261 citations

Quantum integrability and supersymmetric vacua

#8

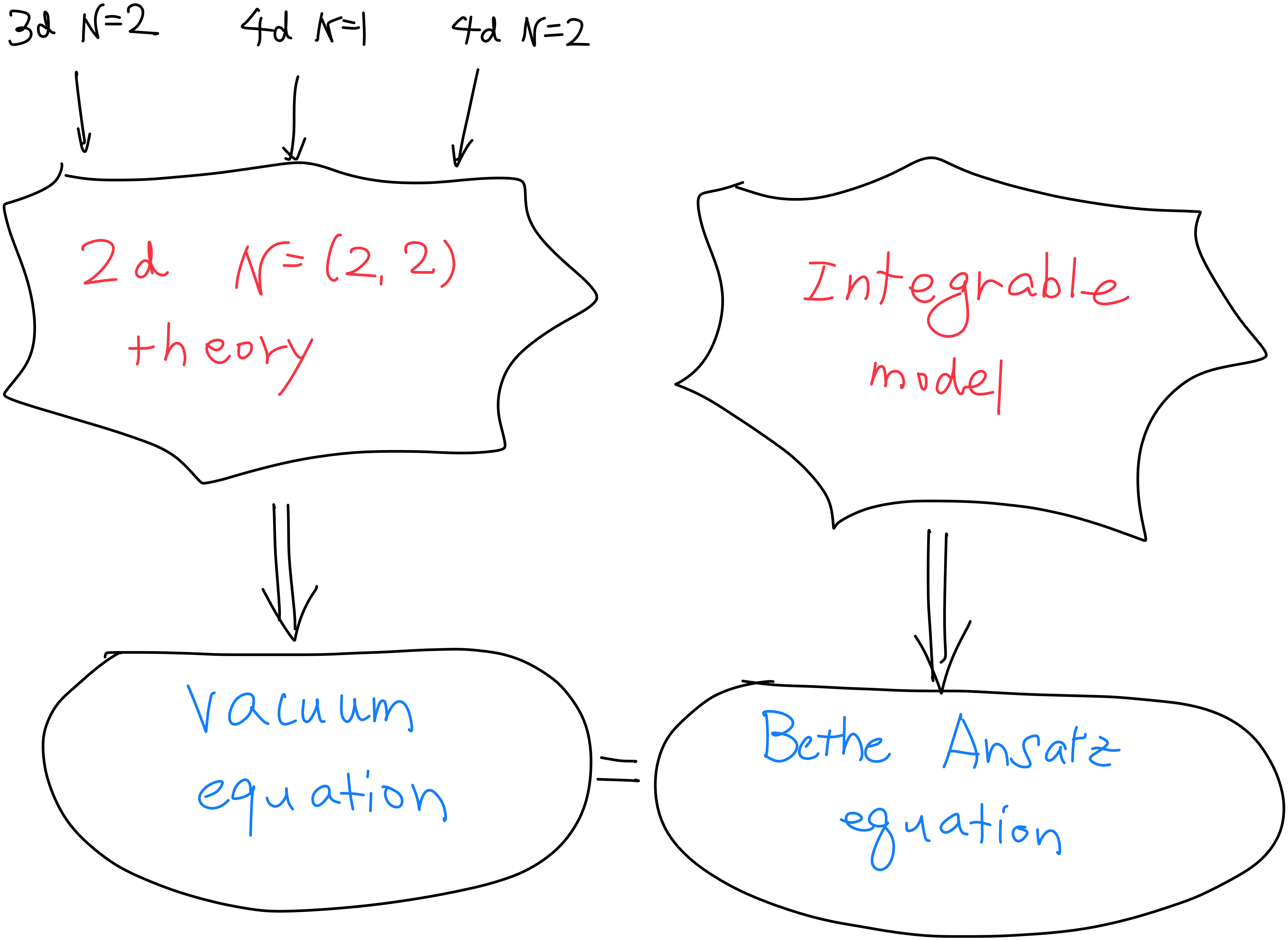
Nikita A. Nekrasov (IHES, Bures-sur-Yvette), Samson L. Shatashvili (IHES, Bures-sur-Yvette and Hamilton Math. Inst., Dublin and Trinity Coll., Dublin) (Jan, 2009)

Published in: *Prog.Theor.Phys.Suppl.* 177 (2009) 105-119 • Contribution to: 30 Years of Mathematical Methods in High Energy Physics (In honor of Professor Tohru Eguchi's 60th Birthday) • e-Print: 0901.4748 [hep-th]

pdf DOI cite 214 citations

Many precursors, e.g.
[Gorsky-Nekrasov, Minahan-Polychronakos, Douglas ('94), Gerasimov (~'93)
Losev/Moore+Nekrasov-Shatashviili ('97-'98)
Gerasimov-Shatashvili ('06-'07)]

Gauge / Bethe 101



3d $\mathcal{N}=2$
4d $\mathcal{N}=1$
4d $\mathcal{N}=2$
2d $\mathcal{N}=(2,2)$ theory
Integrable model
Vacuum equation
=
Bethe Ansatz equation

"Gonge"

e.g. $2d$ $\mathcal{N}=(2,2)$ $U(N_c)$ gauge + N_f flavors

Vector multiplet

$(A_\mu, \sigma, \lambda, \bar{\lambda})$

$0,1$ $\quad$ $A_2 + i A_3$

chiral multiplet

give mass

+ 1 adjoint

---

effective theory after integrating out matters

$$\int d\theta^+ d\bar{\theta}^- \widetilde{W}(\Sigma)$$: effective twisted superpotential

twisted superfield

$$\Sigma = \sigma - i\sqrt{2}\,\theta^+ \bar{\lambda}_+ - i\sqrt{2}\,\bar{\theta}^- \lambda_- + \cdots$$

The vacuum equation

$$\exp\left(\underbrace{\frac{\partial \widetilde{W}(\sigma)}{\partial \sigma}}_{}\right) = 1$$

($2\pi i \mathbb{Z}$; flux sector $\frac{1}{2\pi}\int F \in \mathbb{Z}$)

$$\Downarrow$$

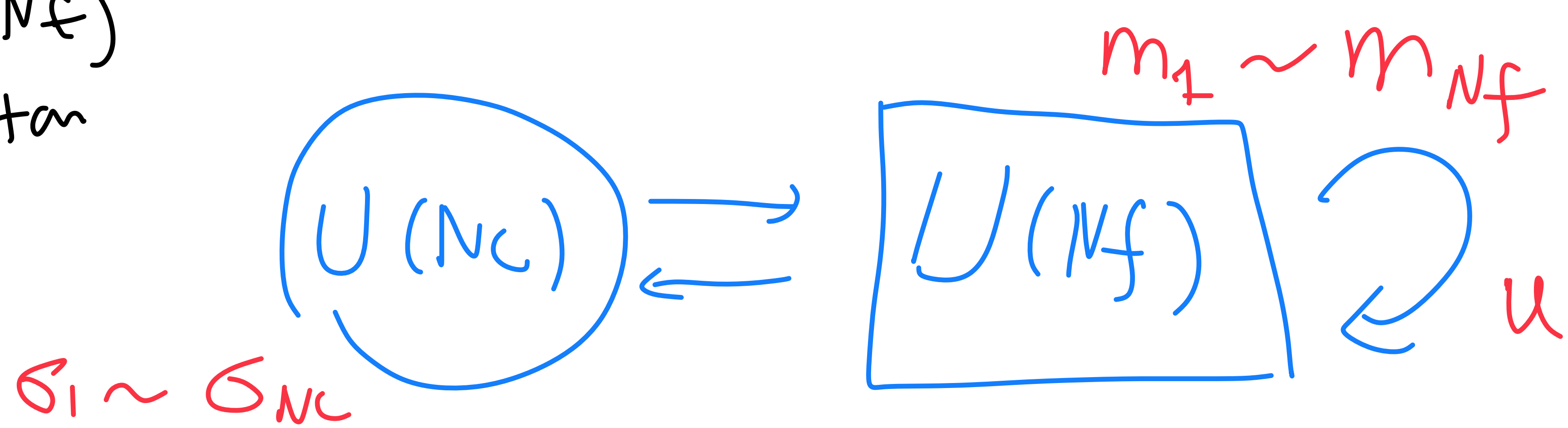
FI param.

$U(1)_R^{\mathcal{N}=4}$ on adj

$$\prod_{a=1}^{N_f} \frac{\sigma_i - m_a + u/2}{\sigma_i - m_a - u/2} = e^{2\pi i t} \prod_{\substack{j \neq i \\ 1 \leq j \leq N_c}} \frac{\sigma_i - \sigma_j + u}{\sigma_i - \sigma_j - u}$$

$U(N_c)$ Cartan

$U(N_f)$ Cartan



"Bethe"

XXX spin chain

$$\hat{H} = -J \sum_{i=1}^{N_f} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}$$

