

Discrete and Continuous **Integrable Models** From **Four Dimensions**

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To appear with
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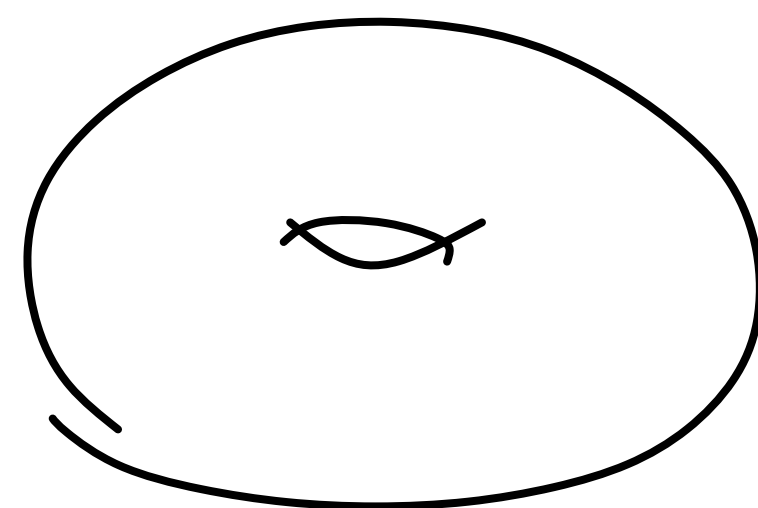
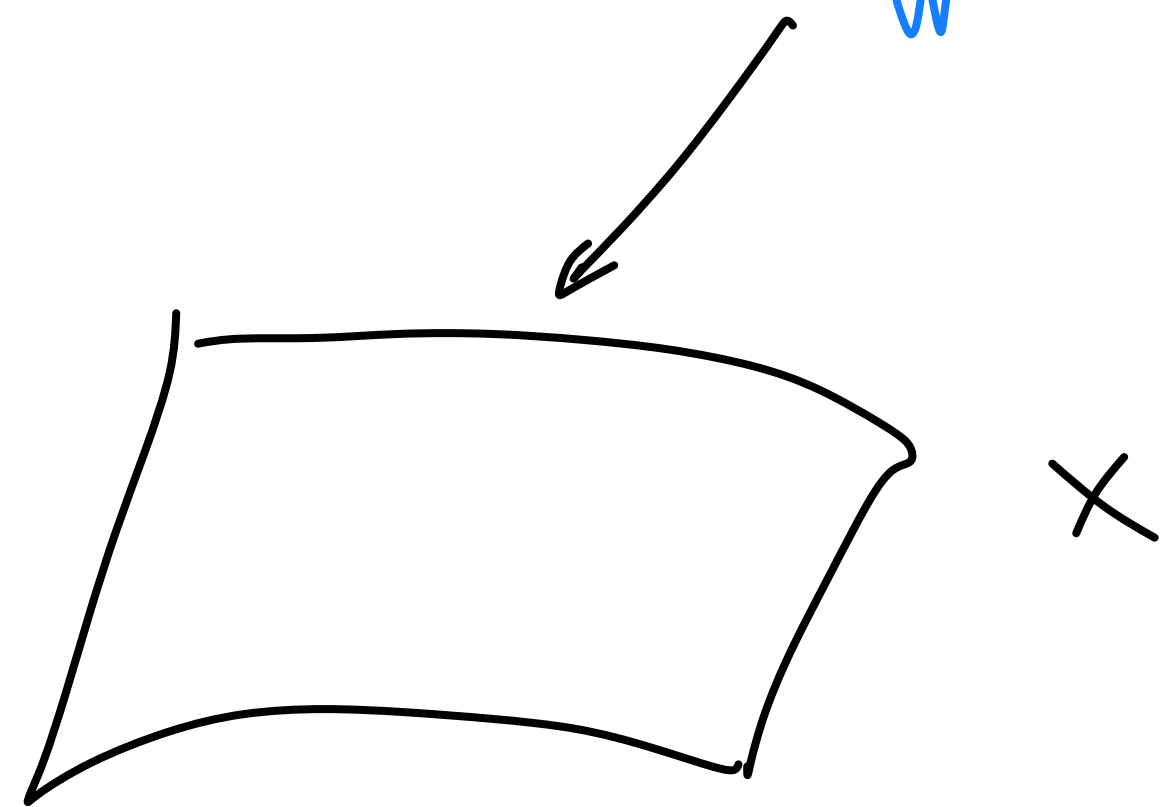
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Revisiting Integrable Models

via 4d Chern-Simons Theory

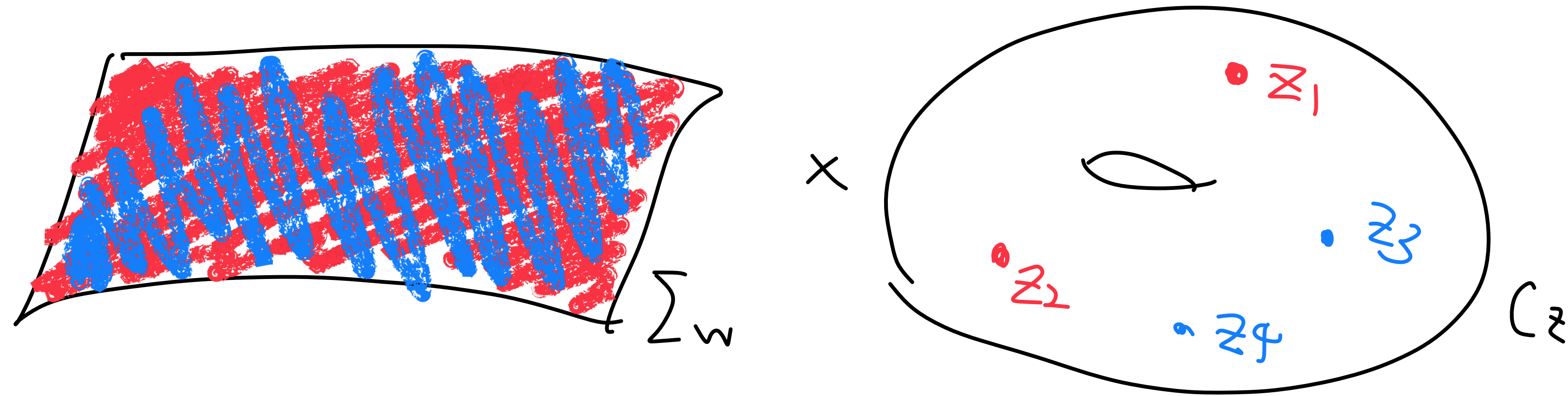
[Costello ('13)
Costello-Witten-MY ('17, '18), Costello-MY ('19)]

$$S = \frac{1}{2\pi\hbar} \int_{\Sigma_W \times C} \omega \wedge \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$



