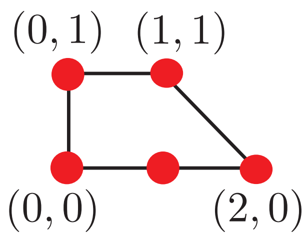
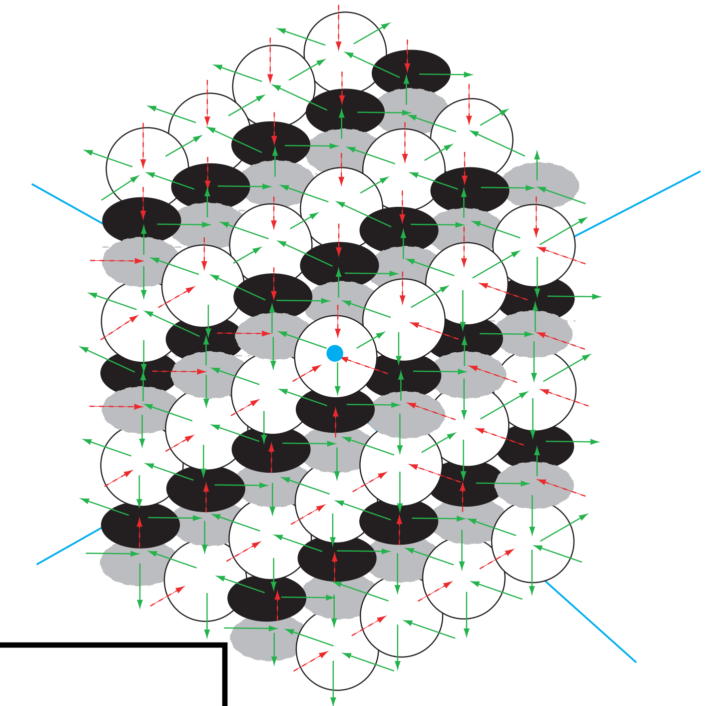


$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,\\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,,\\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,,
\end{aligned}
$$



Quiver Yangians and Crystal Meltings

Masahito Yamazaki

IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

13th Joburg Workshop on String Theory

September 6, 2023

Based on

Wei Li + MY

(2003.08909 [hep-th])

Dimitry Galakhov + MY

(2008.07006 [hep-th])

Dimitry Galakhov+Wei Li + MY

(2106.01230 [hep-th])

(2108.10286 [hep-th])

(2206.13340 [hep-th])

And many works in the literature

Also earlier works, e.g.

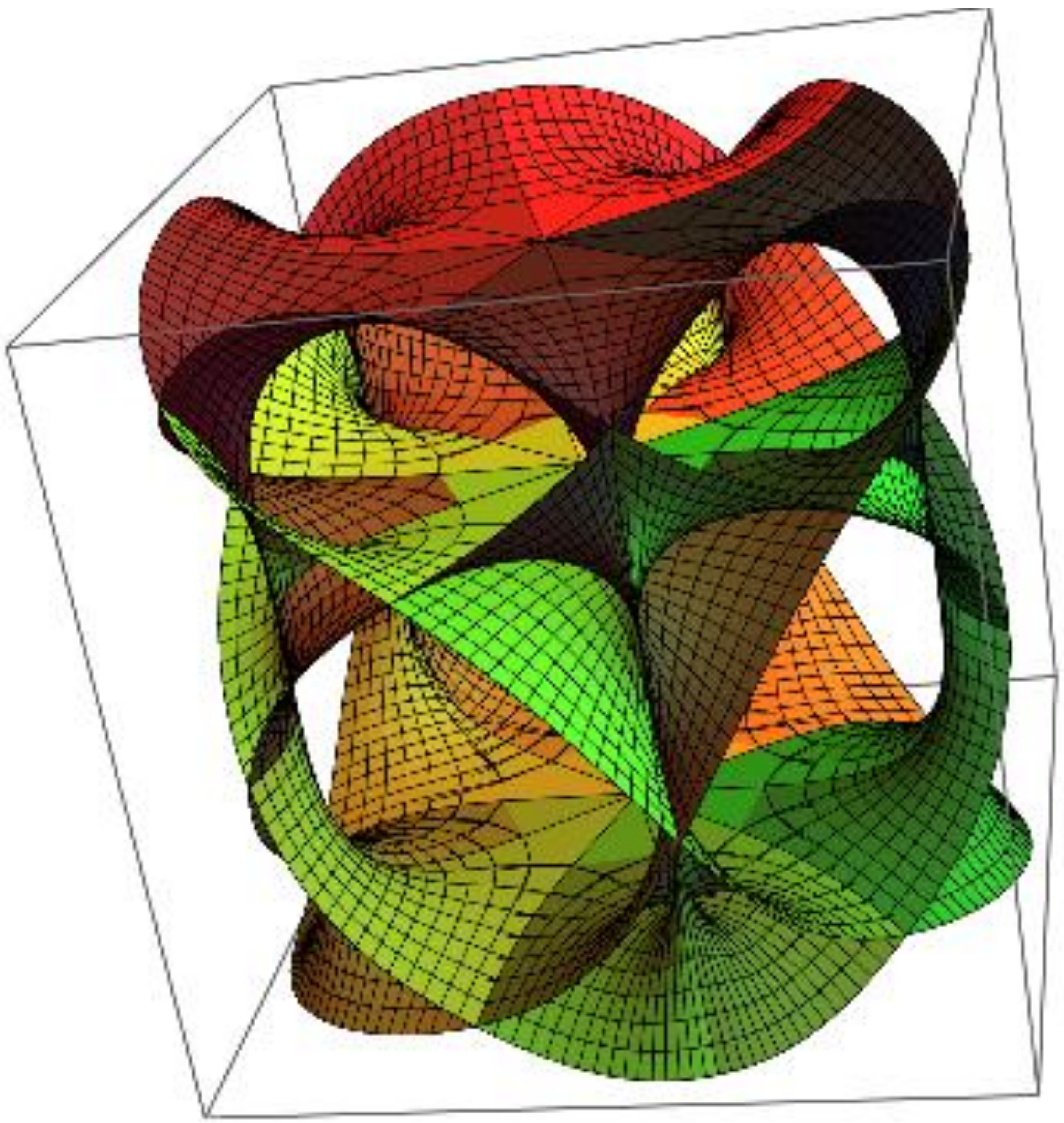
Hirosi Ooguri + MY (0811.2810 [hep-th])

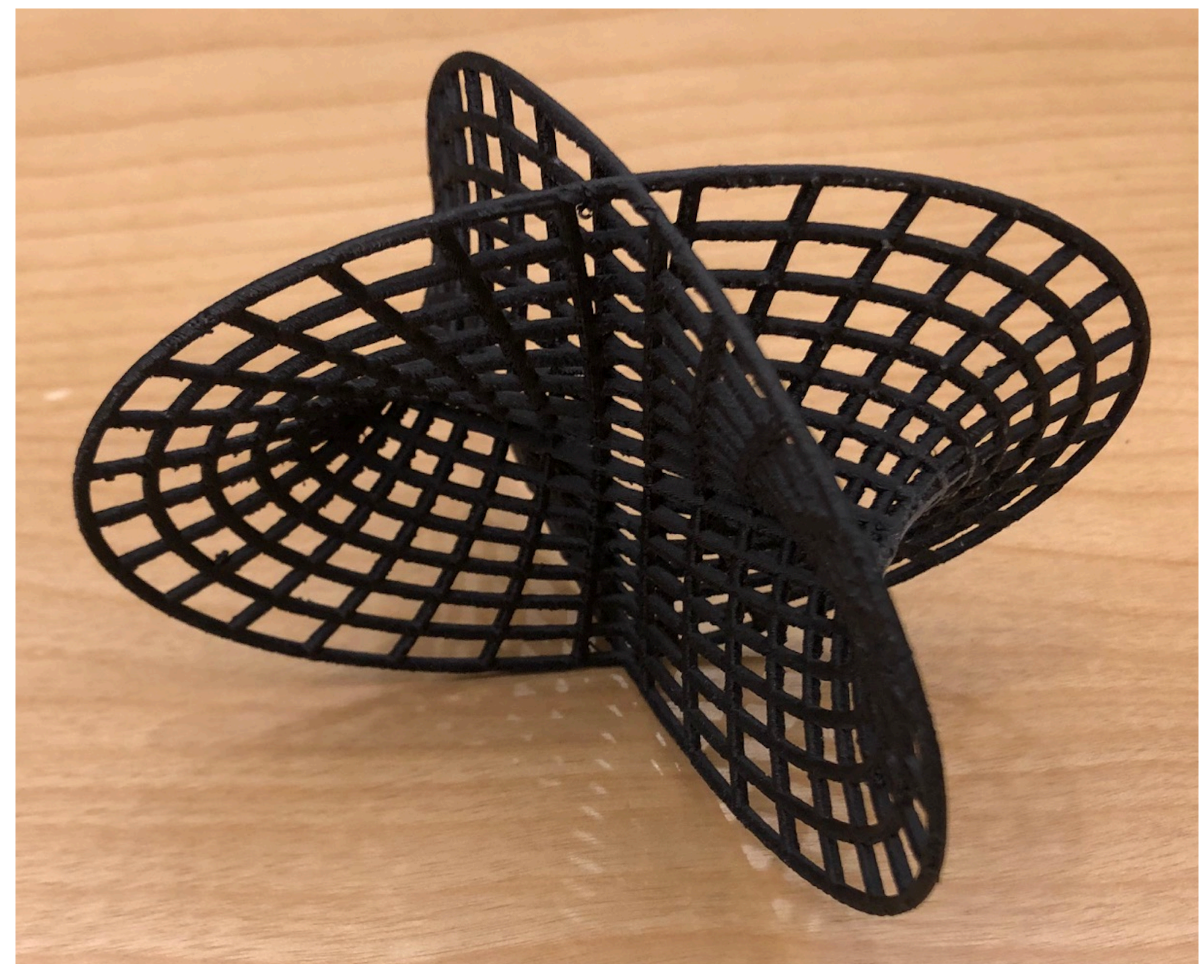
MY (Ph.D. thesis, 1002.1709 [hep-th])

MY (Master thesis, 0803.4474 [hep-th])