spin - 1/2

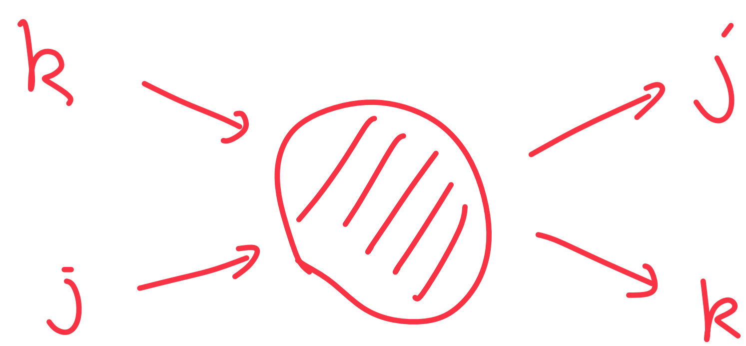
Bethe Ansatz equation

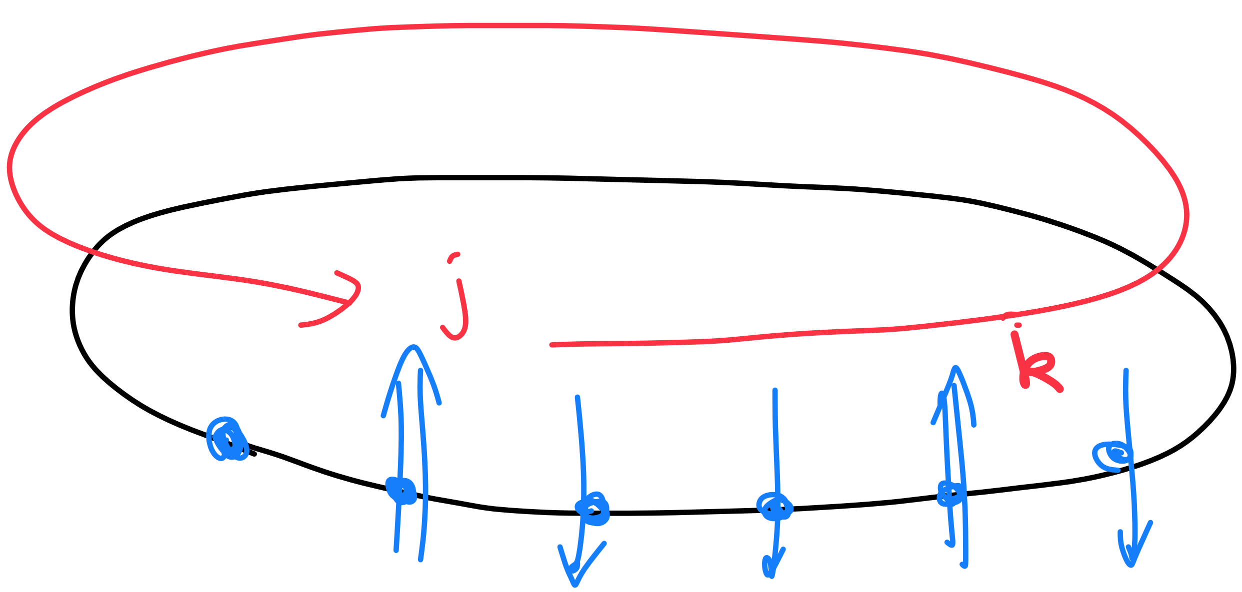
$$\left(\frac{\sigma_j + i}{\sigma_j - i}\right)^{N_f} = \prod_{\substack{k\neq j \\ 1\le k\le N_c}} \frac{\sigma_j - \sigma_k + 2i}{\sigma_j - \sigma_k - 2i}$$

$e^{i p_j N_f}$

S_{jk} : S-matrix

(σ : rapidity)





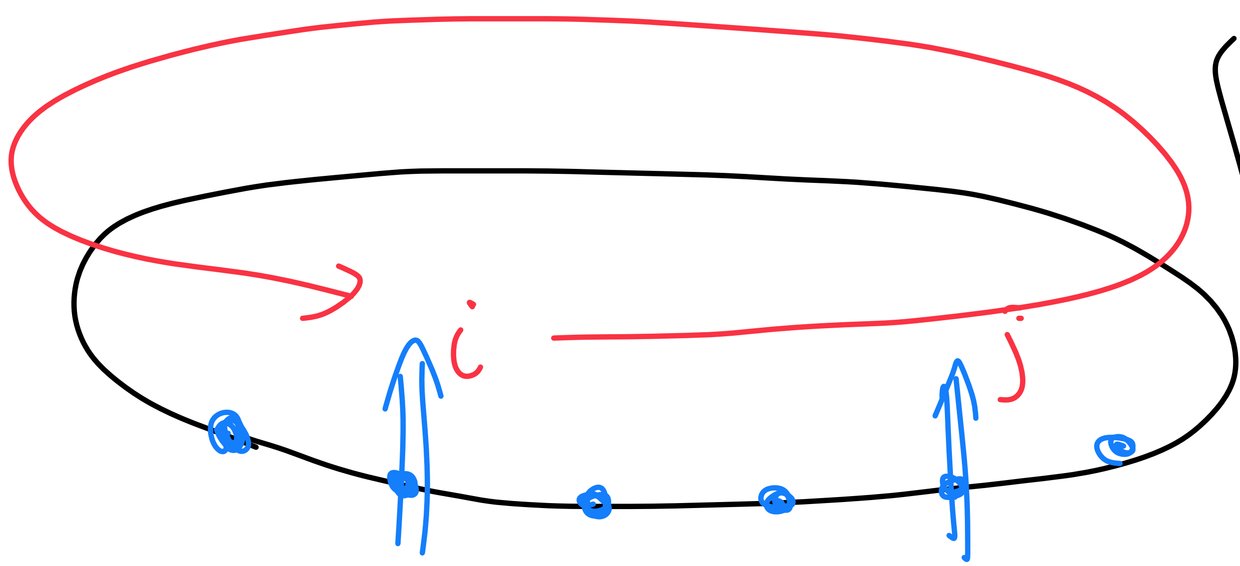
$$\begin{pmatrix} N_c & \uparrow & \text{spins} \\ N_f & \uparrow\downarrow & \text{spins} \end{pmatrix}$$

Bethe Ansatz equation

$$\left(\frac{\sigma_j+i}{\sigma_j-i}\right)^{N_f}=\prod_{\substack{k\neq j\\1\le k\le N_c}}\frac{\sigma_j-\sigma_k+2i}{\sigma_j-\sigma_k-2i}$$

$\Big\downarrow$ inhomogeneity + twist

$$\prod_{a=1}^{N_f}\left(\frac{\sigma_i-m_a+i}{\sigma_i-m_a-i}\right)=e^{t}\left(\prod_{\substack{k\neq j\\1\le k\le N_c}}\frac{\sigma_j-\sigma_k+2i}{\sigma_j-\sigma_k-2i}\right)$$



m_a : inhomogeneity
$\parallel$
twisted mass for $SU(N_f)$

t : twist along chain
$\parallel$
FI param.

Generalizations :

- **2d $\mathcal{N}=(2,2)$**

  $U(N_c) + N_f$ flavors $+$ adj

  $\|$

  XXX chain

  $Y_\hbar(sl_2)$

  $\xleftarrow{S^1}$

  3d $\mathcal{N}=2$

  $U(N_c) + N_f + \text{adj}$

  $\|$

  XXZ chain

  $\mathcal{U}_q(sl_2)$

  $\xleftarrow{S^1}$

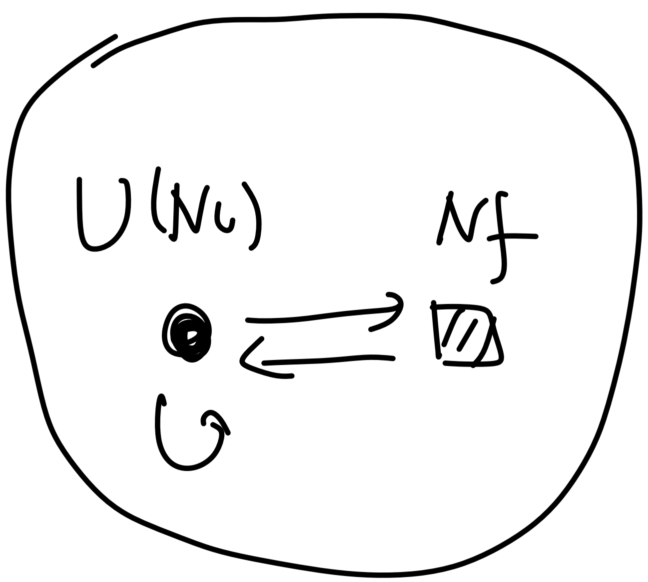
  4d $\mathcal{N}=1$

  $U(N_c) + N_f + \text{adj}$

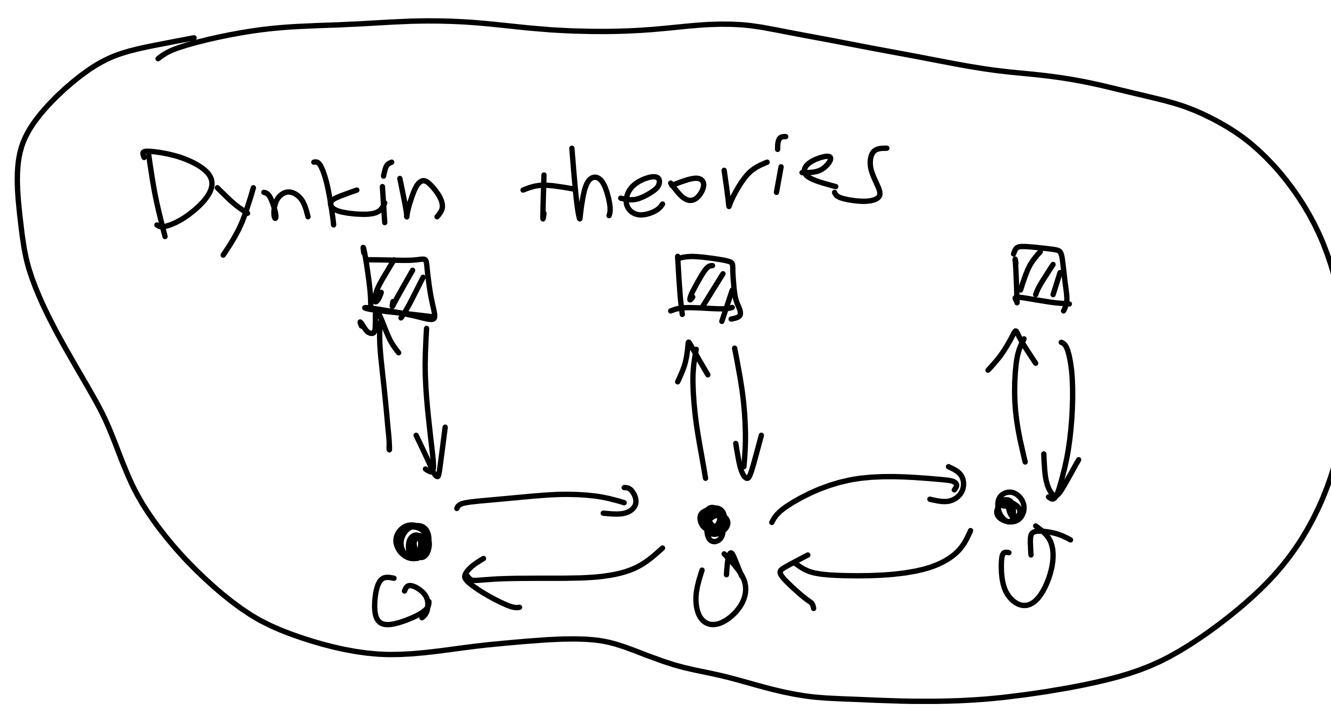
  $\|$

  XYZ chain

  $E_{q,\tau}(sl_2)$

- 

  $\rightsquigarrow$

  $Y_\hbar(\mathfrak{g})$

$\boxed{\text{6d } (2,0)\ A_N \text{ SCFT}}$

on S_2 ⟵ cover ⟵ "IR" {SW curve} = {spectral curve of integrable model}

"UV"

$\boxed{\text{4d } \mathcal{N}=2 \text{ "class S"}}$

on $\mathbb{R}^4_{\epsilon_1, \epsilon_2}$

on $\mathbb{R}^2_{\epsilon_1}$

$\log Z \sim \frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}$

$a_D = \partial \mathcal{F} / \partial a$

$\boxed{\text{2d } \mathcal{N}=(2,2)}$

on $\mathbb{R}^2_{\epsilon_2}$

$\log Z \sim \frac{1}{\epsilon_2} \widetilde{W}$

$\beta = \frac{\partial \widetilde{W}_{oper}}{\partial \alpha}$

"quantize"