$C =$ spectral curve

4d CS + 2d surface defects



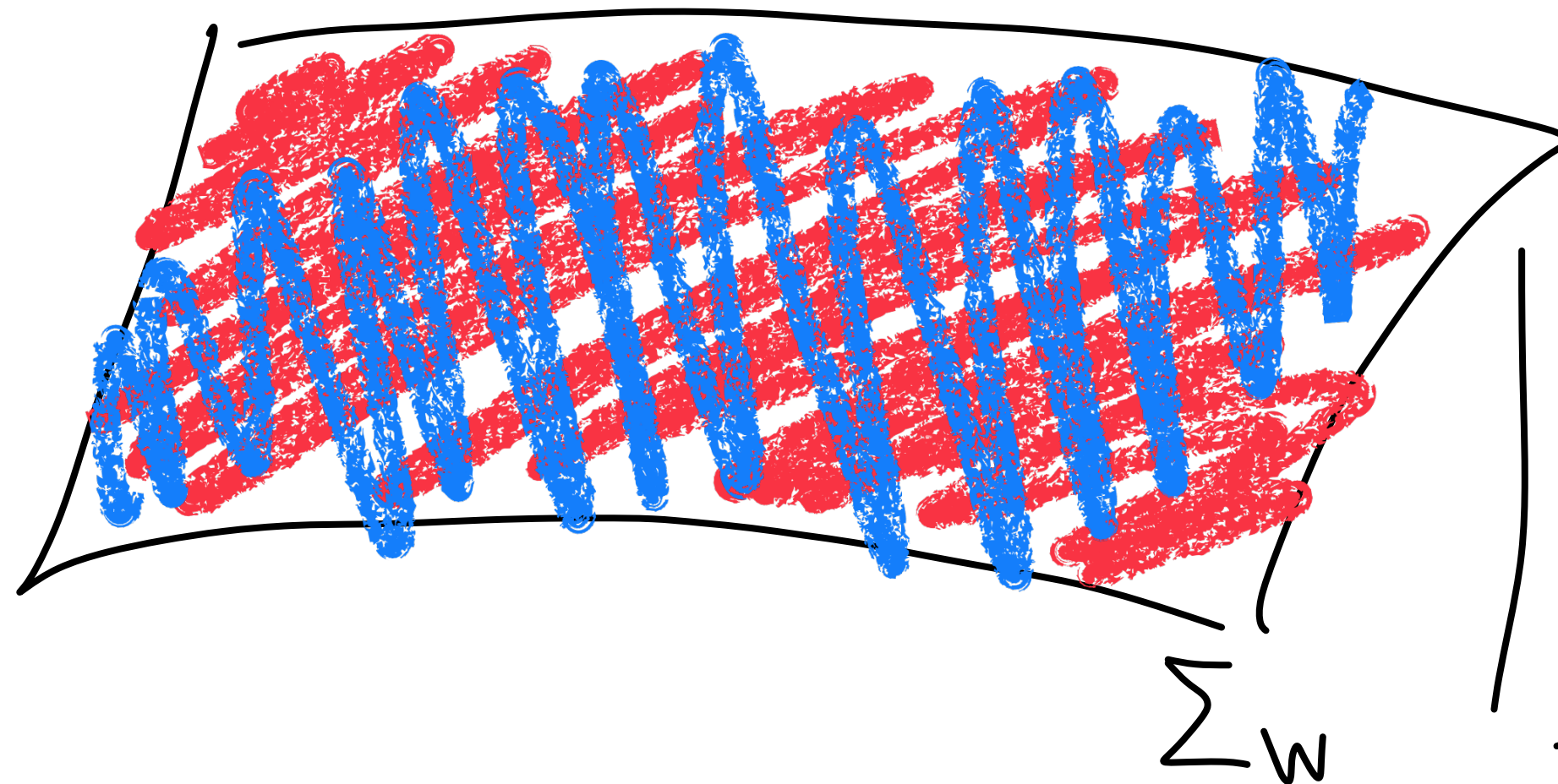
integrate along C

4d gauge field

2d

Integrable

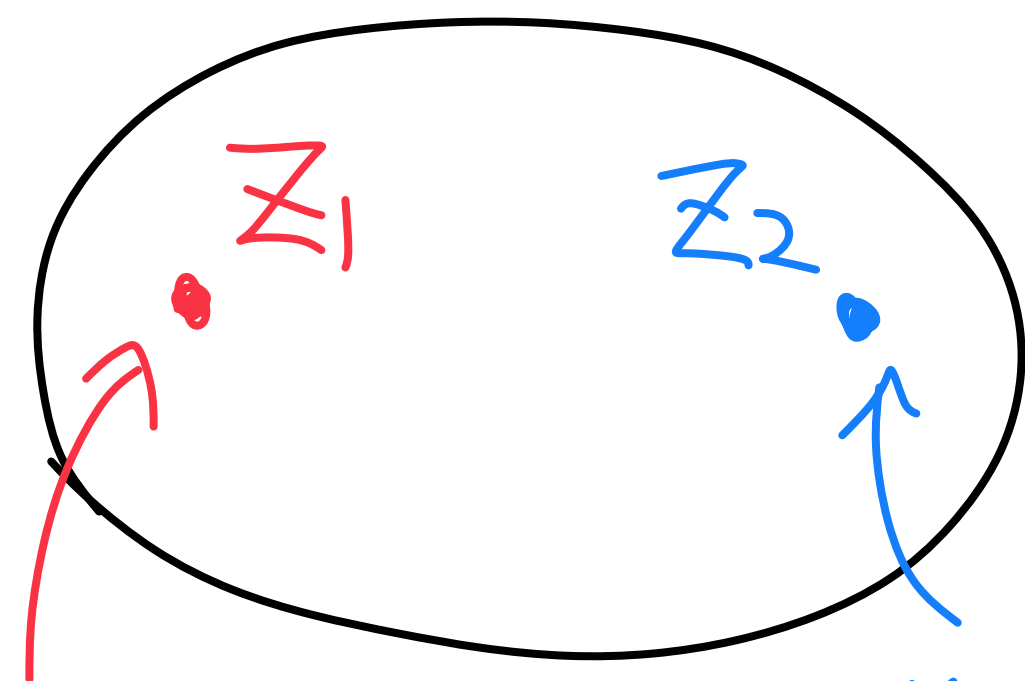
QFTs



2d Lax connection

$$\mathcal{L} = A_w(z) dw + A_{\bar{w}}(z) d\bar{w}$$

eg.