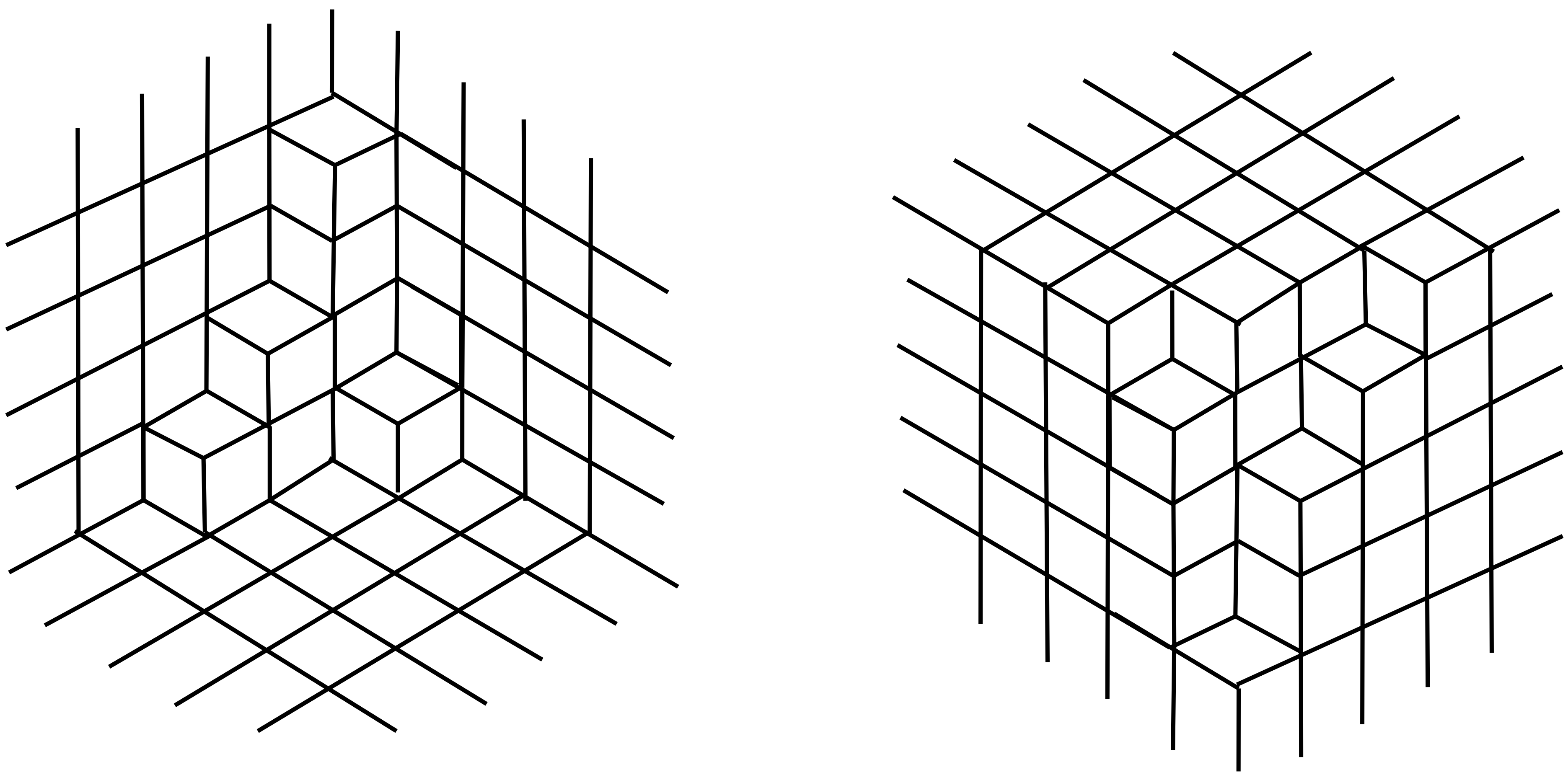
Overview

Calabi - Yau





"Quantum Toric Calabi-Yau"

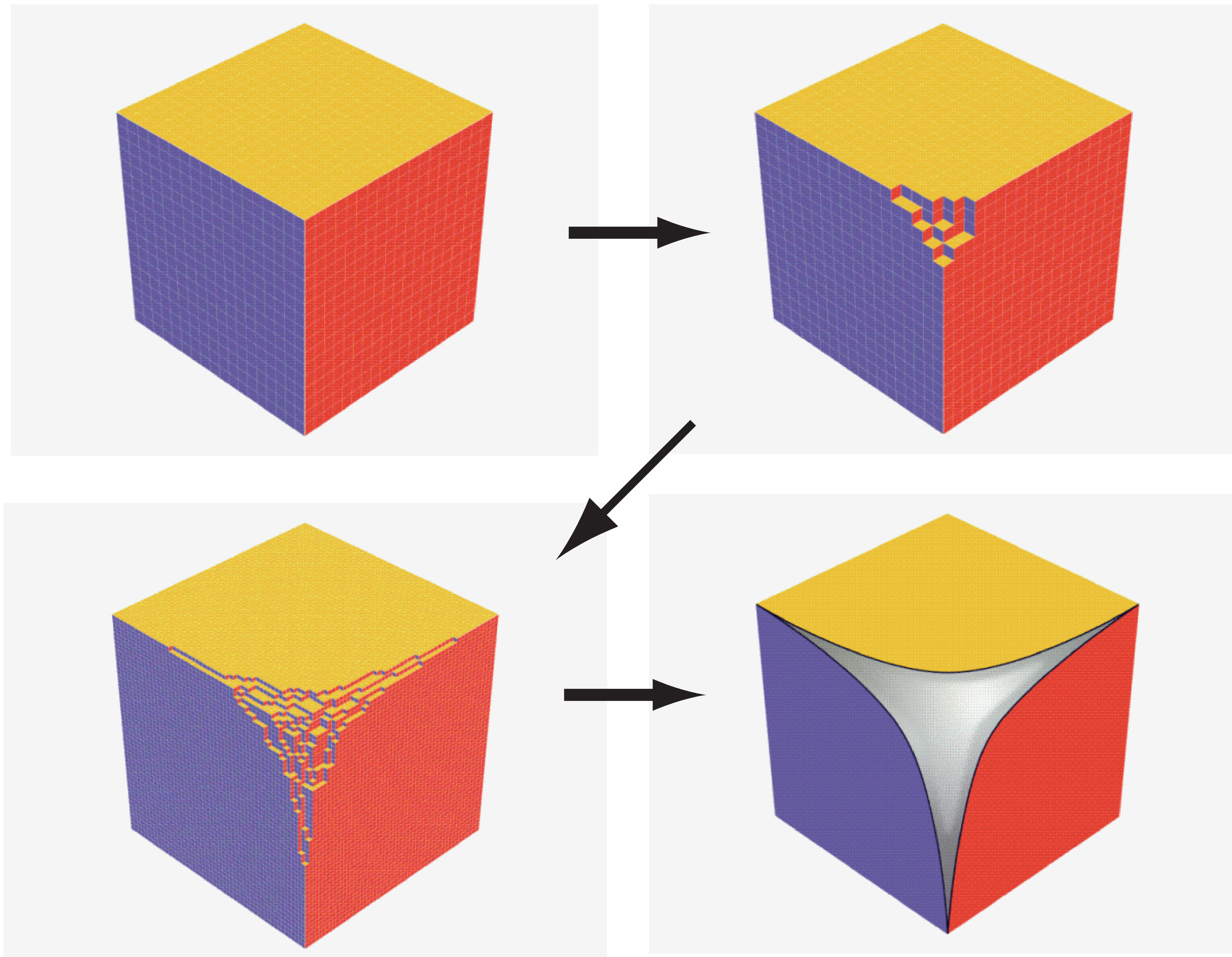


$\mathbb{C}^3$

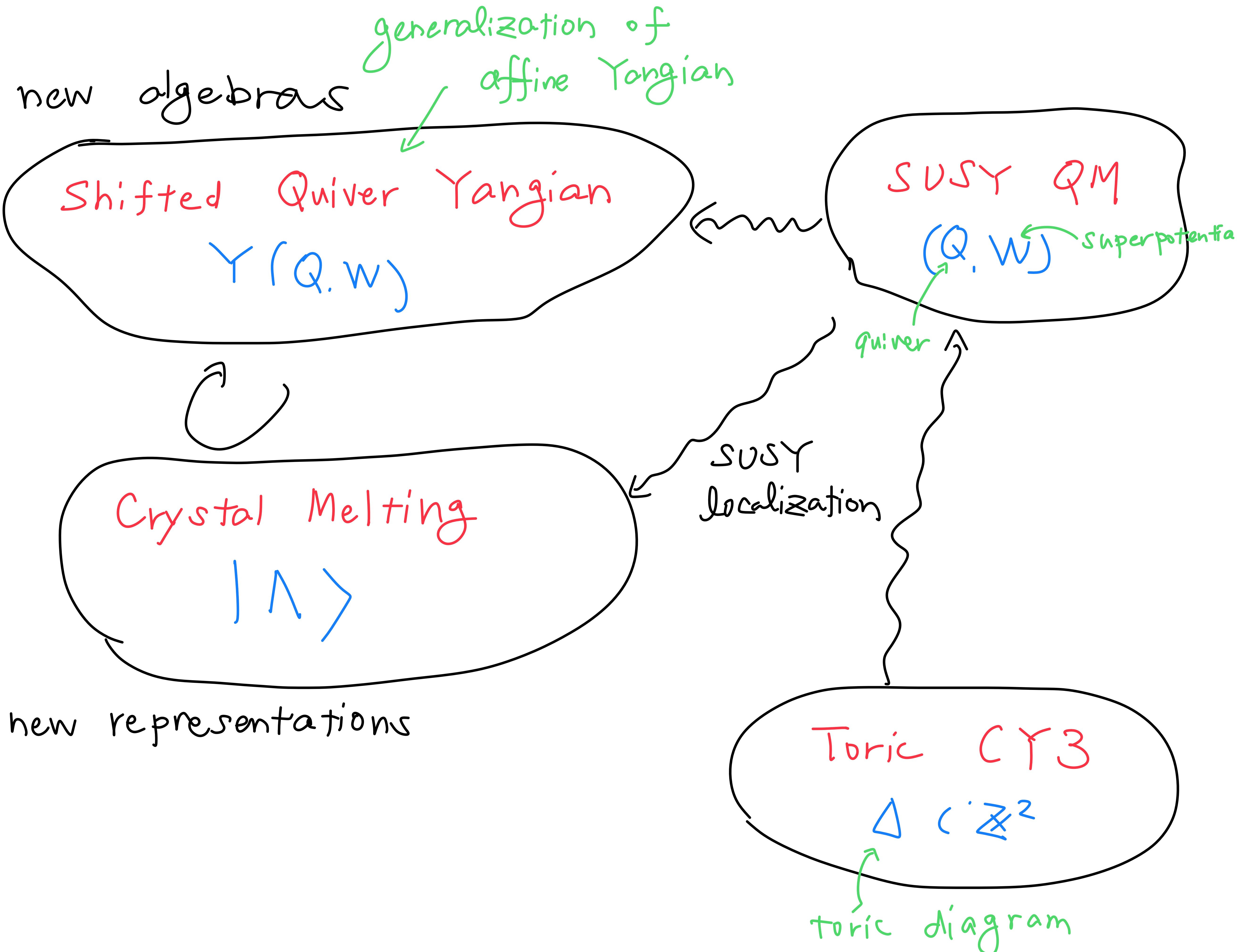
Crystal Melting as Quantum Foam

Okounkov-Reshetikhin-Vafa ('03)