$\widetilde{W} = \widetilde{W}_{oper} - \widetilde{W}^*$

Puzzle? (MY, circa 2008)

2d $\mathcal{N}=(2,2)$

Q, W

quiver

There should exist IM

∞-dim algebra

$\mathcal{Y}_{Q,W} \neq \mathcal{Y}(\mathfrak{g})$

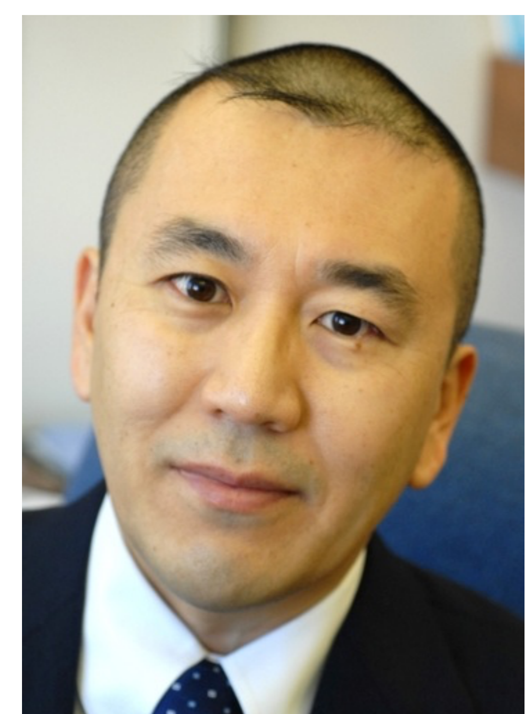
R-matrix, Bethe Ansatz, ...

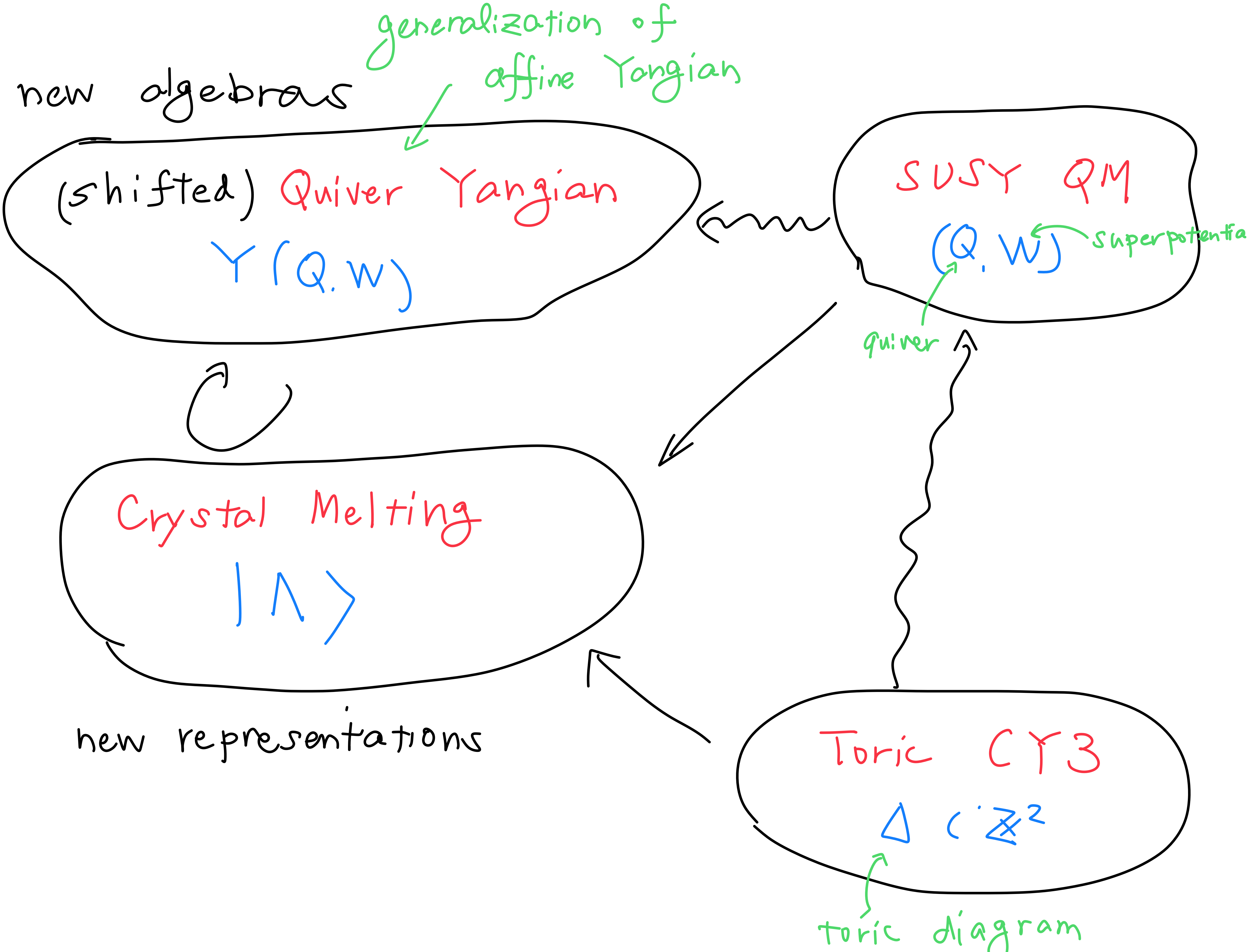
Q: $\mathcal{Y}_{Q,W}$???

Candidate for $\mathcal{Y}_{Q,W}$: Quiver Yangian

Wei Li + MY
(2003.08909 [hep-th])
Dimitry Galakhov + MY
(2008.07006 [hep-th])
Dimitry Galakhov+Wei Li + MY
(2106.01230 [hep-th])
(2108.10286 [hep-th])
(2206.13340 [hep-th])
also works by Noshita, Watanabe, Bao, Negut,⋯

Also earlier works, e.g.
Hirosi Ooguri + MY (0811.2810 [hep-th])
MY (Ph.D. thesis, 1002.1709 [hep-th])
MY (Master thesis, 0803.4474 [hep-th])





generalization of affine Yangian
new algebras
(shifted) Quiver Yangian
Y(Q,W)
SUSY QM
(Q,W)
superpotential
quiver
Crystal Melting
|Λ⟩
new representations
Toric CY3
Δ ⊂ ℤ²
toric diagram

Resolution of the Puzzle?

2d $\mathcal{N}=(2,2)$

Q, W

toric CY_3

$\rightsquigarrow$ $\mathcal{Y}_{Q,W}$ $\neq \mathcal{Y}(\mathfrak{g})$

should be some ∞-dim. algebra

$\mathcal{Y}_{Q,W}$ = quiver Yangian ??

Quiver Yangian

in a nutshell

[See MY 2203.14314 for review]

Generators

(z: spectral parameter)

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}}, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}}, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}},$$

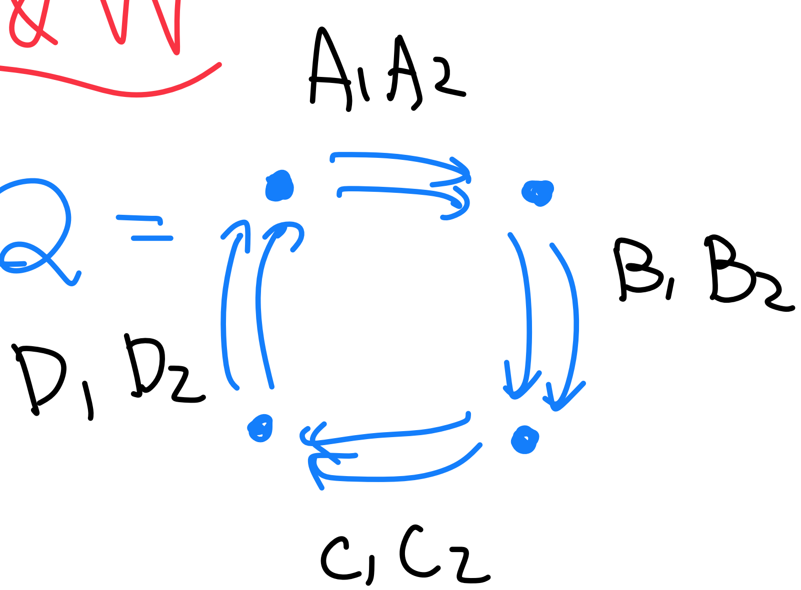
(can truncate to $n=-k$)

a : quiver vertex

$\mathbb{Z}_2$-grading

$$|a| = \begin{cases} 0 & (\exists \text{ edge } I \text{ s.t. } I \text{ starts and ends at } a) \\ 1 & (\text{otherwise}) \end{cases}$$

Q & W

$Q =$ 

$$W = \mathrm{Tr}\,(A_1 B_1 C_1 D_1 - A_1 B_2 C_1 D_2 - A_2 B_1 C_2 D_1 + A_2 B_2 C_2 D_2)$$

$(CY_3 = K_{\mathbb{P}^1 \times \mathbb{P}^1})$

Relations

$\mathcal{Y}(Q, W)$

$$\psi^{(a)}(z)\,\psi^{(b)}(w) = \psi^{(b)}(w)\,\psi^{(a)}(z) \ ,$$
$$\psi^{(a)}(z)\,e^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z) \ ,$$
$$e^{(a)}(z)\,e^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z) \ ,$$
$$\psi^{(a)}(z)\,f^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z) \ ,$$
$$f^{(a)}(z)\,f^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z) \ ,$$
$$\left[e^{(a)}(z), f^{(b)}(w)\right\} \sim -\delta^{a,b}\frac{\psi^{(a)}(z) - \psi^{(b)}(w)}{z-w} \ , \qquad (\Delta = z-w)$$

"$\simeq$" means equality up to $z^n w^{m\geq 0}$ terms

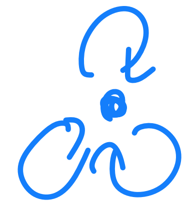
"$\sim$" means equality up to $z^{n\geq 0} w^m$ and $z^n w^{m\geq 0}$ terms

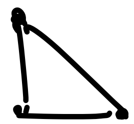
bonding factor

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

flavor charge of arrow = bifundamental

edge

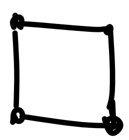
* $\mathbb{C}^3 \rightsquigarrow Q =$  $\rightsquigarrow Y(\widehat{gl_1})$