chiral
free fermion
defects

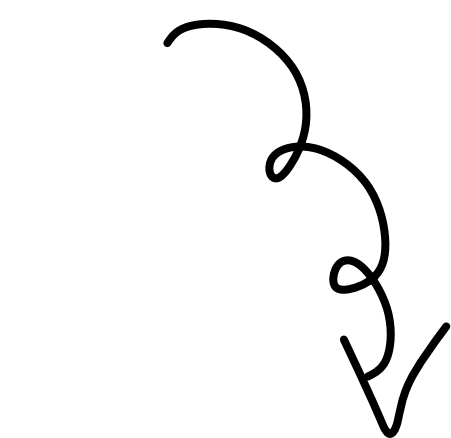


chiral fermion

$$\mathcal{L}_1 = \bar{\Psi}_L (\partial + A_w) \Psi_L$$

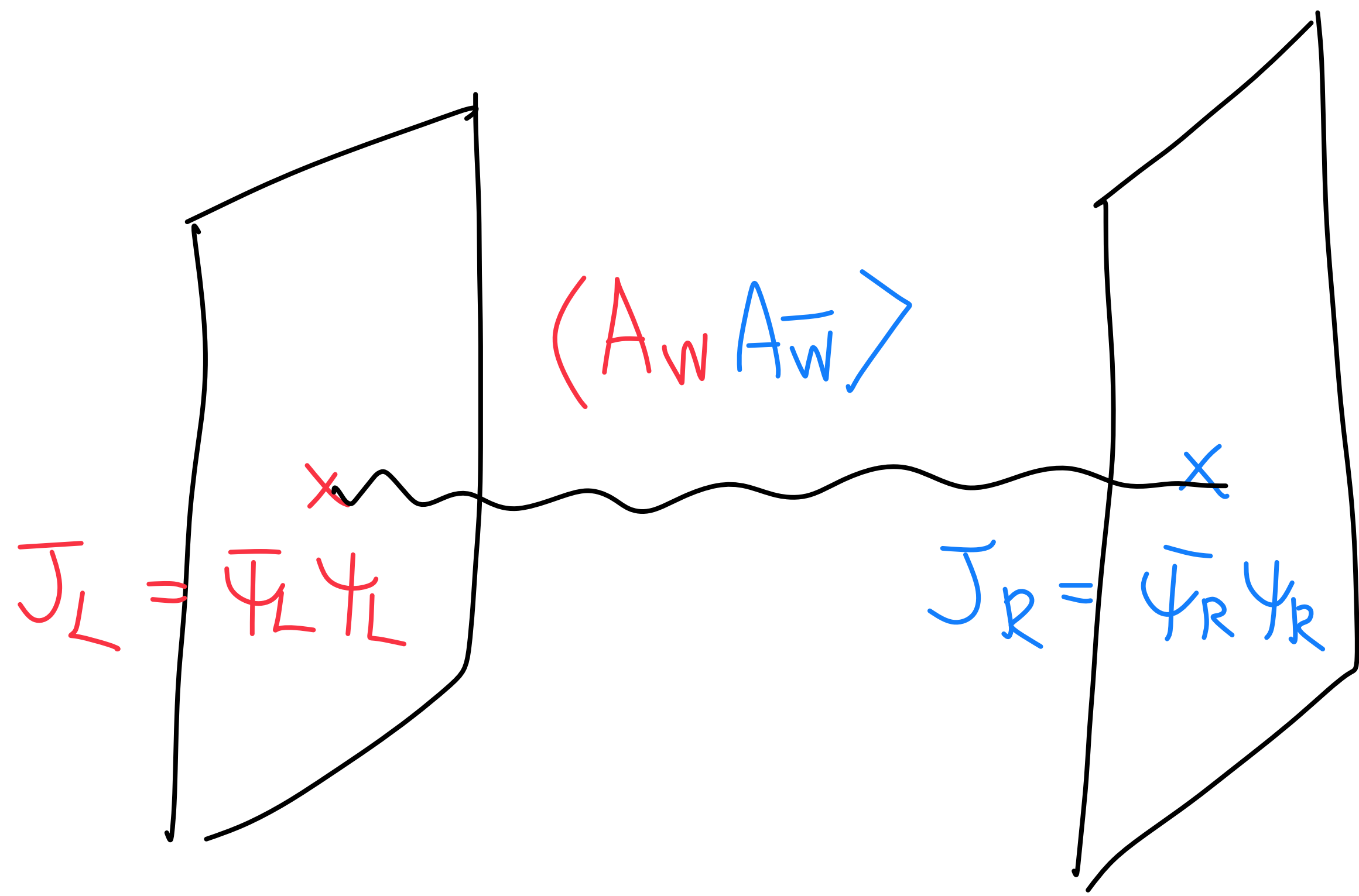
anti-chiral fermion

$$\mathcal{L}_2 = \bar{\Psi}_R (\partial + A_w) \Psi_R$$