Emergence of classical geometry



Okounkov-Reshetikhin-Vafa ('03), ⋯ Ooguri-MY ('09),⋯

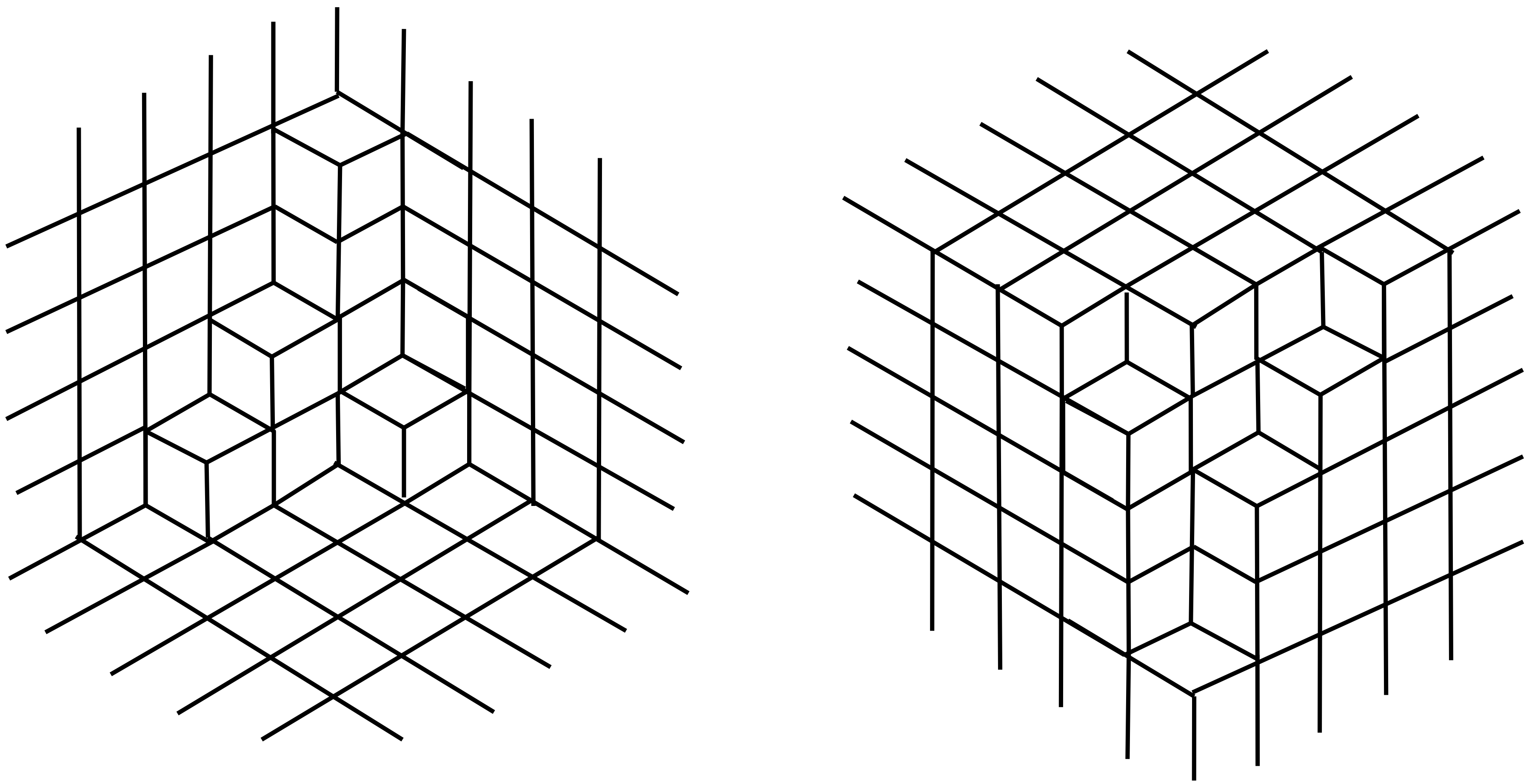


new algebras
generalization of affine Yangian
Shifted Quiver Yangian
$Y(Q, W)$
SUSY QM
(Q, W)
superpotential
quiver
SUSY localization
Crystal Melting
$|\Lambda\rangle$
new representations
Toric CY3
$\Delta \subset \mathbb{Z}^2$
toric diagram

Crystal Melting

[Okounkov-Reshetikhin-Vafa '03, ⋯, Ooguri-MY '08]

$\mathbb{C}^3$: crystal melting [Okounkov-Reshetikhin-Vafa; Iqbal, Nekrasov,...]



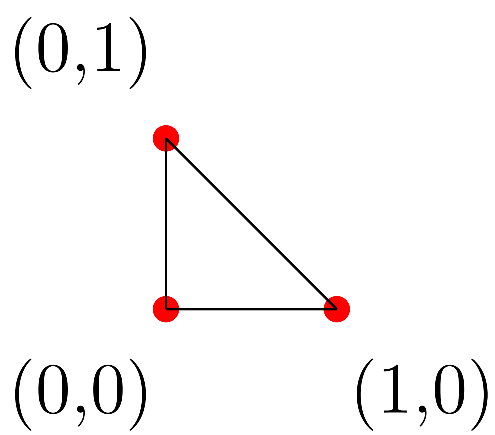
plane partition

$$M(q) \equiv \sum_{\Lambda \in \text{ plane partition}} q^{|\Lambda|} = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}$$

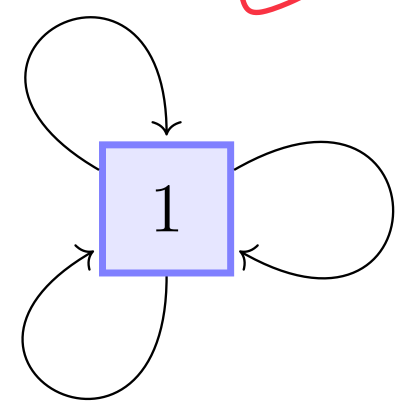
$$= 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \dots ,$$

$= Z^{\mathbb{C}^3}_{\text{top A-model}}$

toric diagram



quiver Q



superpotential

$$+ \left(W = \mathrm{Tr}\, xyz - \mathrm{Tr}\, xzy \right)$$

path algebra

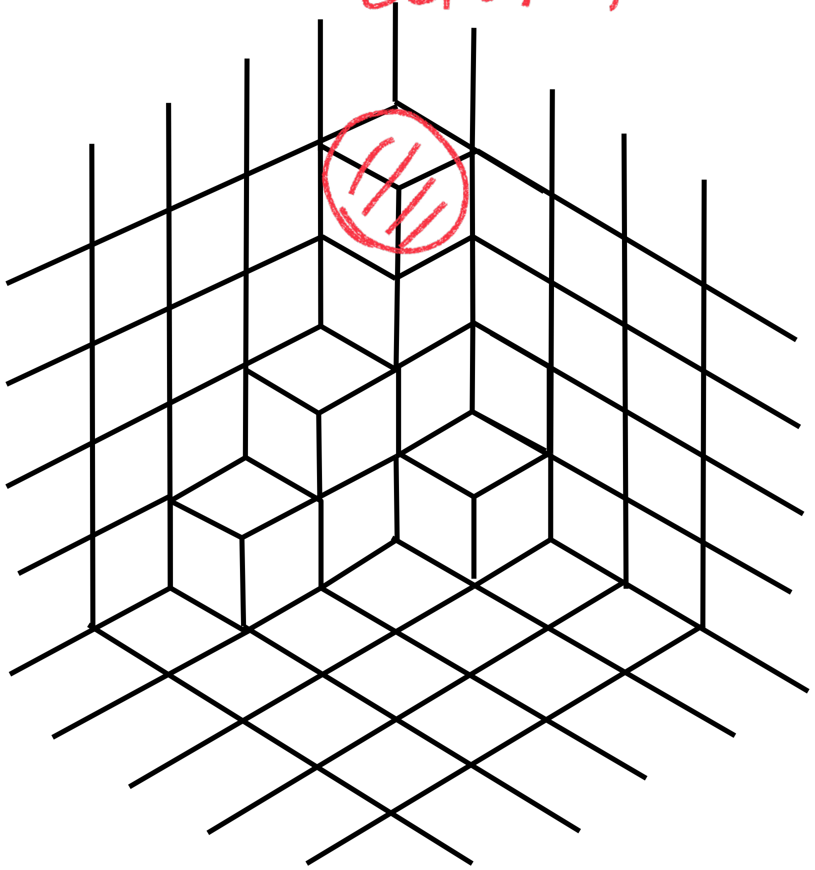
$$\mathbb{C}\langle x, y, z \rangle / (\partial W)$$

$$(\partial W) = (xy - yx,\ yz - zy,\ zx - xz)$$

$$\mathbb{C}\langle x, y, z \rangle / (\partial W) = \mathbb{C}[x, y, z]$$

crystal melting

atom



- "atom" at location (i, j, k):

$$x^i y^j z^k \in \mathbb{C}[x, y, z]$$

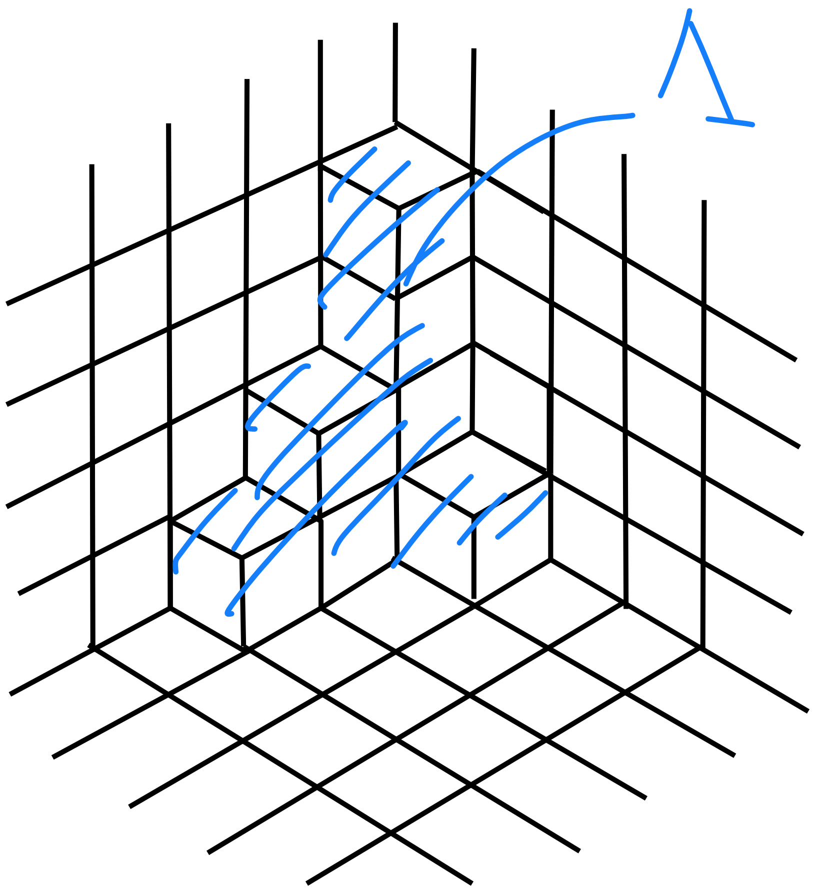
$$\shortparallel$$

$$\mathbb{C}\langle x, y, z\rangle / (\partial W)$$

- atom = element of

$$\mathbb{C}Q/\partial W$$

crystal melting



Λ^c (complement of Λ):

ideal of the path alg.

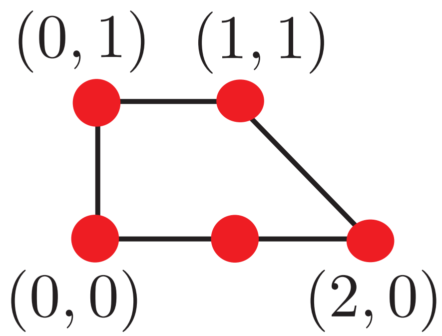
$$\left(\begin{array}{c} I_\Lambda \subset \mathbb{C}[x, y, z] \\ \shortparallel \\ \mathrm{Span}\{x^i y^j z^k \mid (i,j,k) \notin \Lambda\} \\ \\ x \cdot I_\Lambda,\ y \cdot I_\Lambda,\ z \cdot I_\Lambda \subset I_\Lambda \end{array} \right)$$