$W = \mathrm{Tr}(XYZ - XZY)$

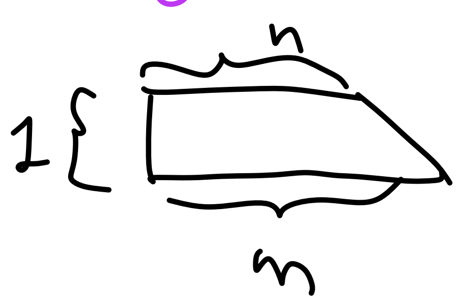
[Miki; Ding-Iohara;⋯
Tsymbaulik; Prochazka;
Gaberdiel, Gopakumar, Li, Peng,⋯]

* conifold $\rightsquigarrow Q =$  $\longrightarrow Y(\widehat{gl_{1|1}})$

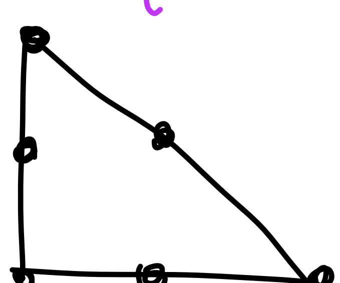


$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

* $xy = z^n w^m \rightsquigarrow Y(\widehat{gl_{m|n}})$ [Bezerra-Mukhin ('19)]



* $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \rightsquigarrow Y(\widehat{D(2,1,\alpha)})$ [Noshita-Watanabe ('21)]



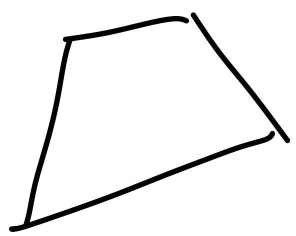
$Y(\hat{g})$ for (non-chiral quiver
toric CY3 w.o. 4-cycle

chiral quiver

toric CY_3 w/ cpt 4-cycle

* general toric CY_3 $\rightsquigarrow$ $Y(Q, W)$

has no "$\mathfrak{g}$"



$\Delta \subset \mathbb{Z}^2$

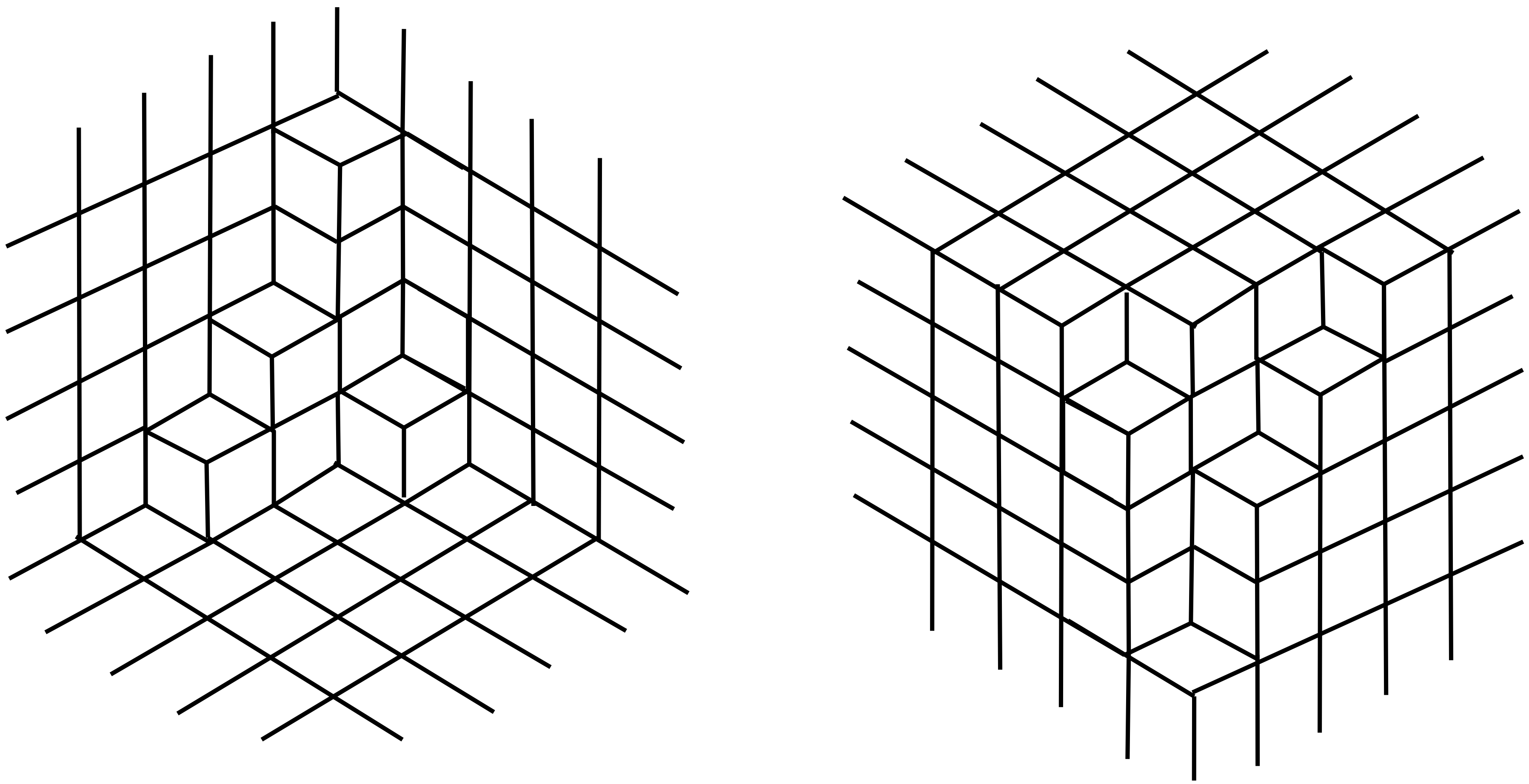
new algebra beyond $Y(\mathfrak{g})$

$Y(\hat{\mathfrak{g}})$!

Representations from Crystal Melting

cf. earlier developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by [Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka; Rapcak; Gaberdiel, Gopakumar; Li, Peng,⋯]

$\mathbb{C}^3$: crystal melting [Okounkov-Reshetikhin-Vafa; Iqbal, Nekrasov,...]



plane partition

$$M(q) \equiv \sum_{\Lambda \in \text{ plane partition}} q^{|\Lambda|} = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}$$

$$= 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \ldots \ ,$$

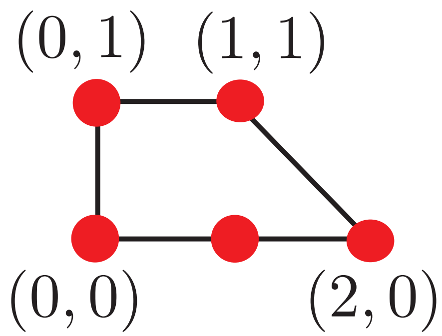
$$= Z^{\mathbb{C}^3}_{\text{top A-model}}$$

The story generalizes to an arbitrary toric CY3

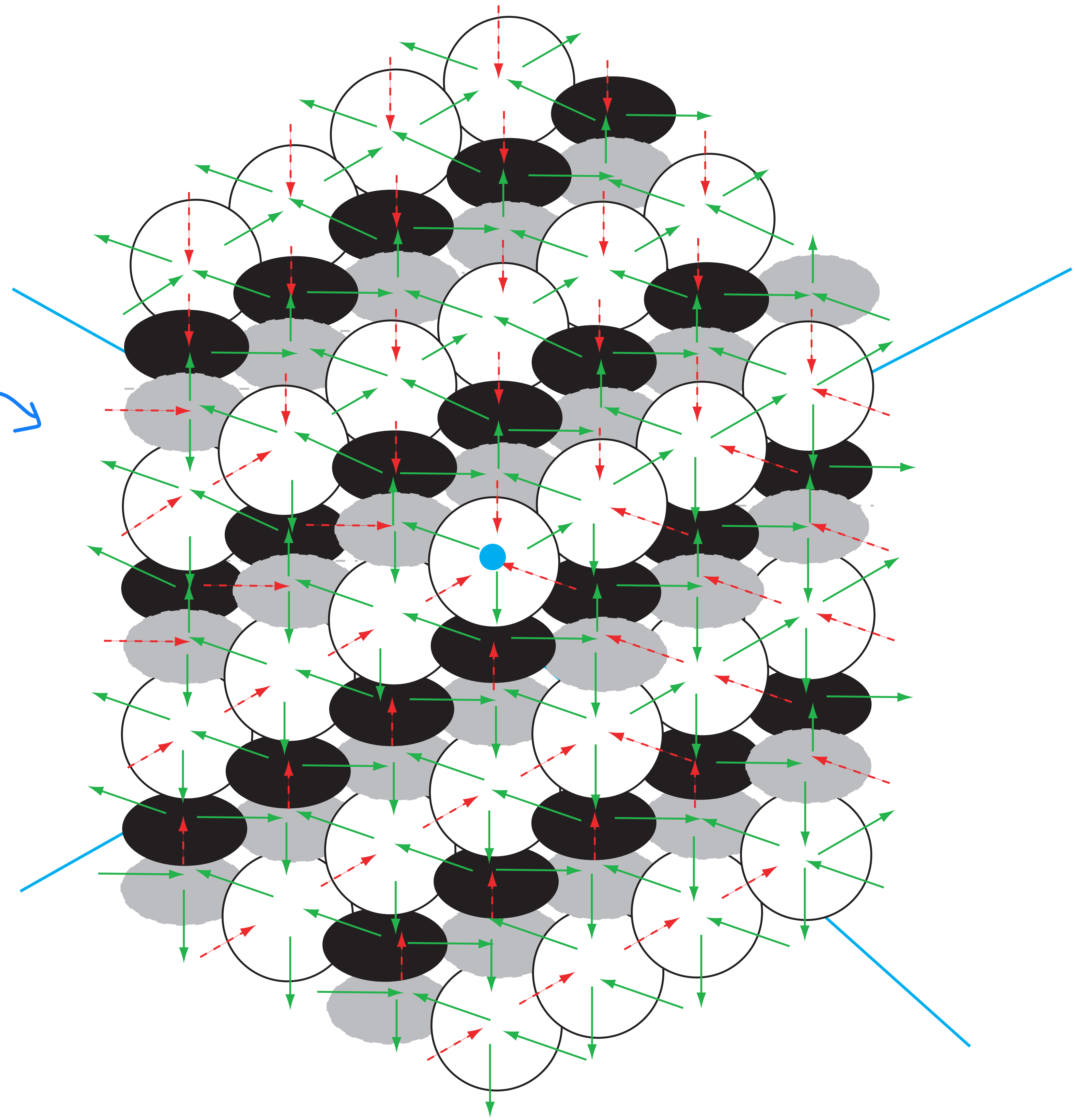
[Ooguri-MY ’08’09]