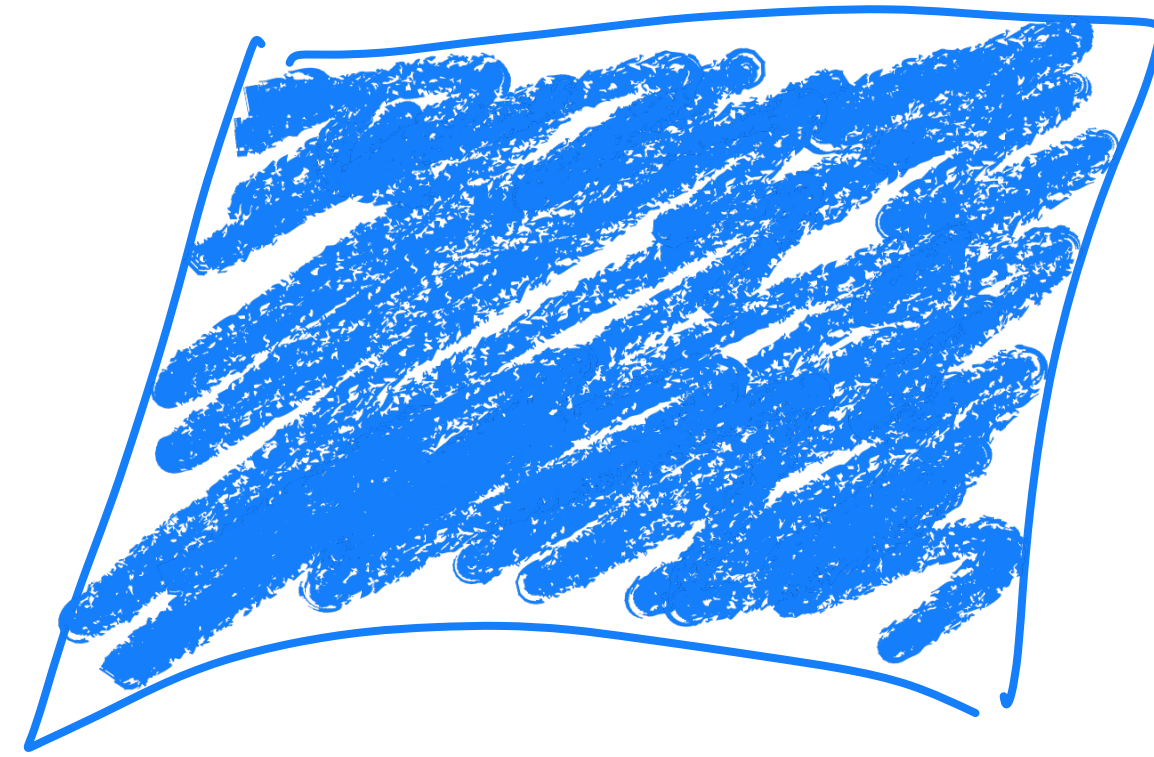
$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \frac{1}{z_1 - z_2} \underbrace{(\bar{\Psi}_L \Psi_L)(\bar{\Psi}_R \Psi_R)}_{4\text{-fermi interaction}}$$

4-fermi
interaction

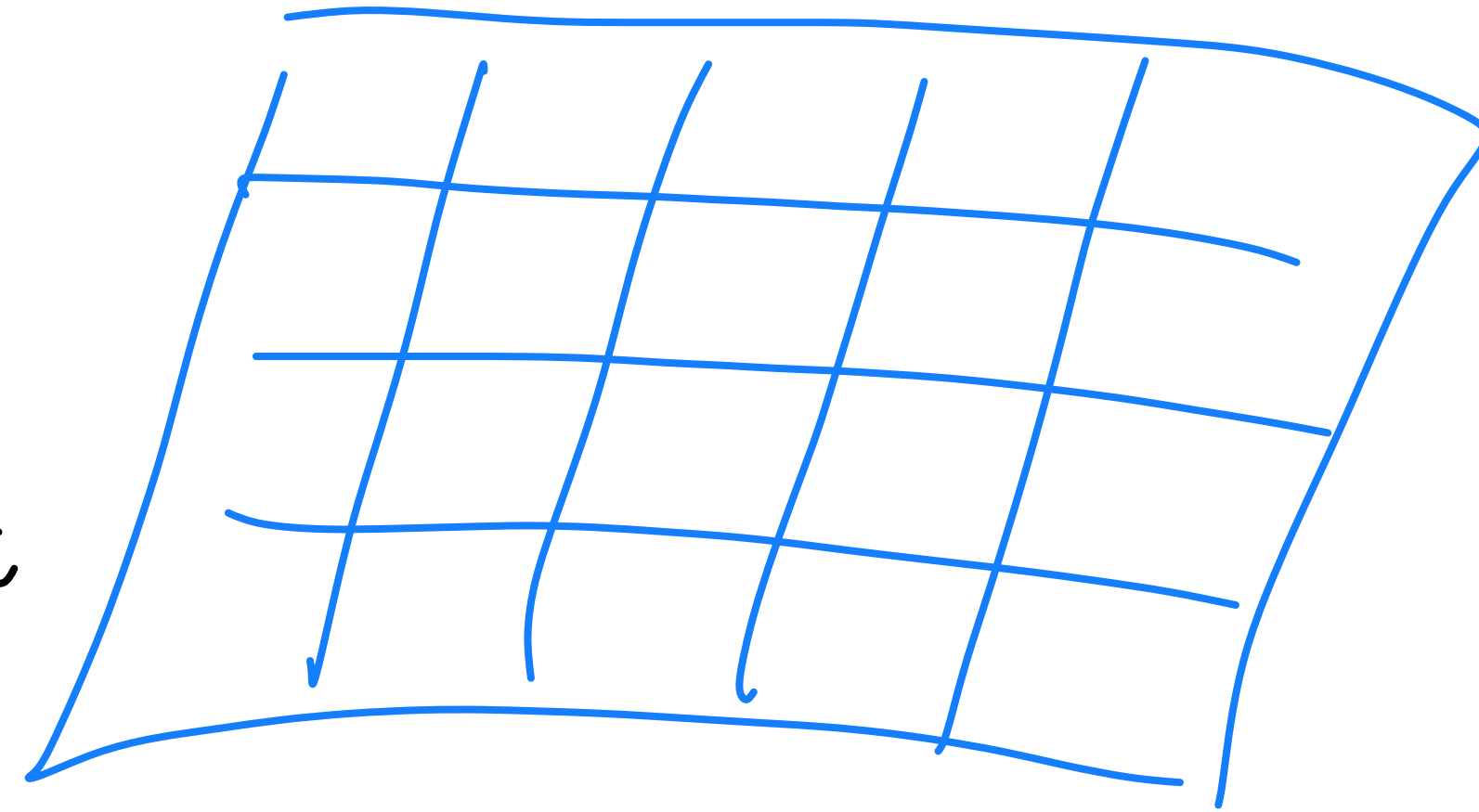


Question 1:

How to discretize integrable field theories?



thermodynamic
limit



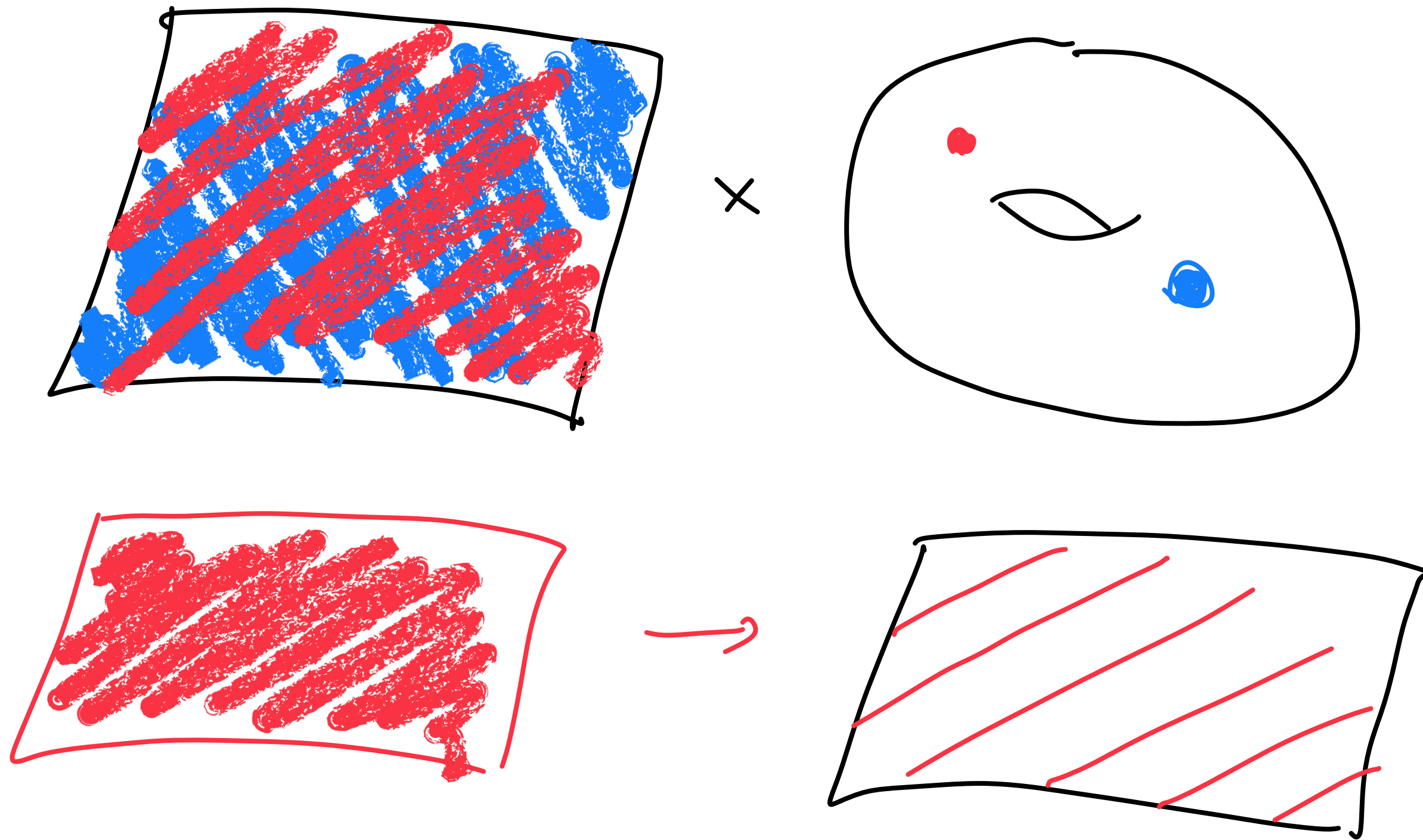
Question 2:

How to describe dualities of IFTs?

e.g. non-Abelian bosonization

$SU(N)$ Thirring \longleftrightarrow $SU(N)$ WZW

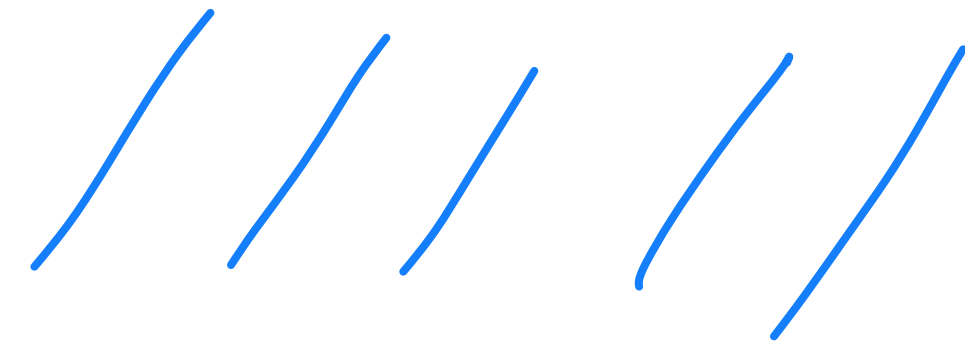
Answer 1:



2d defect \rightsquigarrow lattice of 1d defects
discretize

Chiral defects

$$S = \int d^2\sigma \bar{\Psi} (\not{\partial}_w + A_w) \Psi \rightsquigarrow S = \sum_i \int d\sigma^- \bar{\Psi}_i (\not{\partial}_w + A_w) \Psi_i$$

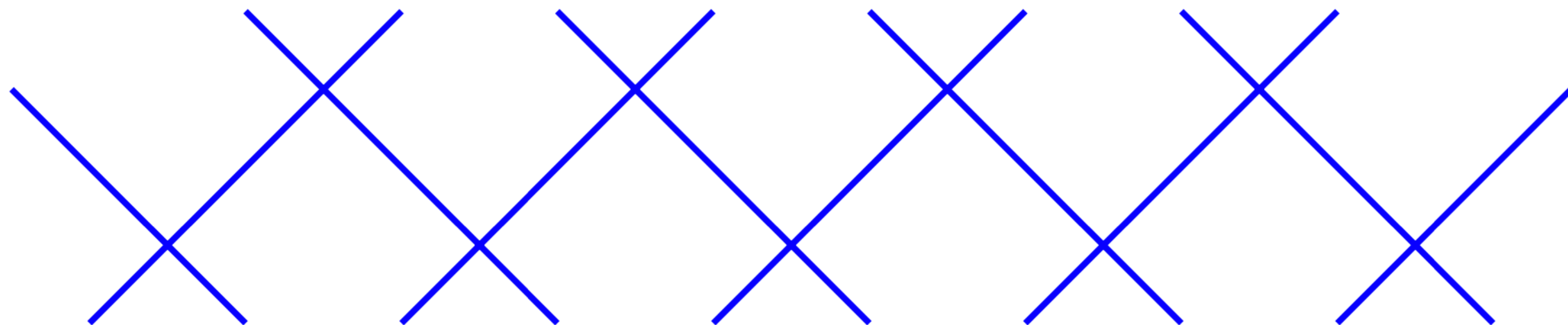


anti-chiral defects

$$S = \int d^2\sigma \bar{\Psi} (\not{\partial}_{\bar{w}} + A_{\bar{w}}) \Psi \rightsquigarrow S = \sum_i \int d\sigma^+ \bar{\Psi}_i (\not{\partial}_{\bar{w}} + A_{\bar{w}}) \Psi_i$$



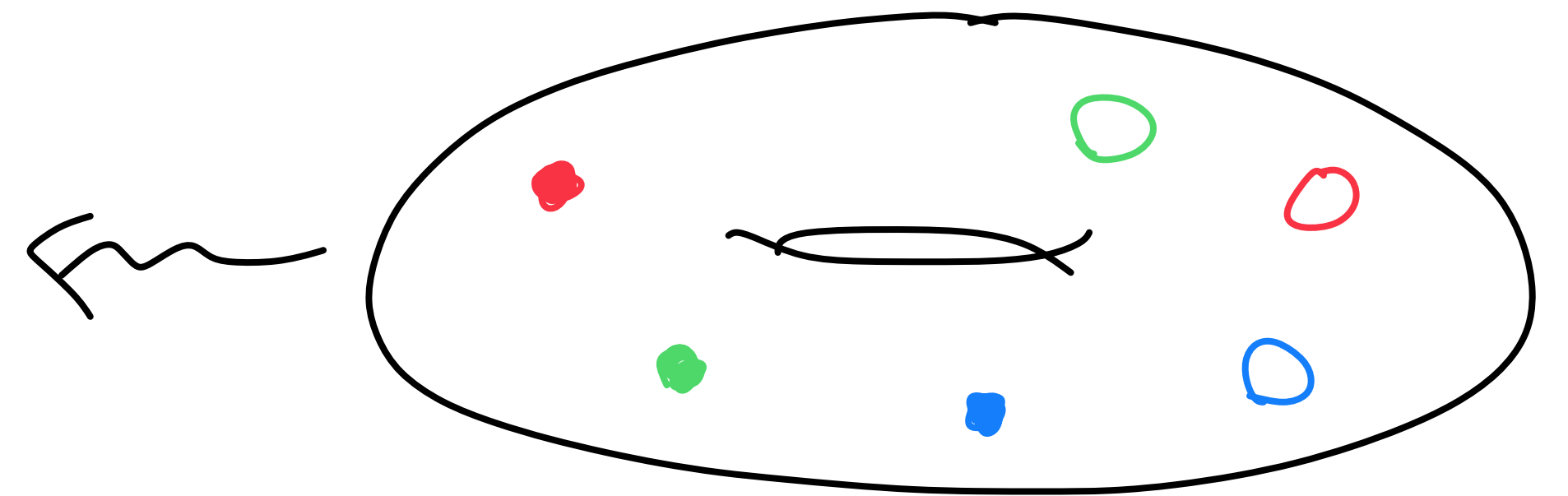
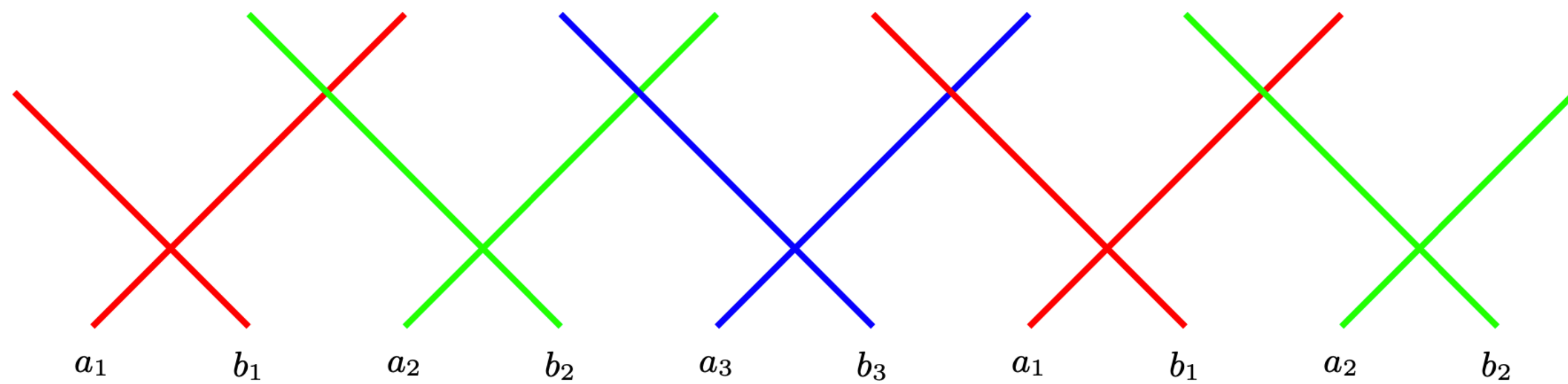
when combined, light-cone lattice of 1d defects