The story generalizes to an arbitrary toric CY3

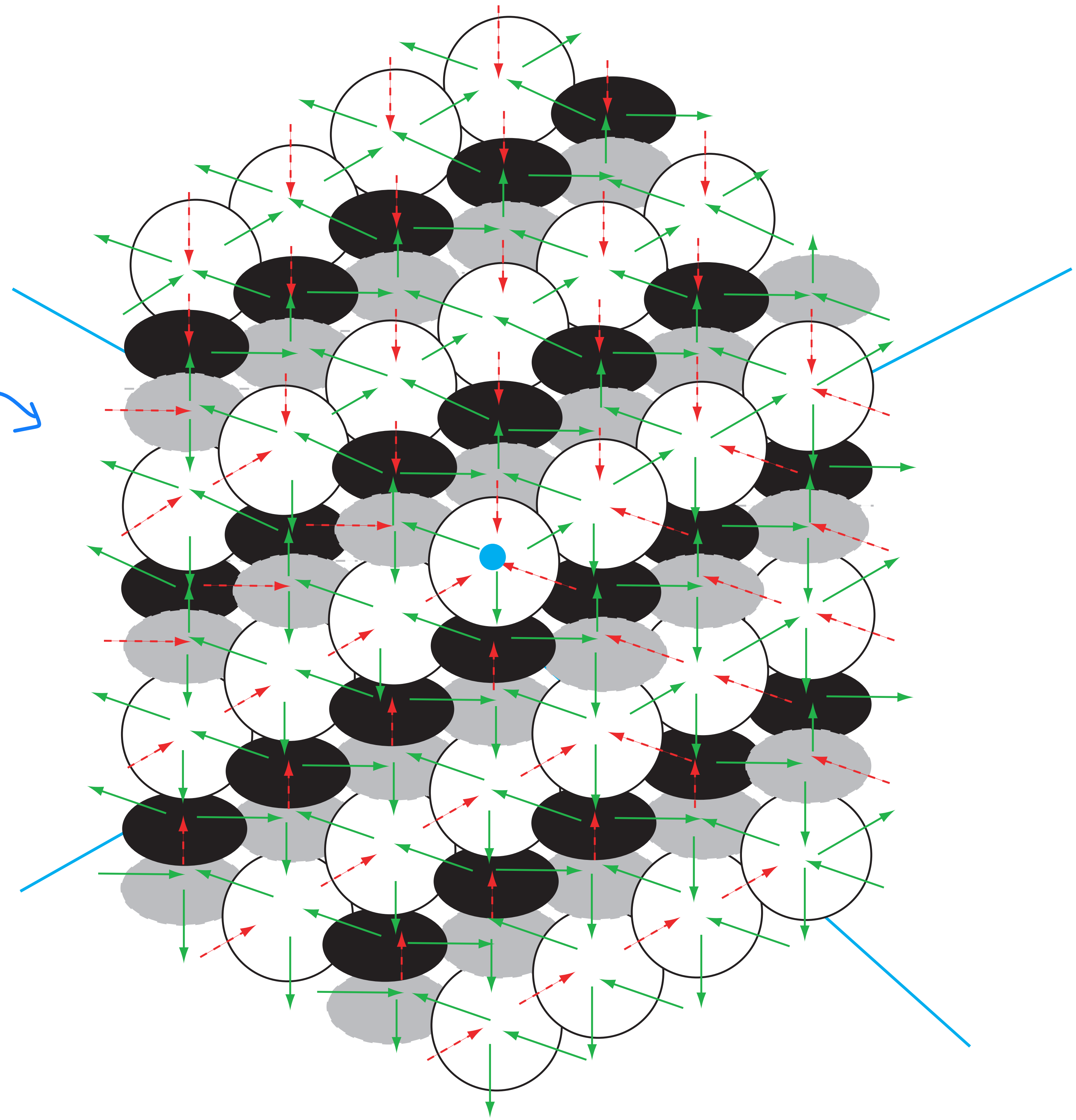
[Ooguri-MY '08'09]

See also [Szendroi; Bryant, Young; Mozgovoy, Reineke; Nagao, Nakajima; Ooguri, MY; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Vafa; ⋯]

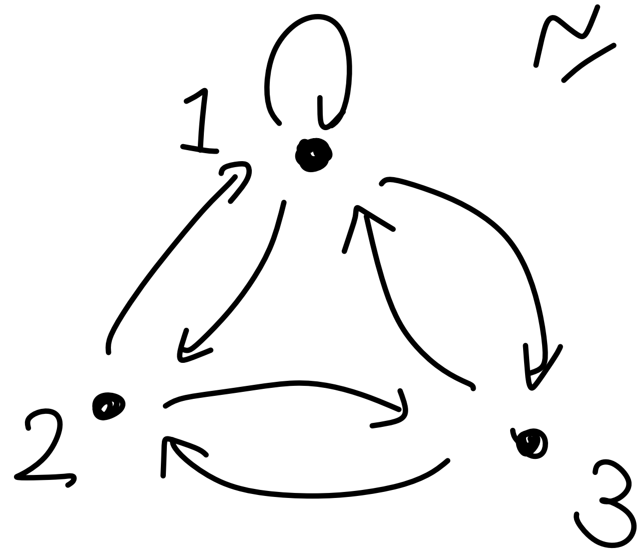
toric diagram $\begin{pmatrix} SPP \\ xy = zw^2 \end{pmatrix}$



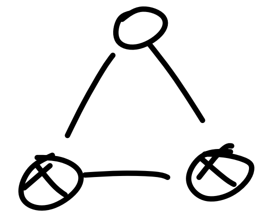
[Ooguri-MY '08]



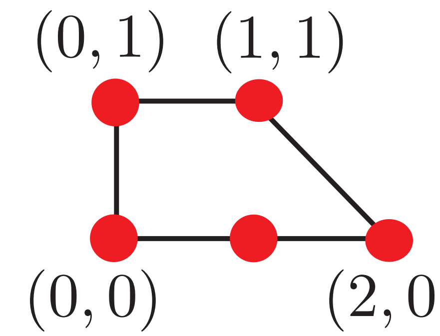
We have an associated SQM



$\simeq$



Dynkin



Q

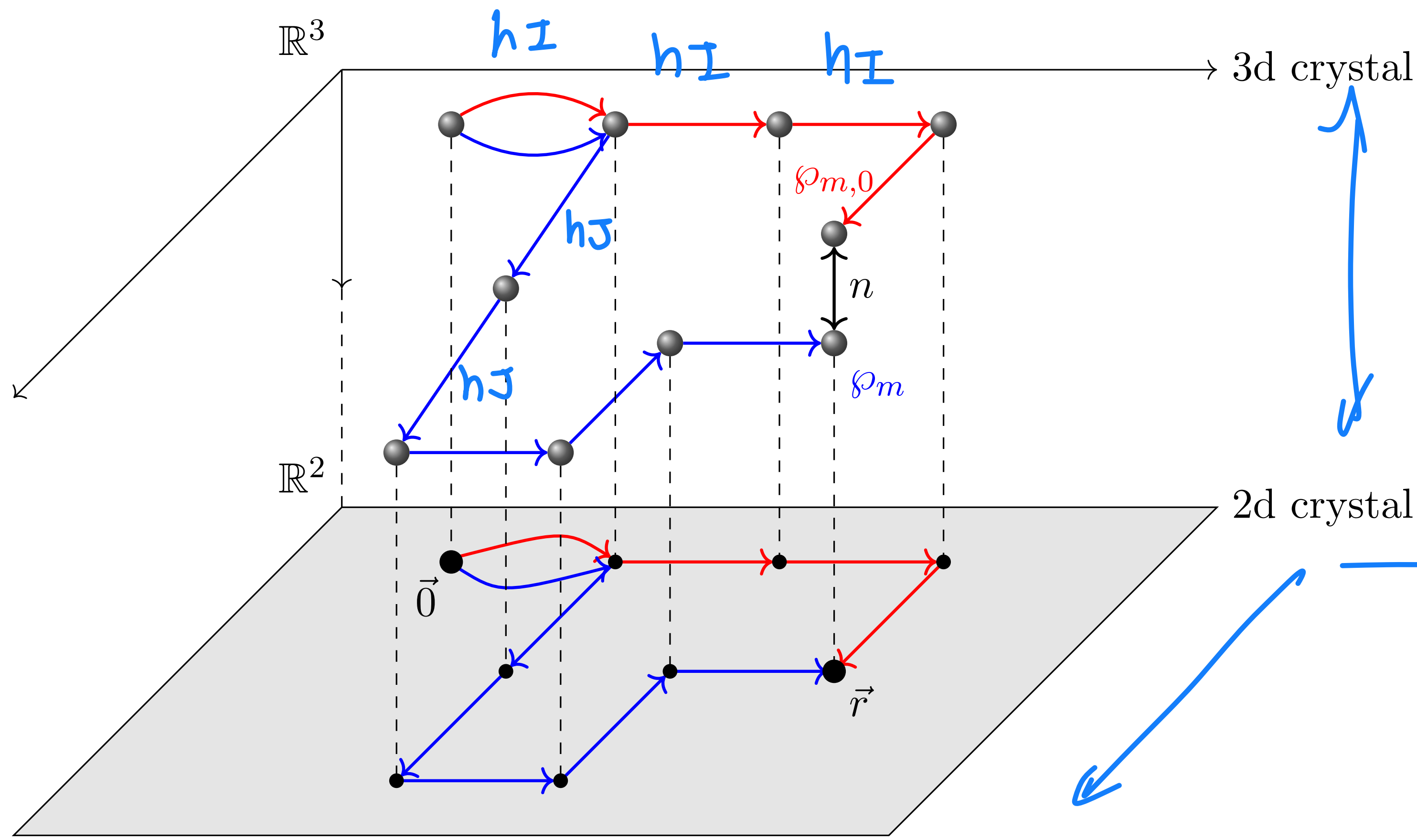
W

$$W = \mathrm{Tr}\left(X_{11} X_{12} X_{21} - X_{11} X_{13} X_{31} - X_{12} X_{21} X_{23} X_{32} + X_{23} X_{31} X_{13} X_{23} \right)$$

$\mathbb{C}Q/(\partial W)$: path algebra (non-commutative in general)

atom in the crystal

We can place the atoms in 3D according to their symmetry charges
(equivariant parameters corresponding to toric isometries)



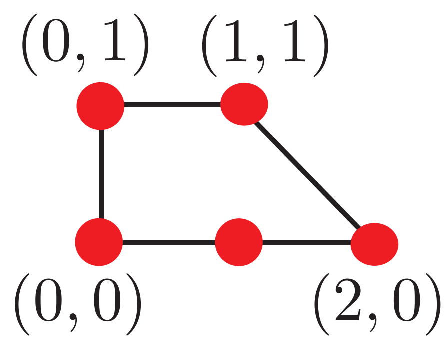
R-sym

flavor sym.

$$h(\boxed{a}) \equiv \sum_{I \in \mathrm{path}[\mathfrak{o} \to \boxed{a}]} h_I$$

equivariant param. of $Y(Q,W)$

toric diagram ($xy = zw^2$)



$\Delta \subset \mathbb{Z}^2$

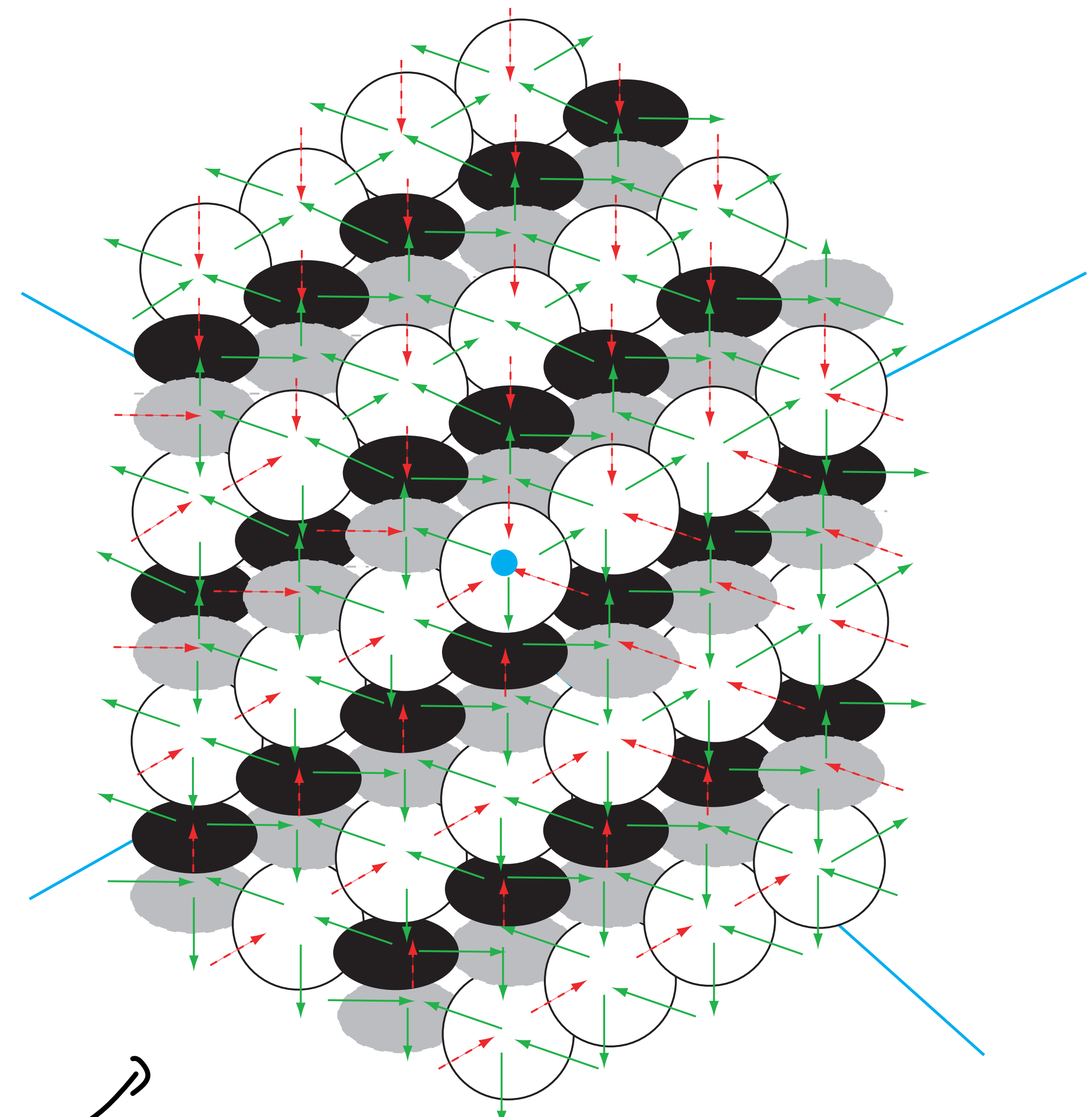
[Ooguri-MY '08]