See also [Szendroi; Bryant, Young; Mozgovoy, Reineke; Nagao, Nakajima; Ooguri, MY; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Vafa; ⋯]

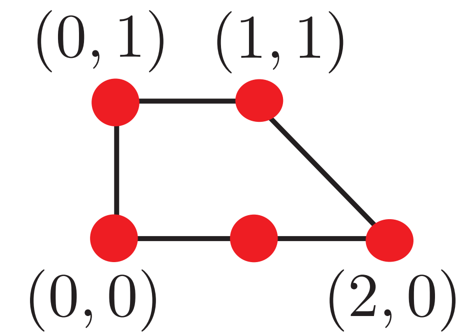
toric diagram $\left(\begin{array}{c}\text{SPP}\\ xy=zw^2\end{array}\right)$

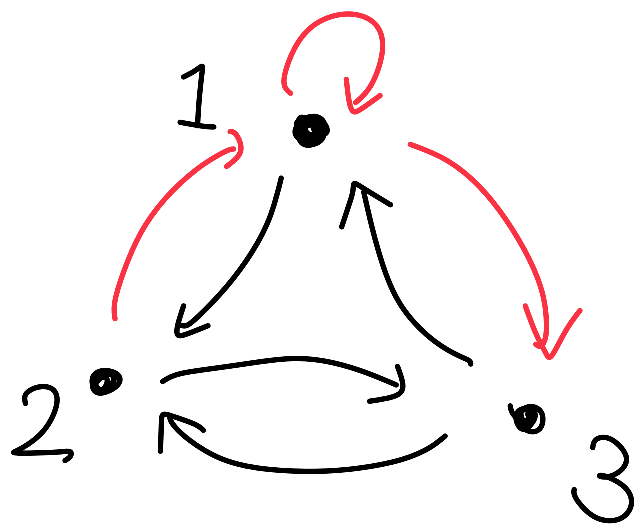


[Ooguri-MY '08]



We have an associated SQM



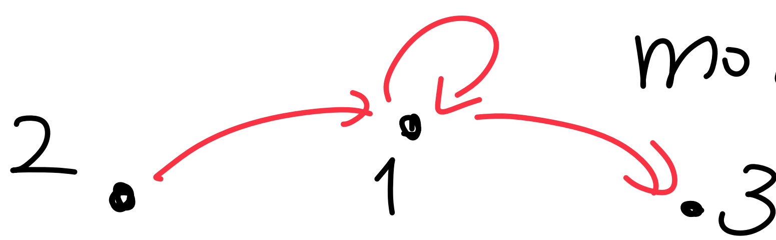


Q

{atom in the crystal}

= {(open path) starting at a vertex}

module F-term ($\partial W = 0$)



= {"chiral ring operator"}

We can place the atoms in 3D according to their R + flavor charges

Representations from Crystal Melting

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\,,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle\,,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle\,,$$

add/remove an atom

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \ ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to \mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \ ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to \mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \ ,$$

poles for atom $\boxed{a}$

add/remove an atom

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi_{\mathrm{K}}^{(a)}(z)|\mathrm{K}\rangle ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to \mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to \mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle ,$$

poles for atom $\boxed{a}$

$$h(\boxed{a}) \equiv \sum_{I\in \mathrm{path}[\mathfrak{o}\to \boxed{a}]} h_I .$$

$\Psi_{\mathrm{K}}^{(a)}$:

$$\Psi_{\mathrm{K}}^{(a)}(u) = \psi_0^{(a)}(z) \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{b\Rightarrow a}(u-h(\boxed{b})) ,$$

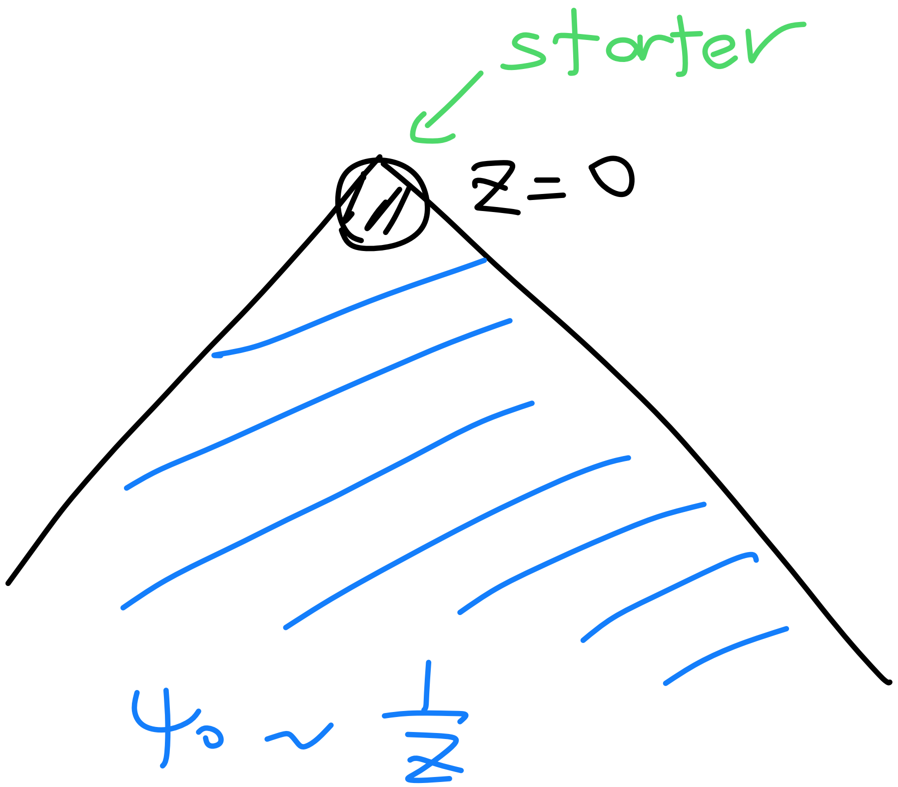
$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

$E^{(a)}/F^{(a)}$: $E^{(a)}/F^{(a)} = \sqrt{\pm \operatorname*{Res}_{u=h(\boxed{a})} \Psi_{\mathrm{K}}^{(a)}(u)}$

$$\psi^{(a)}(z)|\emptyset\rangle = \psi_0^{(a)}(z)|\emptyset\rangle$$

[Galakhov-Li-MY '21]

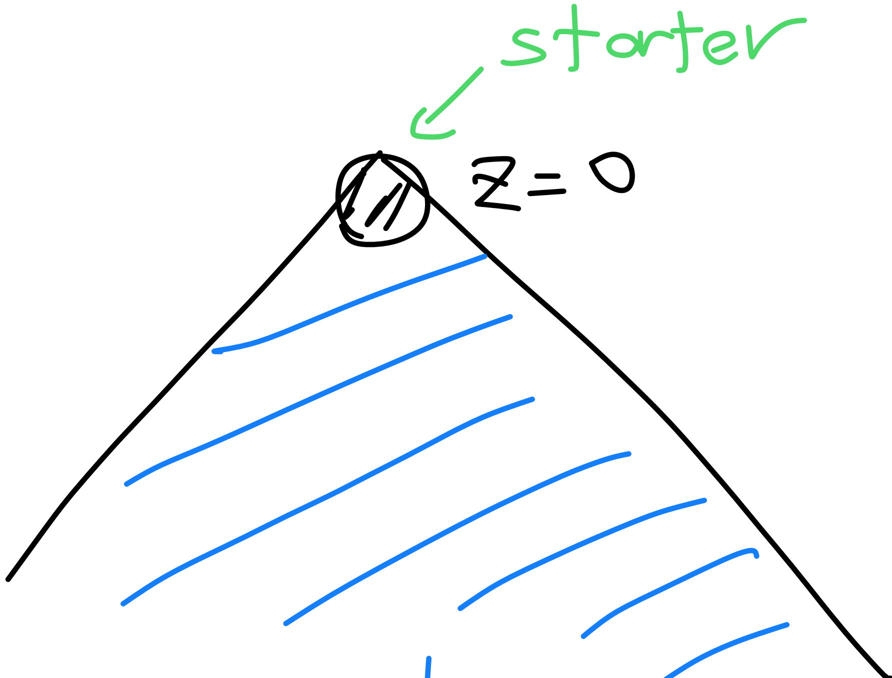
vacuum charge function $\longleftrightarrow$ representation



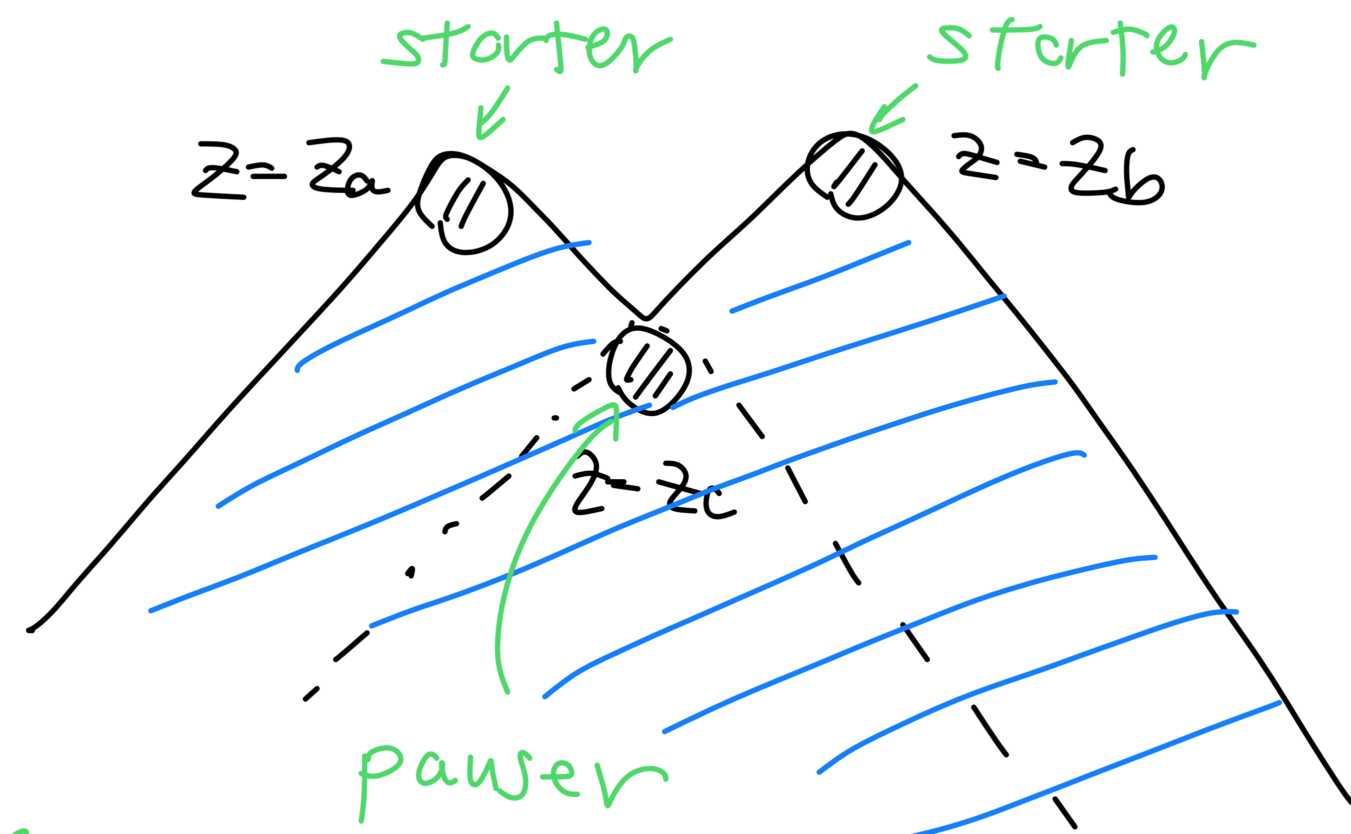
$$\psi^{(a)}(z)|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation

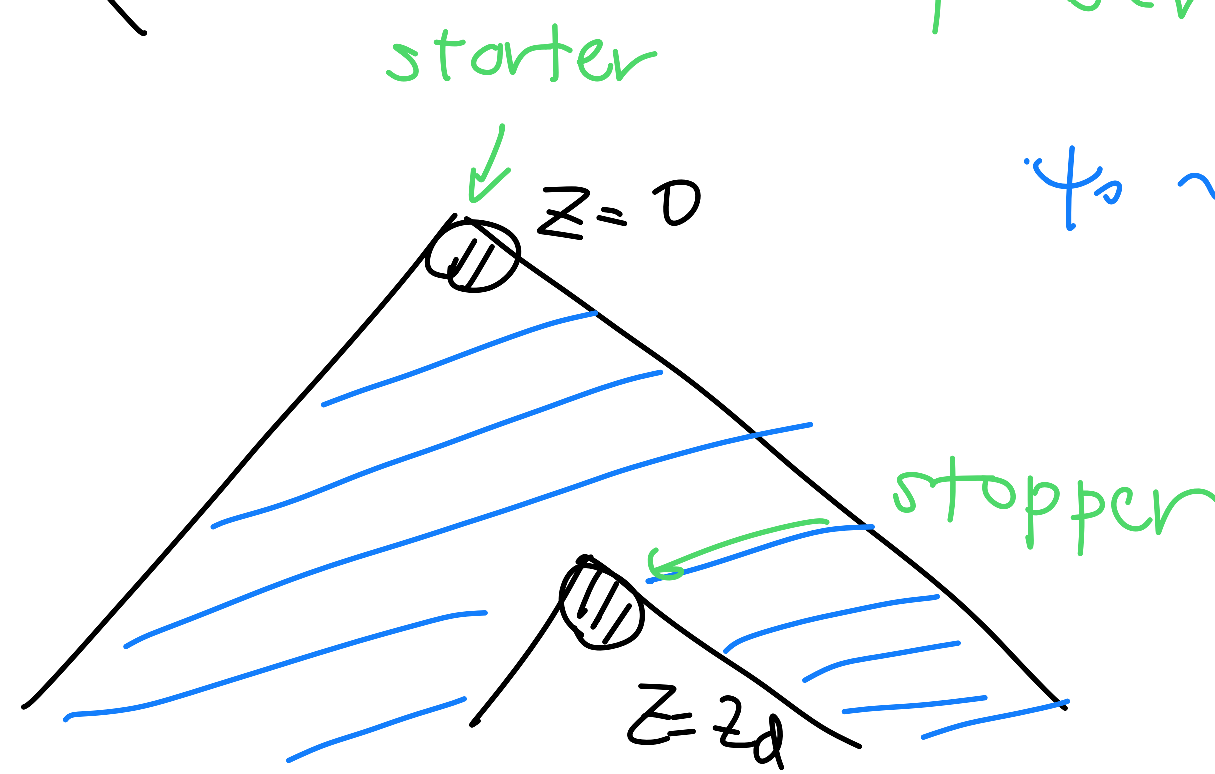


$\psi_0 \sim \frac{1}{z}$



$\psi_0 \sim \frac{z-z_c}{(z-z_a)(z-z_b)}$

$\psi_0 \sim \frac{z-z_d}{z}$



R – matrix

Bethe Ansatz

[Galakhov-Li-Y ('22)]

We now have an algebra Y

and a representation $\mathcal{C}$ from crystal

$\rightsquigarrow$

Natural to consider

"crystal chains" $\mathcal{C}^{\otimes n}$

&

R-matrix, BAE, ...

Vacuum equation = "Would-be BAE"

exp(FI param.)

$$1 = \mathbf{BAE}_i^{(a)}\left(\vec{\sigma}, \vec{u}, \vec{\mathfrak{q}}\right) := \mathfrak{q}_a^{-1} \prod_{\substack{1 \le j \le N_a \\ j \ne i}} \varphi^{a \Leftarrow a}\left(\sigma_i^{(a)} - \sigma_j^{(a)}\right) \times$$

$$\times \prod_{\substack{b \in Q_0 \\ b \ne a}} \prod_{k=1}^{N_b} \varphi^{a \Leftarrow b}\left(\sigma_i^{(a)} - \sigma_k^{(b)}\right) \prod_{\mathfrak{f}} \varphi^{a \Leftarrow \mathfrak{f}}\left(\sigma_i^{(a)} - u_{\mathfrak{f}}\right)$$

net deg $\neq 0$ in general $\rightarrow$

$$\varphi^{a \Rightarrow b}(u) \equiv \frac{\prod_{I \in \{b \to a\}}(u + h_I)}{\prod_{I \in \{a \to b\}}(u - h_I)}$$

"Yes - Go"

[Galakhov-Li-Y ('22)]
See also [Feigin-Jimbo-Miwa-Mukhin ('15)]
[Litvinov-Vilkovisky ('20)] [Chistyakova-Litvinov-Orlov ('21)]
[Kolyaskin, A. Litvinov, and A. Zhukov ('22)] [Bao ('22)]

We can choose shift $= 0$

* For 2d-crystal repr. (Fock module) of $Y(\hat{\mathfrak{g}})$ w/ $\mathfrak{g} = \mathfrak{gl}_{m|n}, D(2,1;\alpha)$

We can derive BAE

and verify Gauge/Bethe!

"No - Go"

[Galakhov-Li-Y ('22)]

shift $\neq 0$

* For $Y(Q, W)$ without underlying $\mathfrak{g}$

[chiral quiver / toric CY_3 with 4-cycle]

We have obstructions (under some assumptions)

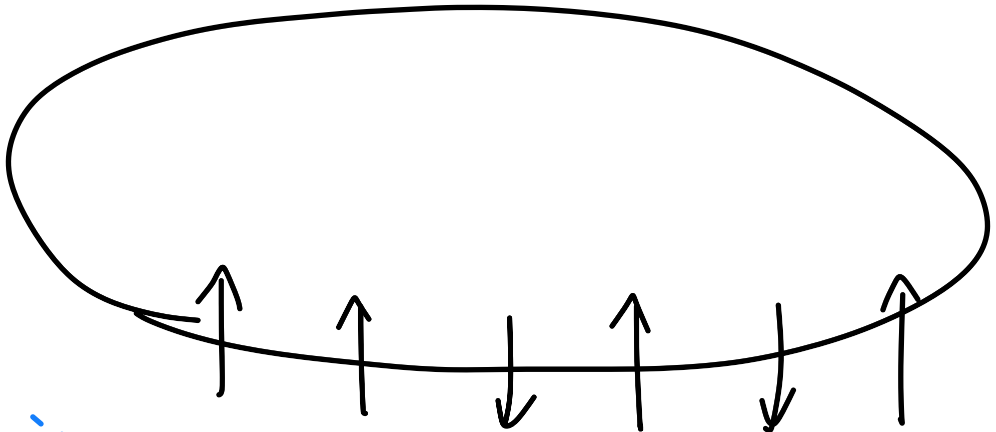
to finding consistent Δ / R

whose BAE matches vacuum eqn.

Derivation / Assumptions

[Galakhov-Li-Y ('22)]

spin chain



$\cdots \; V_1 \otimes V_2 \otimes V_3 \; \cdots\cdots$

need tensor product rep. $V_1 \otimes V_2$

R-matrix

$R_{12} : V_1 \otimes V_2 \to V_2 \otimes V_1$

$\left(\text{rep} \quad \rho_i : A \to V_i \right)$

coproduct

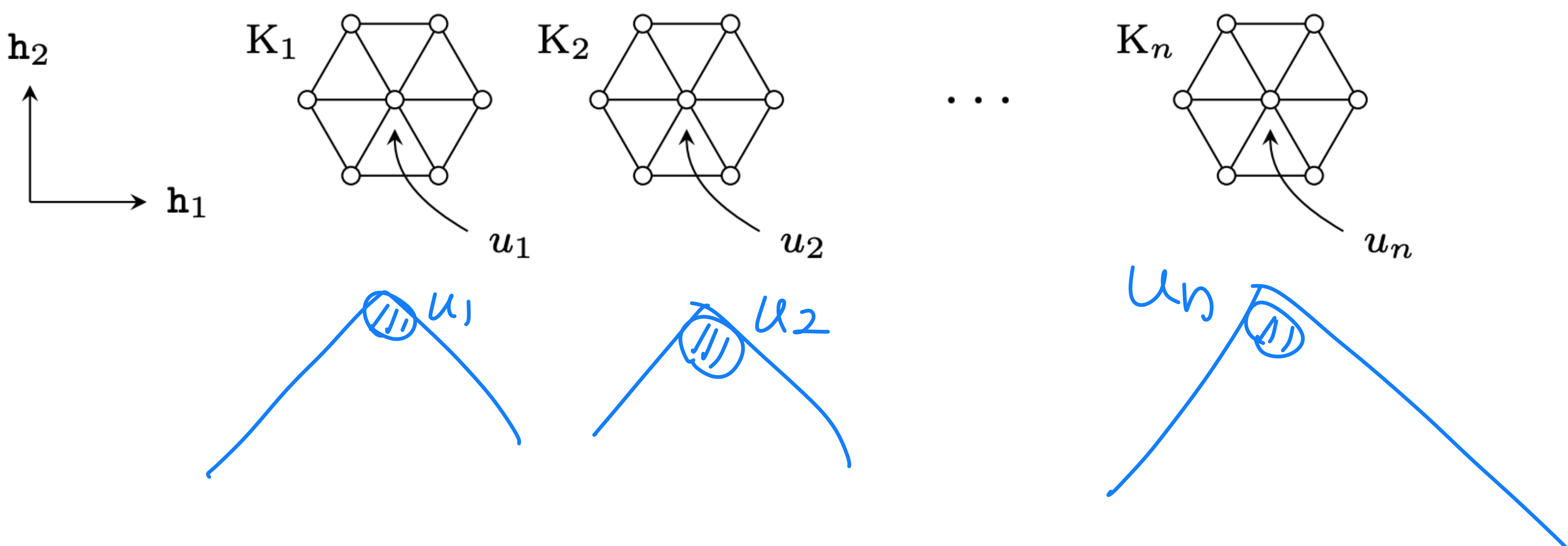
$\Delta : A \to A \otimes A$

$$\rho_1 \otimes \rho_2 : \; A \xrightarrow{\Delta} \begin{matrix} A \\ \otimes \\ A \end{matrix} \begin{matrix} \xrightarrow{\rho_1} \\ \otimes \\ \xrightarrow[\rho_2]{} \end{matrix} \begin{matrix} V_1 \\ \otimes \\ V_2 \end{matrix}$$