integrable light-cone discretization

[Many papers by Destri-de Vega
also Faddeev, Reshetikhin, Volkov, ...]

with inhomogeneities



We can discuss discretizations of

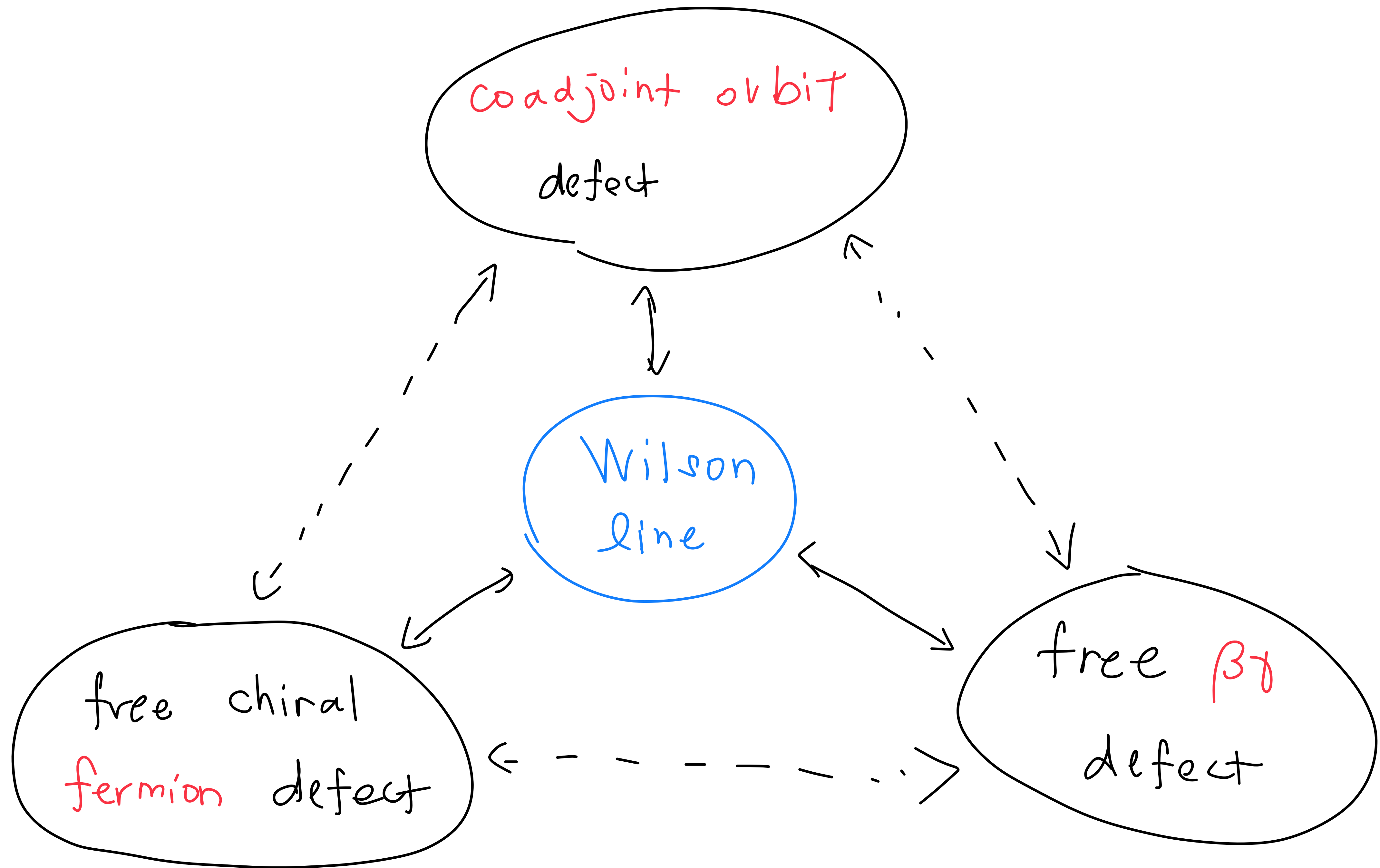
- Free Chiral Fermion $\bar{\Psi} (\not{\partial} + \not{A}) \Psi$
- Coadjoint Orbit Defect $\text{Tr} (\wedge g^{-1} (\partial + A) g)$
- Free $\beta\gamma$ defect $\beta (\partial + A) \gamma$
- Curved $\beta\gamma$ defect $\gamma: \mathbb{C} \rightarrow X \quad \beta \in \Omega^{1,0}(\mathbb{C}, \gamma^* T^* X)$

all related to Wilson lines

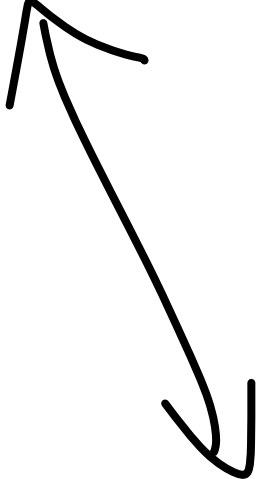
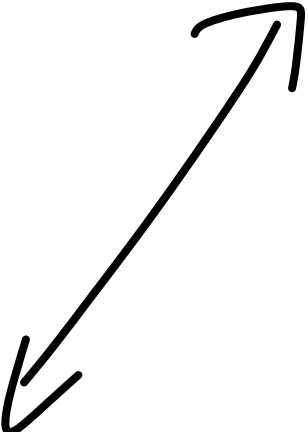
[Borel-Weil-Bott, ...]