Type IIA on toric CY$_3$

+ D6/D4/D2/D0

$\mathcal{N}=4$ SQM

$\mathcal{M}_{VAC} \circlearrowleft$ torus action



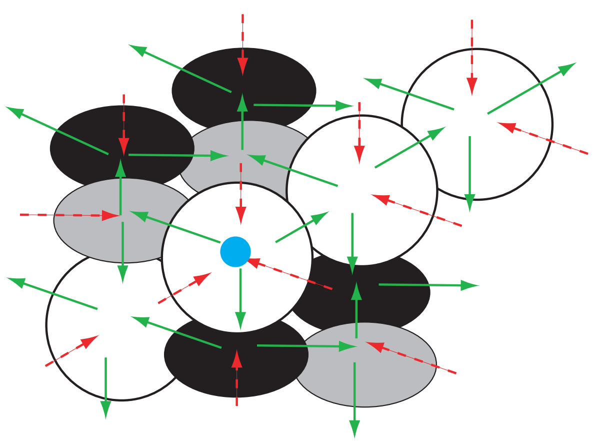
torus fixed point

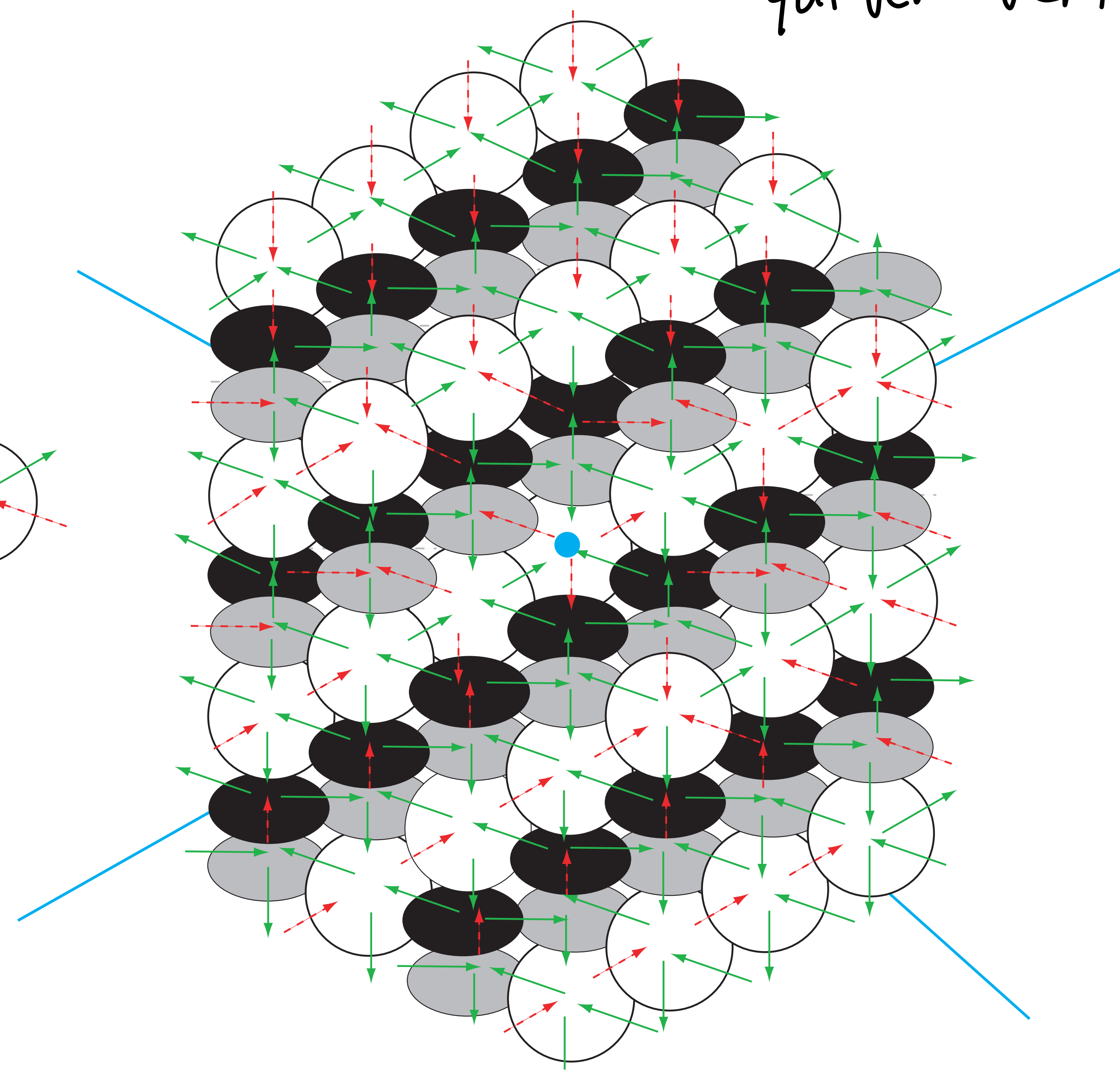
$$\left[\ \text{“}\chi(\mathcal{M}_{VAC})\text{”} = \begin{pmatrix}\text{BPS}\\ \text{degeneracy}\end{pmatrix} = \begin{pmatrix}\text{Donaldson-Thomas}\\ \text{inv.}\end{pmatrix}\ \right]$$

BPS partition function

$$Z(q_1, \ldots, q_{|Q_0|}) = \sum_{\mathrm{K}} \prod_{a \in Q_0} q_a^{|\mathrm{K}(a)|}$$

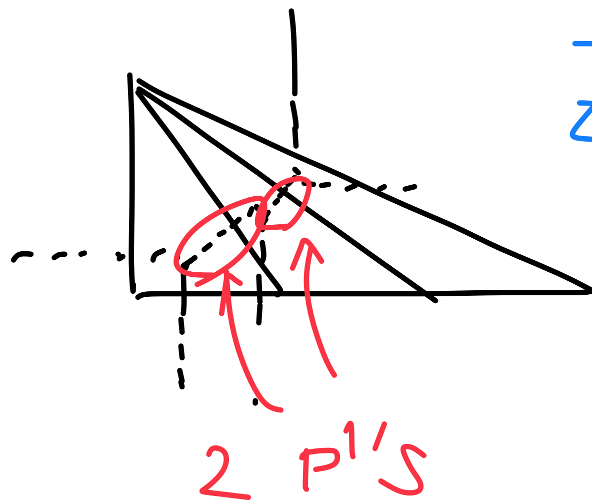
formal variable for each quiver vertex





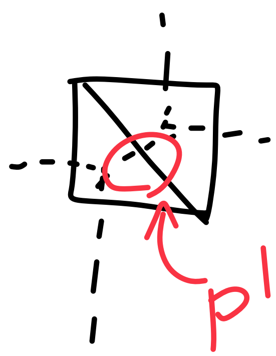
Infinite-product forms discussed
in [Szendroi, Young, Nagao, Aganagic-Ooguri-Vafa-MY,⋯]

$(\mathbb{C}^2/\mathbb{Z}_3)\times\mathbb{C}$



$$Z_{top}\sim\prod_n \frac{1}{(1-q^nQ_1)^n}\frac{1}{(1-q^nQ_2)^n}\frac{1}{(1-q^nQ_1Q_2)^n}$$

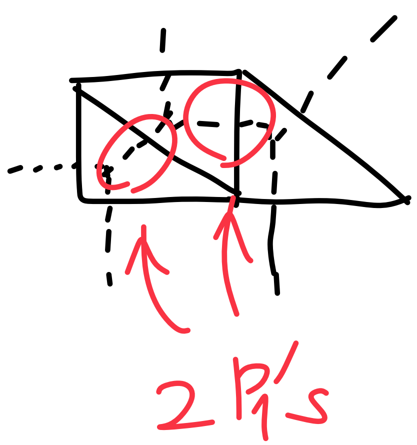
conifold



$$Z_{top}\sim\prod_n (1-q^nQ)^n$$

$$Z_{BPS}\sim Z_{top}(q,Q)\times Z_{top}(q,Q^{-1})$$

SPP



$$Z_{top}\sim\prod_n (1-q^nQ_1)^n(1-q^nQ_1Q_2)^n\frac{1}{(1-q^nQ_2)^n}$$

characters of $Y(\hat{\mathfrak{g}})$ in modern terms!!

Shifted

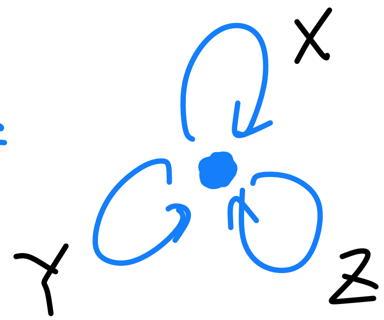
Quiver Yangian

Wei Li + MY (20)

Dimitry Galakhov + Wei Li + MY (21)