$$\left(\begin{array}{ccc} & (\Delta \otimes 1) \circ \Delta & (A \otimes A) \otimes A \\ A & & \cong \\ & (1 \otimes \Delta) \circ \Delta & A \otimes (A \otimes A) \end{array} \right)$$

We can make "crystal chains" by bringing together 2d crystals in spectral-parameter plane

[Galakhov-Y, Galakhov-Li-Y ('21)]



$$|\mathrm{K}_1, {}^\sharp\mathcal{C}_1\rangle_{u_1} \otimes |\mathrm{K}_2, {}^\sharp\mathcal{C}_2\rangle_{u_2} \otimes \ldots \otimes |\mathrm{K}_n, {}^\sharp\mathcal{C}_n\rangle_{u_n} \,.$$

We can derive representations

[Galakhov-Y, Galakhov-Li-Y ('21)]

$$\mathbf{\Delta}_0^{(n)}(\psi(z)) \bigotimes_{i=1}^{n} |\mathrm{K}_i\rangle_{u_i} = \prod_i \Psi_{\mathrm{K}_i}(z-u_i) \times \bigotimes_i |\mathrm{K}_i\rangle_{u_i}\,,$$

$$\mathbf{\Delta}_0^{(n)}(e(z)) \bigotimes_{i=1}^{n} |\mathrm{K}_i\rangle_{u_i} = \sum_i \sum_{\square\in\mathrm{Add}(\mathrm{K}_i)} \prod_{j<i} \Psi_{\mathrm{K}_j}(u_i+h_\square-u_j) \times \frac{[\mathrm{K}_i\to\mathrm{K}_i+\square]}{z-(u_i+h_\square)} \times \bigotimes_{j<i}|\mathrm{K}_j\rangle_{u_j} \otimes |\mathrm{K}_i+\square\rangle_{u_i} \otimes \bigotimes_{k>i}|\mathrm{K}_k\rangle_{u_k}\,,$$

$$\mathbf{\Delta}_0^{(n)}(f(z)) \bigotimes_{i=1}^{n} |\mathrm{K}_i\rangle_{u_i} = \sum_i \sum_{\square\in\mathrm{Rem}(\mathrm{K}_i)} \prod_{k>i} \Psi_{\mathrm{K}_k}(u_i+h_\square-u_k) \times \frac{[\mathrm{K}_i\to\mathrm{K}_i-\square]}{z-(u_i+h_\square)} \times \bigotimes_{j<i}|\mathrm{K}_j\rangle_{u_j} \otimes |\mathrm{K}_i-\square\rangle_{u_i} \otimes \bigotimes_{k>i}|\mathrm{K}_k\rangle_{u_k}\,,$$

※ not inv. under permutations

and "standard coproduct"

$$\mathbf{\Delta}_0 e = e\otimes 1 + \psi \overset{\rightarrow}{\otimes} e\,,$$
$$\mathbf{\Delta}_0 f = 1\otimes f + f \overset{\leftarrow}{\otimes} \psi\,,$$
$$\mathbf{\Delta}_0 \psi = \psi\otimes\psi\,.$$

However,

$$\Delta_0 e = e \otimes 1 + \psi \overset{\rightarrow}{\otimes} e\,,$$
$$\Delta_0 f = 1 \otimes f + f \overset{\leftarrow}{\otimes} \psi\,,$$
$$\Delta_0 \psi = \psi \otimes \psi\,.$$

- Δ_0 does NOT reproduce R-matrix needed for (BAE) = (vacuum equation)
- For rational/Yangian case does NOT come from a coproduct $\Delta_0 : Y \to Y \otimes Y$

[Prochazka ('15)] [Galakhov-Li-Y ('22)]

We need to search for "correct" coproduct

$$\Delta \neq \Delta_0$$

Assumption 1

$$\Delta = \mathcal{U}\,\Delta_0\,\mathcal{U}^{-1}$$

true Δ evaluated in crystal rep.

Δ_0 obtained by bringing together crystal

cf. stable envelope of [Maulik-Okounov]

Physically:

The R-matrix involves adiabatic continuation
in parameter space
(Berry connection)

We need to solve the BPS flow eqn

$\rightsquigarrow$ there are BPS solitions
interpolating different vacua

Counting by "Mirror"

crystal basis $\longrightarrow$ Lefshetz thimble basis

K $\qquad\qquad$ $\mathcal{L}$

in GLSM $\qquad\qquad$ in Landau-Ginzburg $\mathcal{W}$

Morse flow eqn.

$$\frac{\partial \sigma}{\partial t} \sim \overline{\frac{\partial \mathcal{W}}{\partial \sigma}}$$

$$\Delta = \mathcal{U} \Delta_0 \mathcal{U}^{-1}$$

Assumption 2

$\mathcal{U}$ is lower-triangular

cf. stable envelope of [Maulik-Okounov]

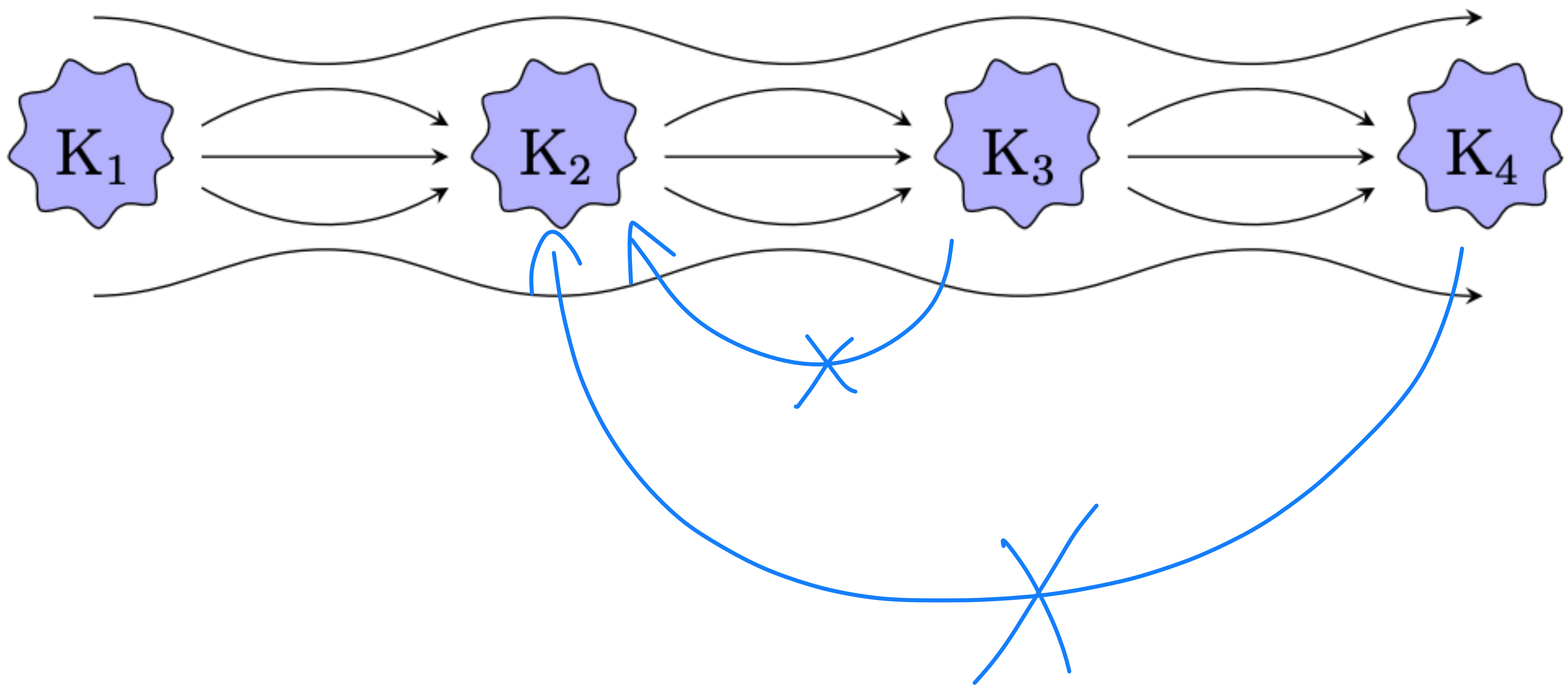
Gauss decomposition of universal R-matrix

$$\mathcal{U} = 1 \otimes 1 + \sum_{k=1}^{\infty} S_k \qquad \deg S_k = 2k$$

$$\left(\begin{array}{lll} \deg(e \otimes 1) = +1 & \deg(f \otimes 1) = +1 & \deg(\psi \otimes 1) = 0 \\ \deg(1 \otimes e) = -1 & \deg(1 \otimes f) = -1 & \deg(1 \otimes \psi) = 0 \end{array} \right)$$

Physically:

ordering from downword Morse flow



Assumption 3

$$\left(e^{a}_{(z)}|k\rangle_u = \sum_{\substack{\boxed{a} \\ \in \\ Add(k)}} \frac{[k\to k+\boxed{a}]}{z-(u+h_{\boxed{a}})}\, |k+\boxed{a}\rangle_u \right)$$