↪ [Costello-Witten-MY '17]

standard spin chains



Generalized
Faddeev-Reshetikhin
Model



Constrained
Fermionic Massless
Thirring Model

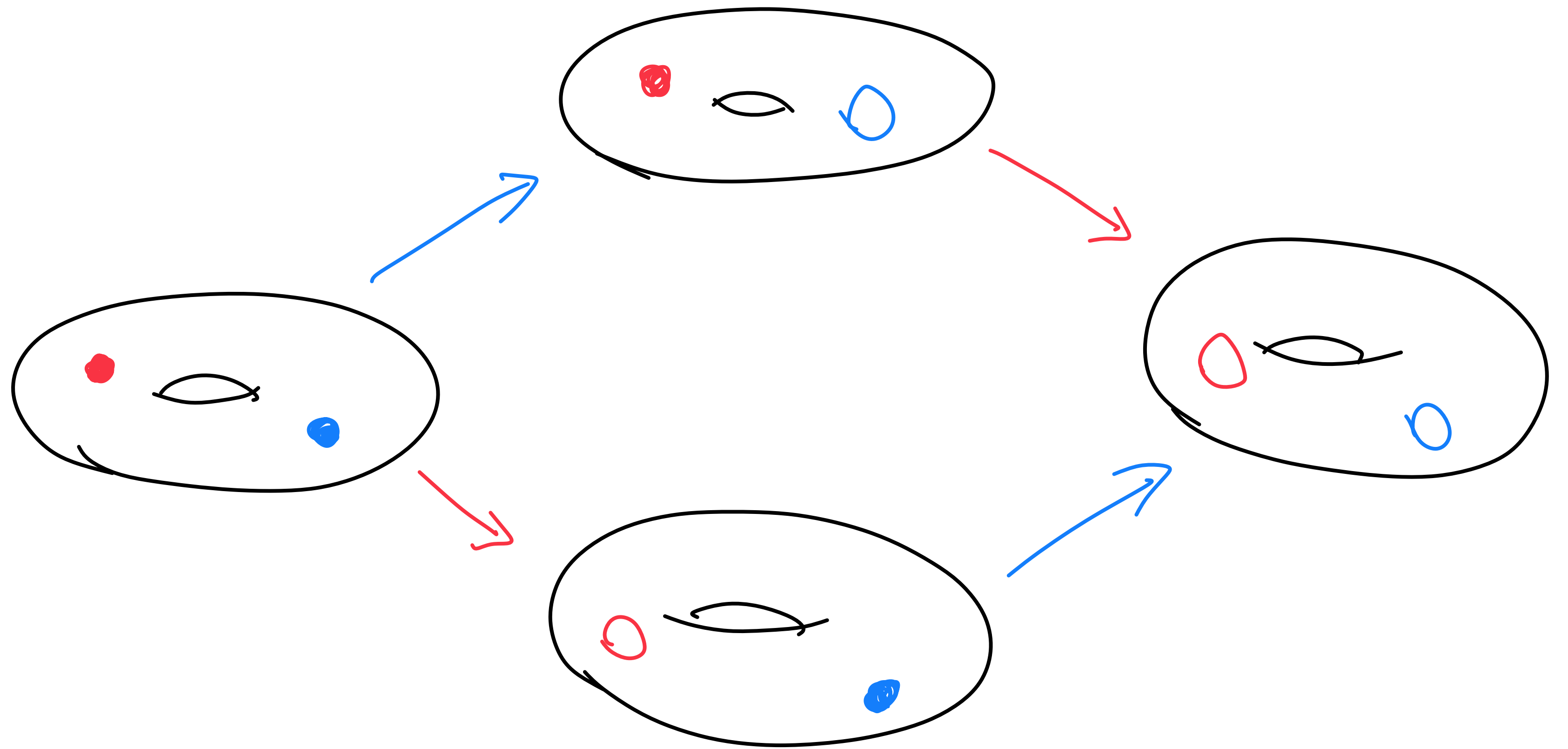


Constrained
Bosonic Massless
Thirring Model

Answer 2:

dualities among defects
(e.g. bosonization)

● = ○
one 2d defect = another 2d defect



"duality web" for multiple defects

"Seed duality" \rightsquigarrow many new bosonizations

bosonization ('80s)

Free Fermion

$$\sum_{j=1}^{N_F} \bar{\Psi}_j i \underbrace{D_R}_{=} \Psi_j$$

$$\mathcal{D} + A \frac{1-\gamma_5}{2}$$

WZW model

$$N_F S_{WZW}[g_-]$$

$$+ \text{Tr}[A + J_-]$$

$$J_- := \frac{N_F}{2\pi} g_-^{-1} \partial_- g_-$$

bosonization

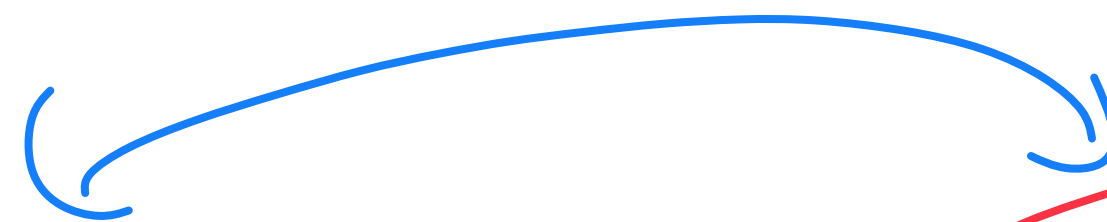
Free Fermion
Defect



WZW
defect

multiple
defects }
↓

new
bosonization



} multiple
defects
↓

massless
Thirring-type model

coupled WZW
model

Summary

• Discretization: 2d defects

→ 1d line defects

(often converted to Wilson lines)

• Duality: dualities among 2d defects

→ a huge web of 2d dualities
among IFTs

1d duality