Quiver Q & Superpotential W $\leftarrow$ toric CY_3

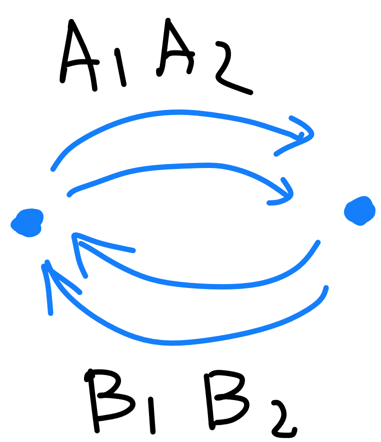
* $Q =$



$$W = \mathrm{Tr}(XYZ - XZY)$$

$(CY_3 = \mathbb{C}^3)$

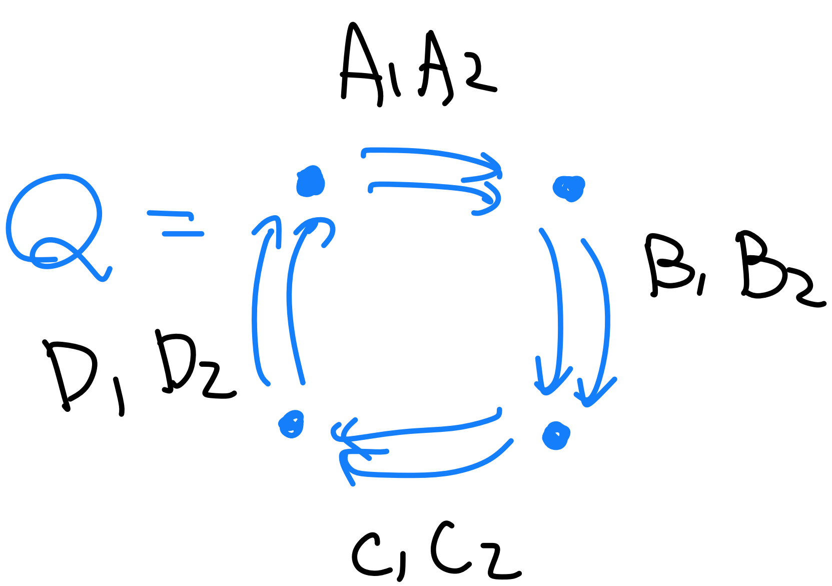
* $Q =$



$$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$$

$(CY_3 = \text{conifold})$

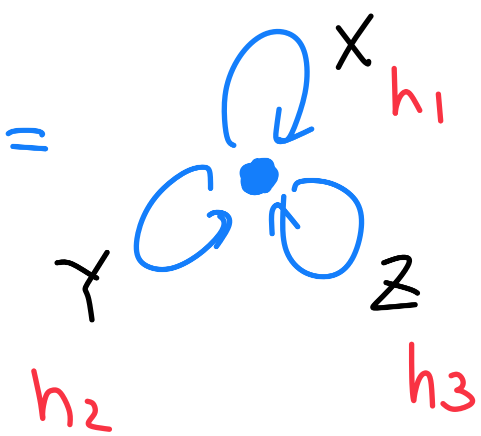
* $Q =$



$$W = \mathrm{Tr}(A_1 B_1 C_1 D_1 - A_1 B_2 C_1 D_2 - A_2 B_1 C_2 D_1 + A_2 B_2 C_2 D_2)$$

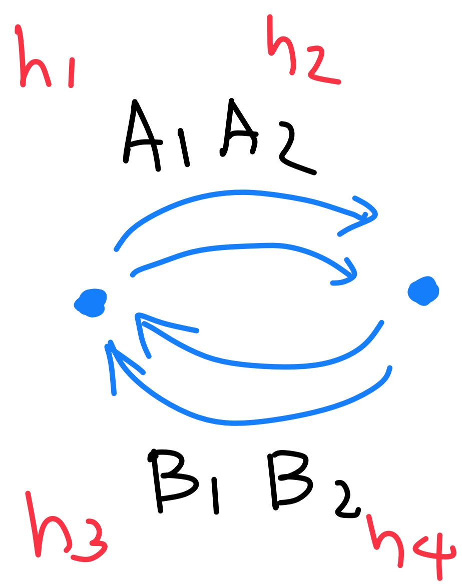
$(CY_3 = K_{\mathbb{P}^1 \times \mathbb{P}^1})$

Quiver Q & Superpotential W ⟵ toric CY_3

* $Q =$ 

$W = \mathrm{Tr}(XYZ - XZY)$ $\quad (CY_3 = \mathbb{C}^3)$

$h_1 + h_2 + h_3 = 0$

* $Q =$ 

$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

$(CY_3 = \text{conifold})$

$h_1 + h_2 + h_3 + h_4 = 0$

* Assign equivariant parameters h_I consistent w/ W

edge

Generators

(z: spectral parameter)

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}}, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}}, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}},$$

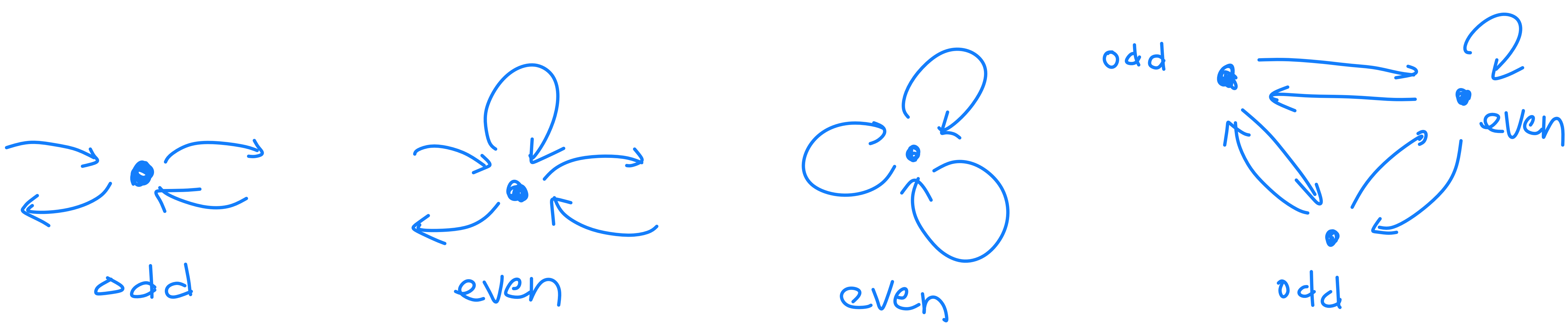
$n = -k$

"k-shifted Quiver Yangian"

a : quiver vertex

$\mathbb{Z}_2$-grading

$$|a| = \begin{cases} 0 & (\exists \text{ edge } I \text{ s.t. } I \text{ starts and ends at } a) \\ 1 & (\text{otherwise}) \end{cases}$$



Relations

$Y(Q, W)$

$$\psi^{(a)}(z)\,\psi^{(b)}(w) = \psi^{(b)}(w)\,\psi^{(a)}(z)\,,$$

$$\psi^{(a)}(z)\,e^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,$$

$$e^{(a)}(z)\,e^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,,$$

$$\psi^{(a)}(z)\,f^{(b)}(w) \simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,,$$

$$f^{(a)}(z)\,f^{(b)}(w) \sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,,$$

$$\left[e^{(a)}(z), f^{(b)}(w)\right\} \sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,, \qquad (\Delta = z-w)$$

"$\simeq$" means equality up to $z^n w^{m\geq 0}$ terms

"$\sim$" means equality up to $z^{n\geq 0} w^m$ and $z^n w^{m\geq 0}$ terms

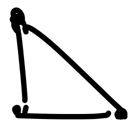
bonding factor

equivariant weight

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

edge

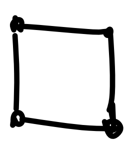
* $\mathbb{C}^3 \rightsquigarrow$ $Q=$  $\rightsquigarrow Y(\widehat{gl}_1)$



$W = \mathrm{Tr}(XYZ - XZY)$

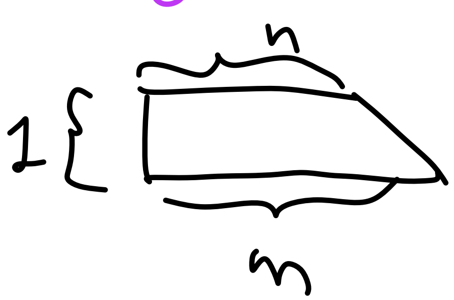
[Miki; Ding-Iohara;⋯
Tsymbaulik; Prochazka;
Gaberdiel, Gopakumar, Li, Peng,⋯]

* conifold $\rightsquigarrow$ $Q=$  $\rightsquigarrow Y(\widehat{gl}_{1|1})$

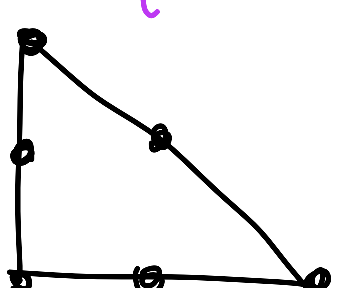


$W = \mathrm{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$

* $xy = z^n w^m \rightsquigarrow Y(\widehat{gl}_{m|n})$ [Bezerra-Mukhin ('19)]



* $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2) \rightsquigarrow Y(\widehat{D(2,1,\alpha)})$ [Noshita-Watanabe ('21)]

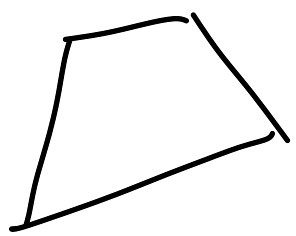


$Y(\hat{g})$ for (non-chiral quiver
toric CY3 w.o. 4-cycle

chiral quiver

toric CY_3 w/ cpt 4-cycle

* general toric CY_3 $\rightsquigarrow$ $Y(Q, W)$



$\Delta \subset \mathbb{Z}^2$

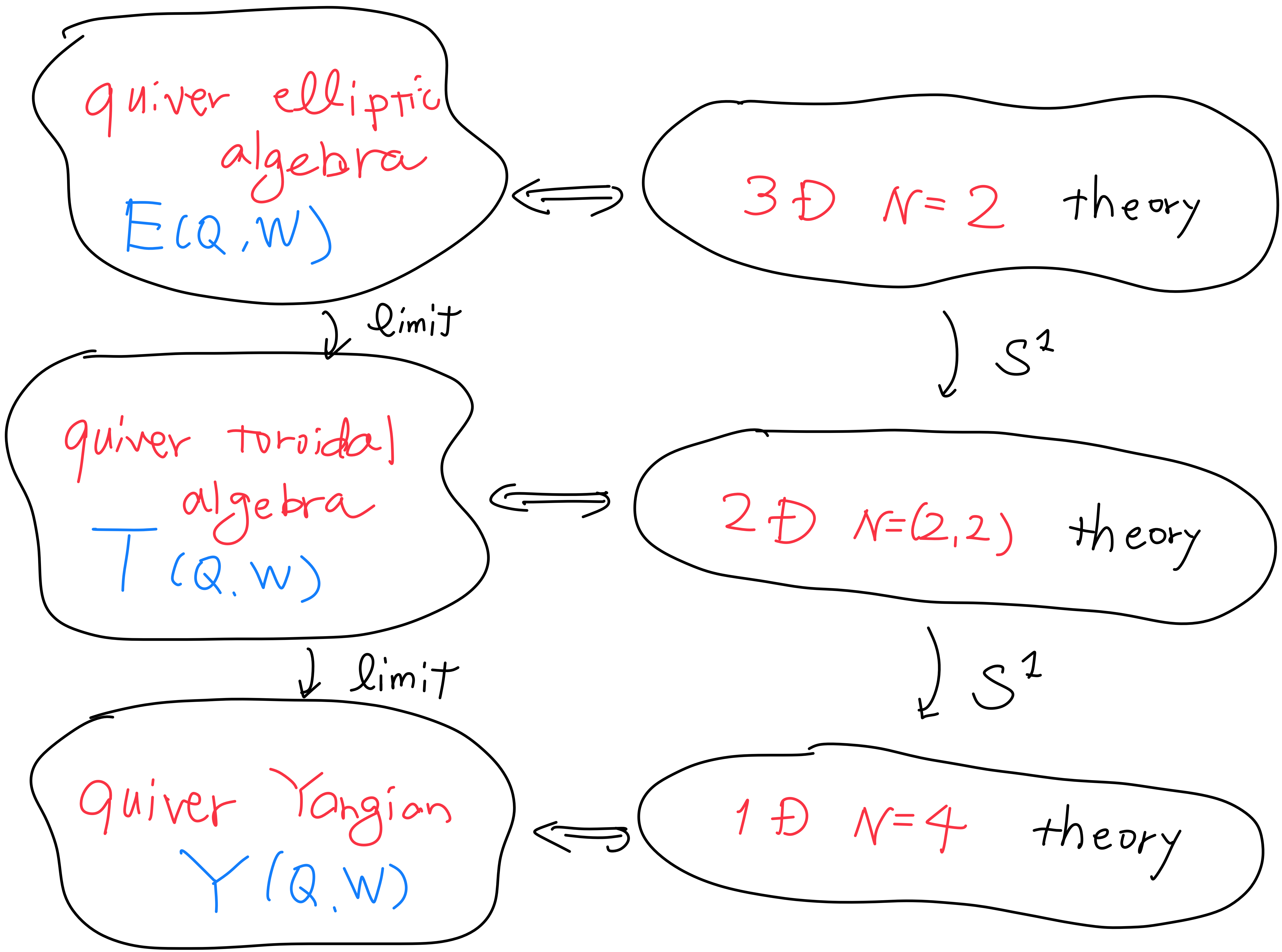
has no "$\mathfrak{g}$"