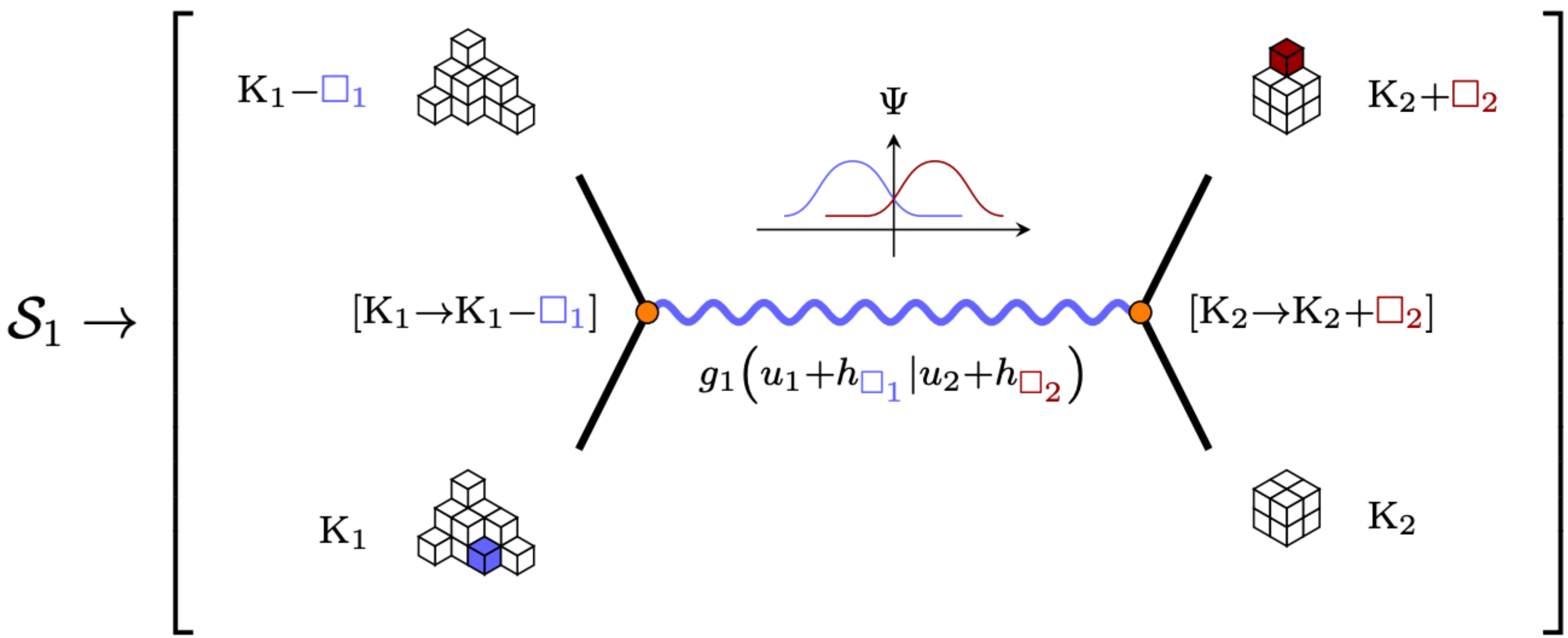
$$S_1\, |k_1\rangle_{u_1} \otimes |k_2\rangle_{u_2}$$

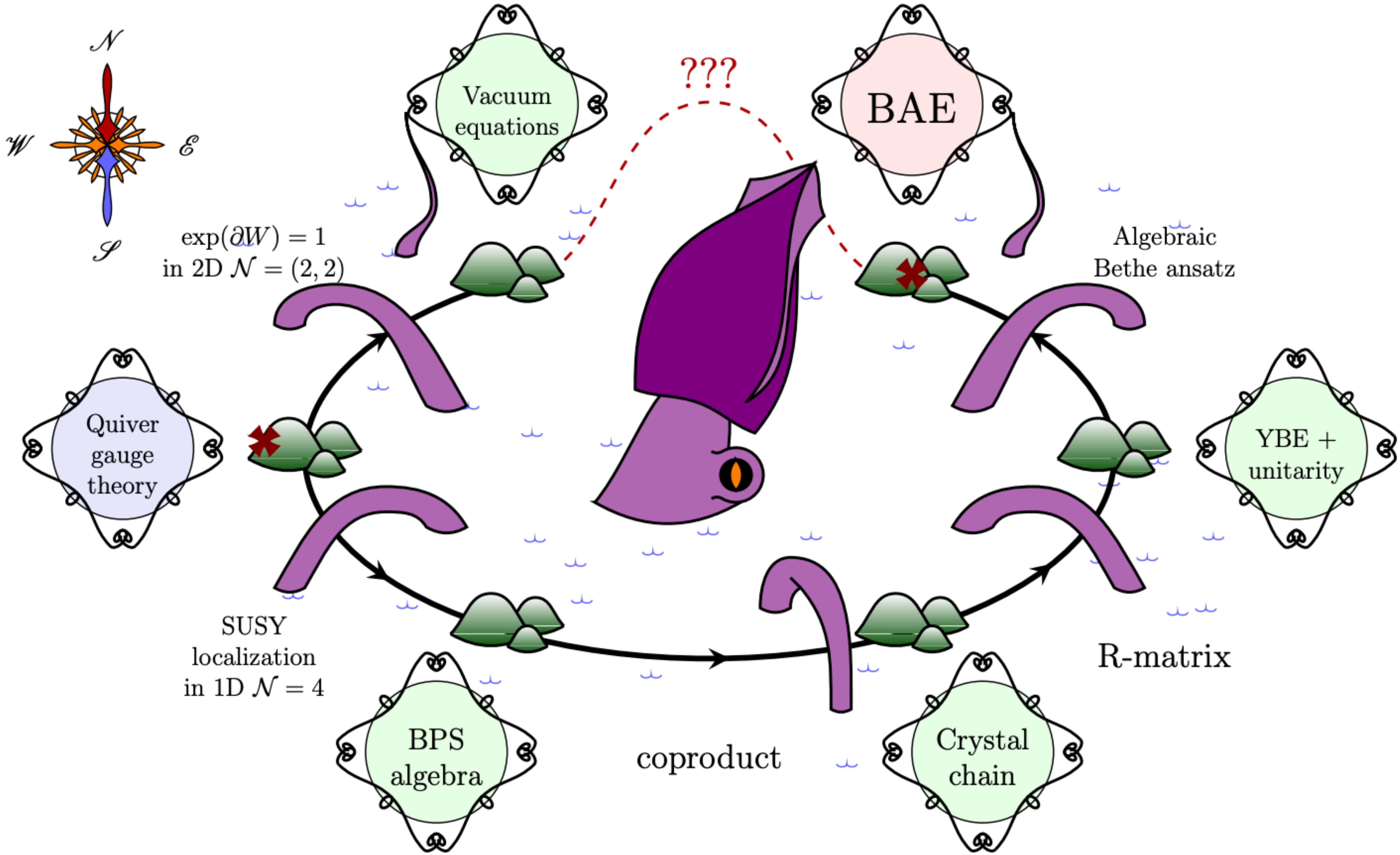
$$= \sum_{a\in Q_1} (-1)^{|a|+1} \sum_{\boxed{a}_1 \in Rem(k_1)} \sum_{\boxed{a}_2 \in Add(k_2)} \frac{[k\to k-\boxed{a}_1][k'\to k'+\boxed{a}_2]}{z+h_{\square}-h_{\square'}}\, |k-\boxed{a}_1\rangle_{u_1} |k'+\boxed{a}_2\rangle_{u_2}$$

(S_1 above satisfies all consistency conditions for coproduct)

Physically:

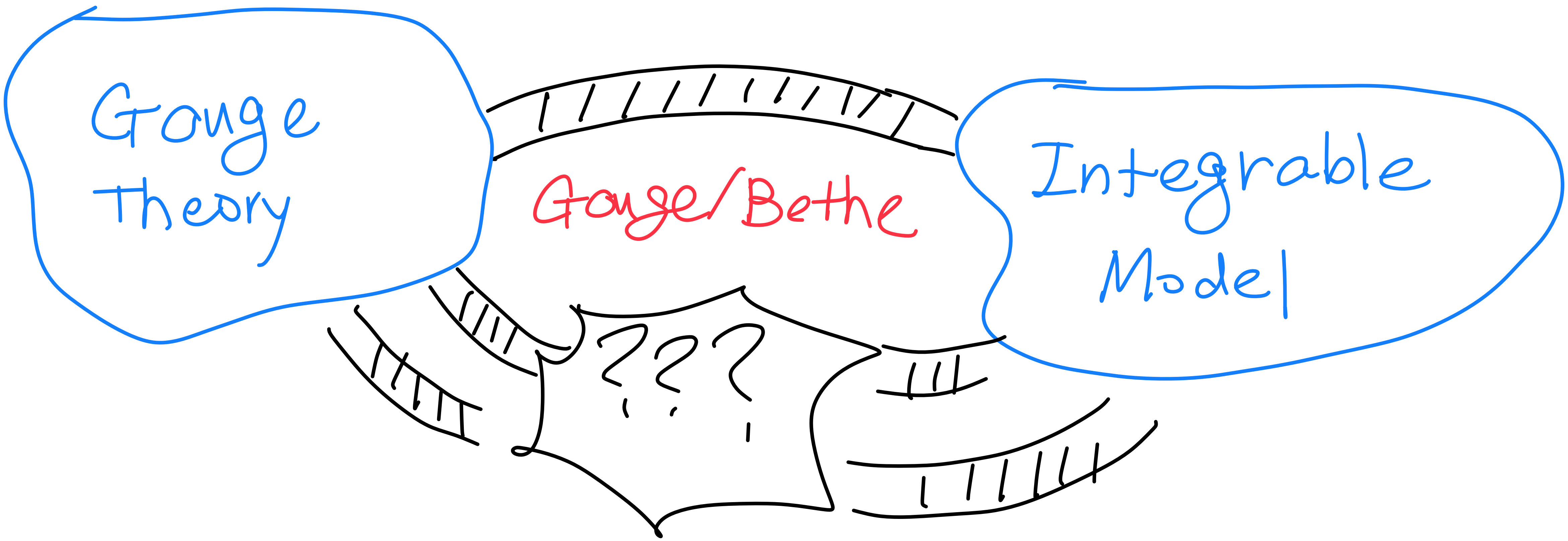
simple t-channel process





$\mathscr{N}$
$\mathscr{W}$
$\mathscr{E}$
$\mathscr{S}$
Vacuum equations
???
BAE
$\exp(\partial W) = 1$ in 2D $\mathcal{N} = (2,2)$
Algebraic Bethe ansatz
Quiver gauge theory
YBE + unitarity
SUSY localization in 1D $\mathcal{N} = 4$
R-matrix
BPS algebra
coproduct
Crystal chain

Summary



- Still foundational issues to be solved
- "beyond $Y(\mathfrak{g})$"

  quiver Yangian $Y(Q,W)$ provides new clue