new algebra

beyond $Y(\mathfrak{g})$

$Y(\hat{\mathfrak{g}})$

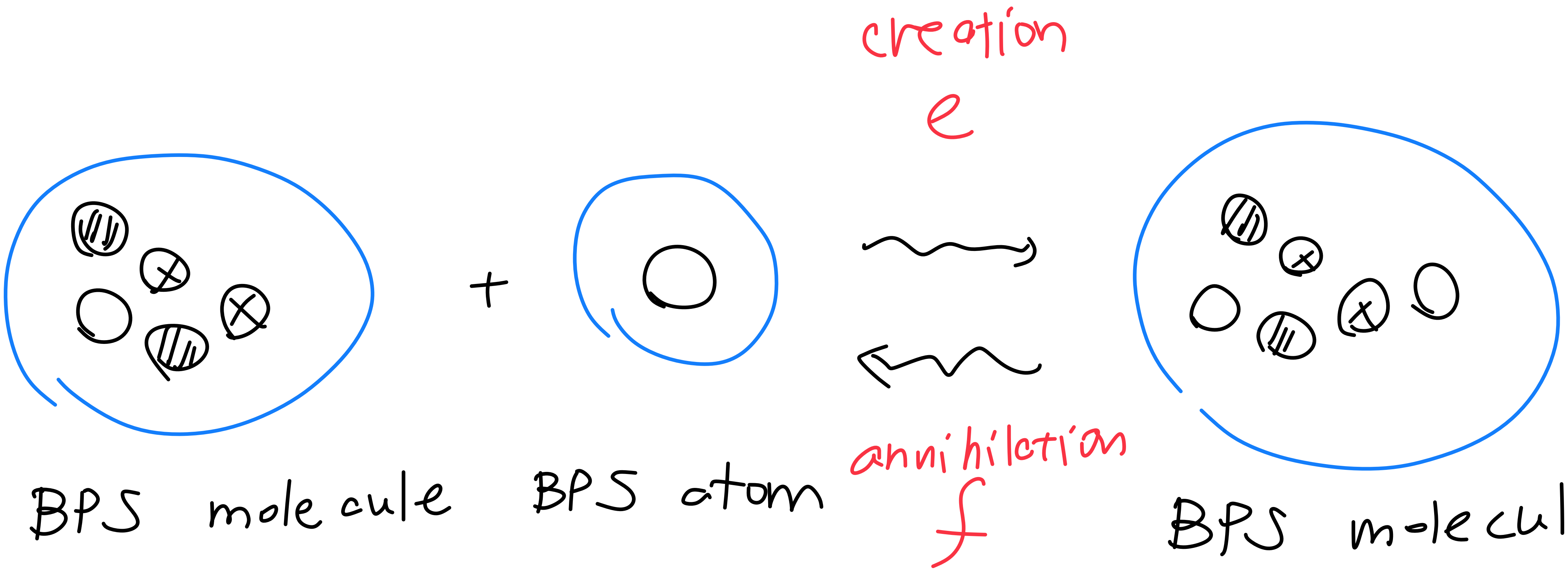
!


quiver elliptic algebra
E(Q,W)
3D N=2 theory
limit
S1
quiver toroidal algebra
T(Q,W)
2D N=(2,2) theory
limit
S1
quiver Yangian
Y(Q,W)
1D N=4 theory

Representations from Crystal Melting

cf. earlier developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by [Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka; Rapcak; Gaberdiel, Gopakumar; Li, Peng,⋯]

Basic Idea



BPS molecule + BPS atom ⇌ BPS molecule

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

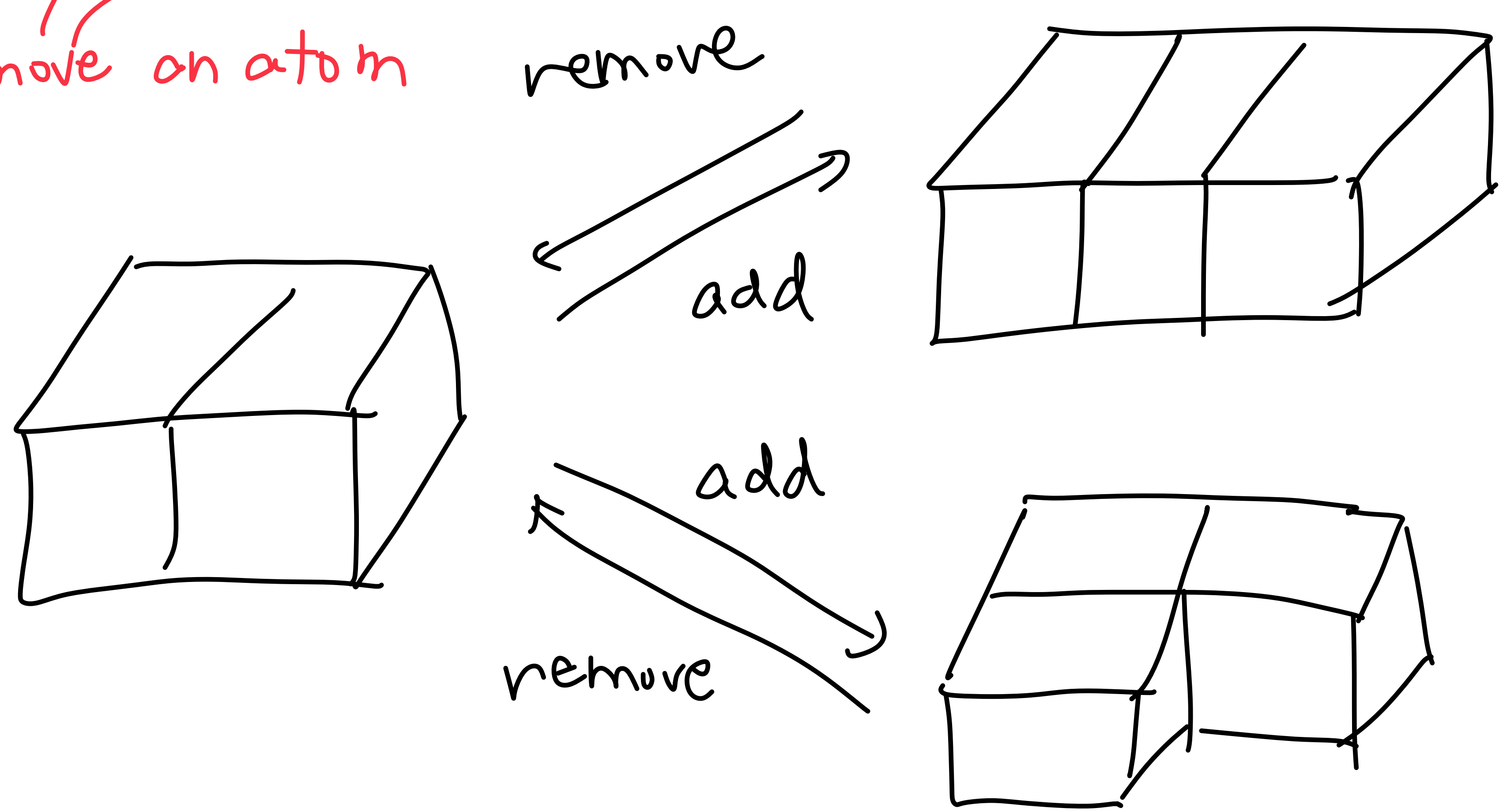
crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in\mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in\mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle,$$

add/remove an atom



Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

crystal

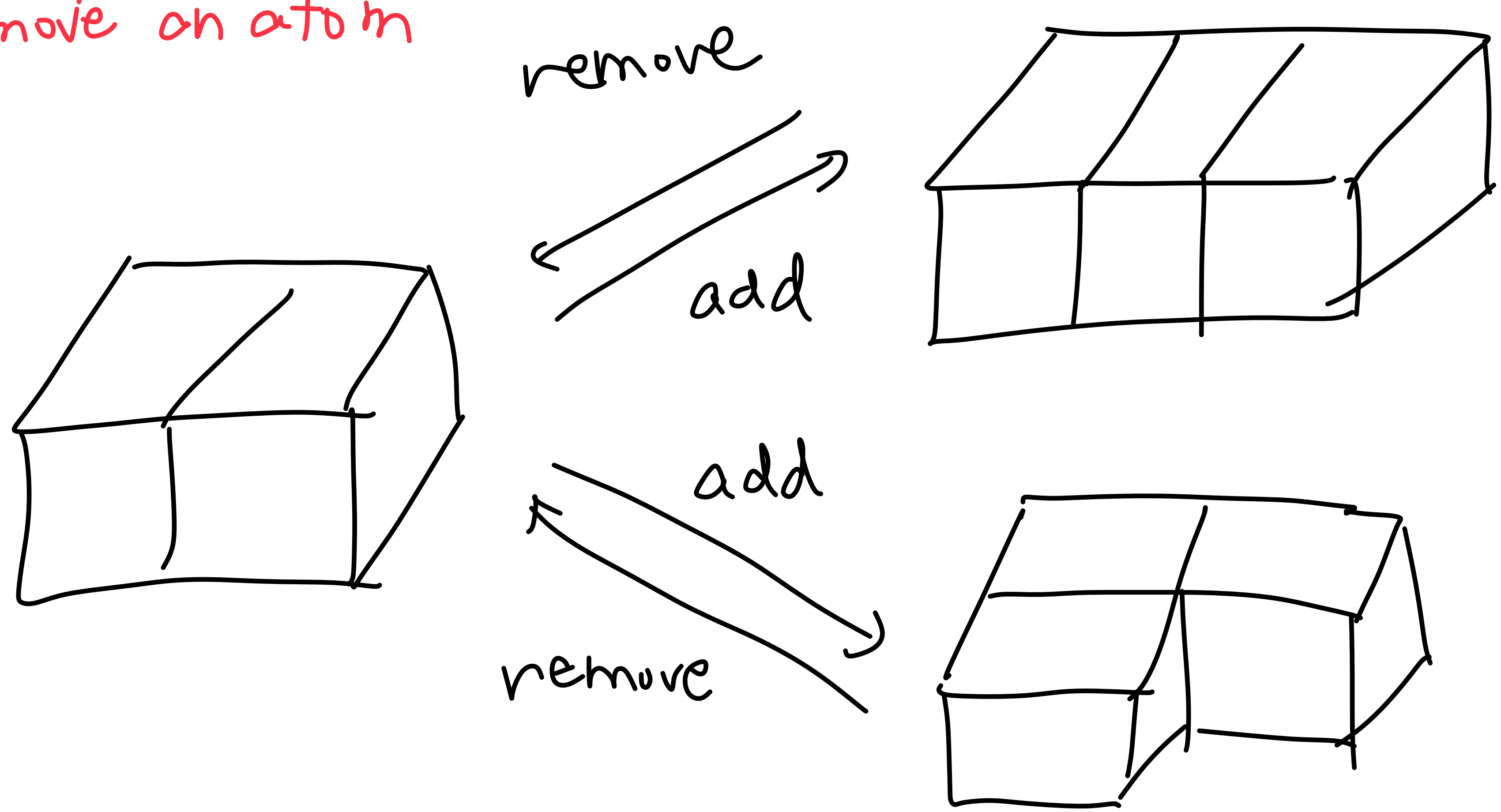
$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\ ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle\ ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle\ ,$$

poles for atom $\boxed{a}$

add/remove an atom



Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\ ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle\ ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle\ ,$$

poles for atom $\boxed{a}$

$$h(\boxed{a}) \equiv \sum_{I\in \mathrm{path}[\mathfrak{o}\to\boxed{a}]} h_I\ .$$

$\Psi^{(a)}_{K}$:
$$\Psi^{(a)}_{\mathrm{K}}(u) = \psi^{(a)}_0(z) \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{b\Rightarrow a}(u-h(\boxed{b}))\ ,$$

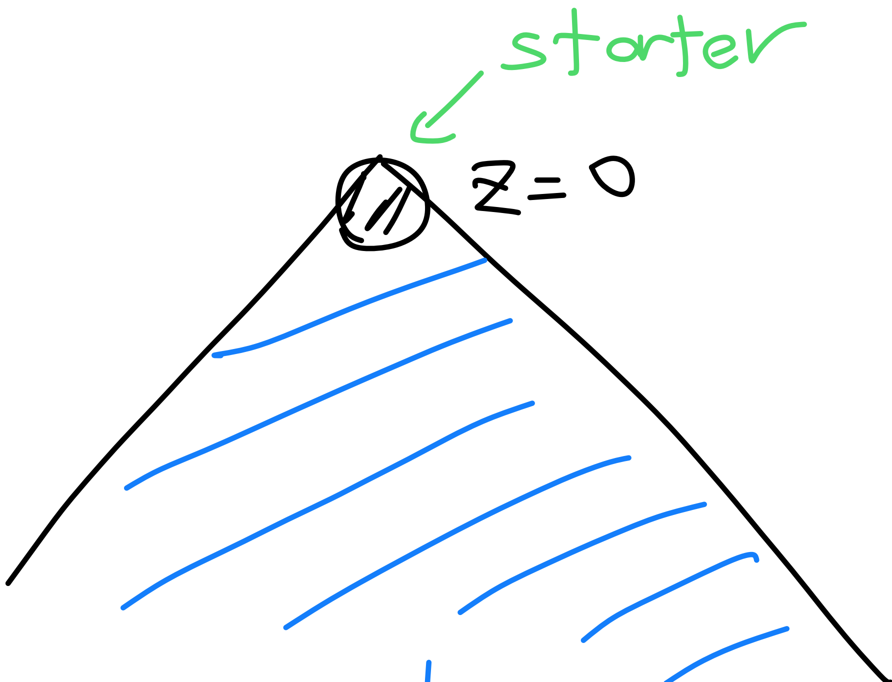
$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

$E^{(a)}/F^{(a)}$: $E^{(a)}/F^{(a)} = \sqrt{\pm \operatorname*{Res}_{u=h(\boxed{a})} \Psi^{(a)}_{K}(u)}$

$$\psi^{(a)}(z)|\emptyset\rangle = \psi_0^{(a)}(z)|\emptyset\rangle$$

[Galakhov-Li-MY '21]

vacuum charge function $\longleftrightarrow$ representation

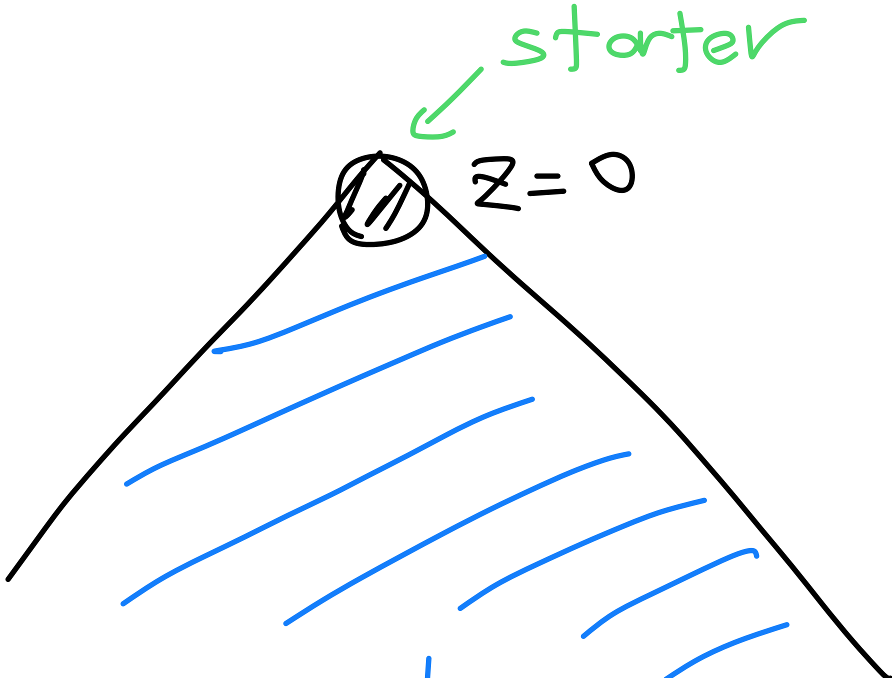


$$\psi_0 \sim \frac{1}{z}$$

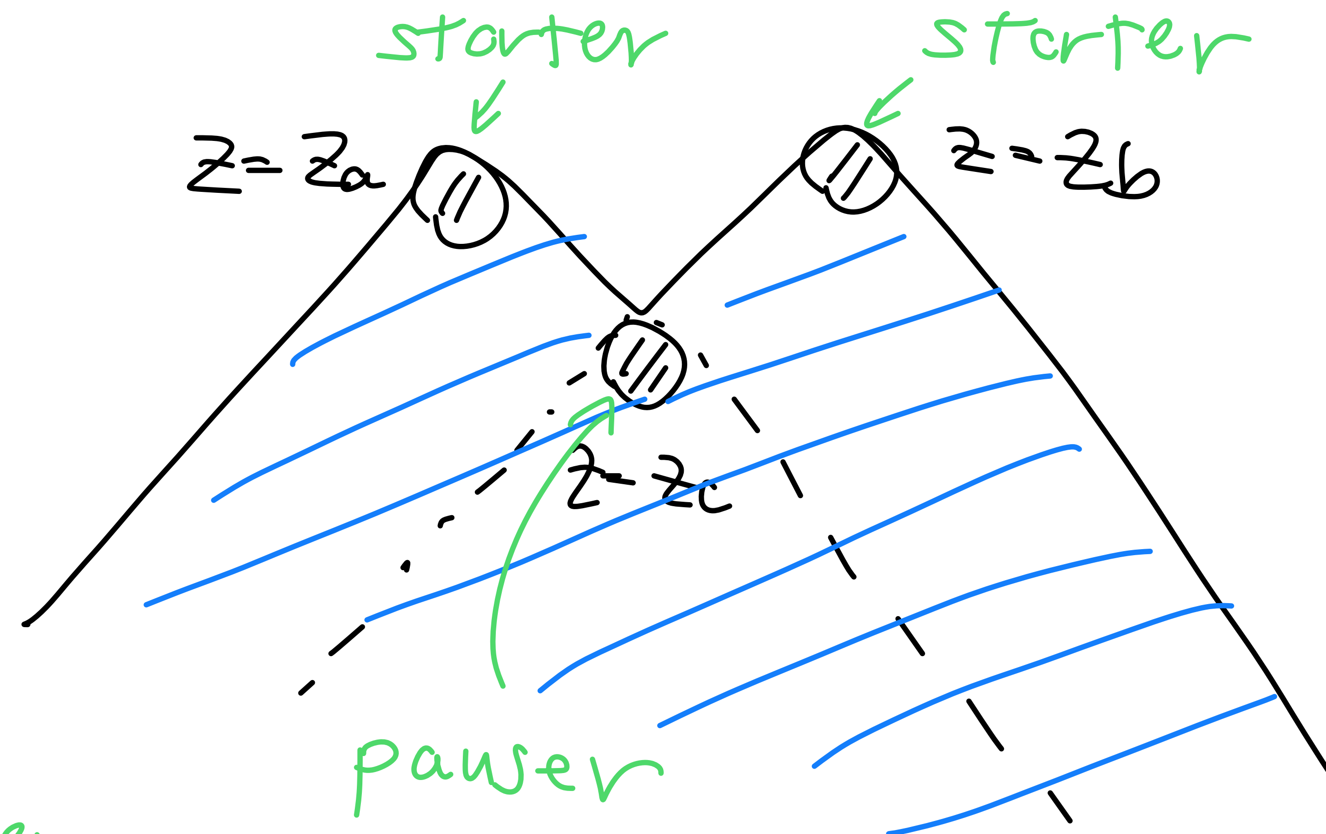
$$\psi^{(a)}(z)|\emptyset\rangle = \underbrace{\psi_0^{(a)}(z)}_{\text{vacuum charge function}}|\emptyset\rangle$$

[Galakhov-Li-MY '21]

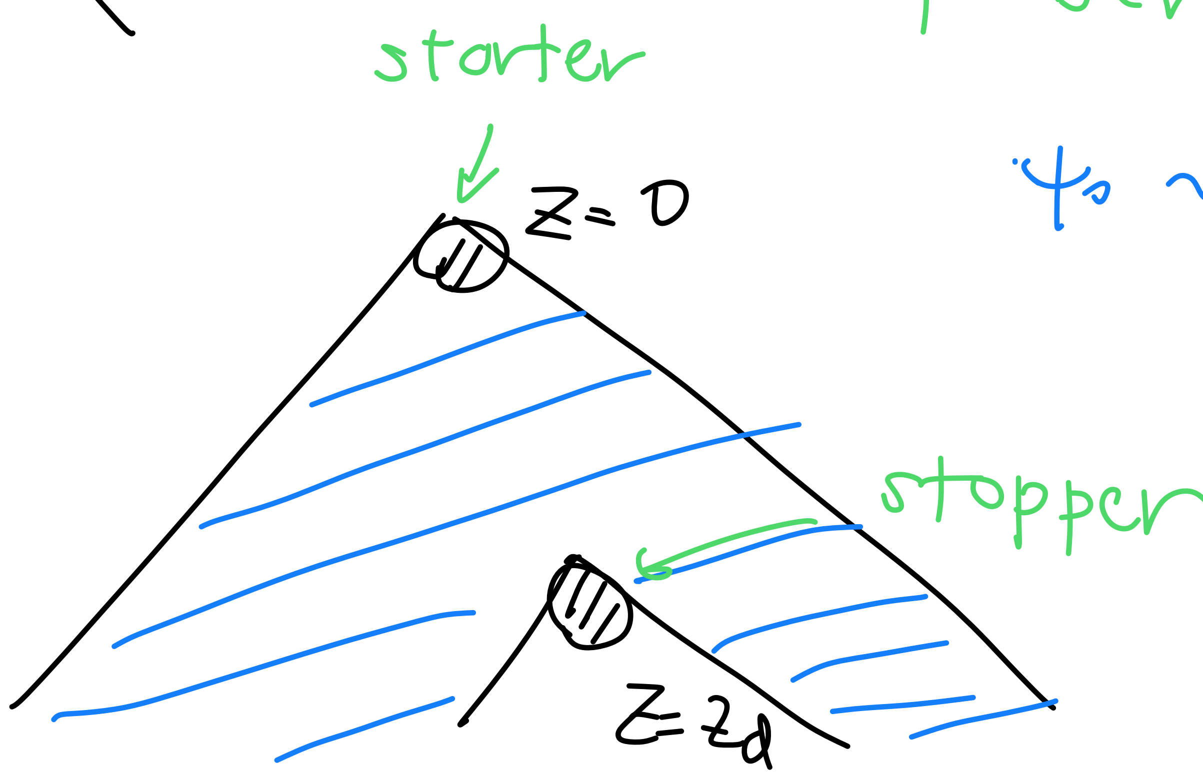
vacuum charge function $\longleftrightarrow$ representation



$\psi_0 \sim \frac{1}{z}$

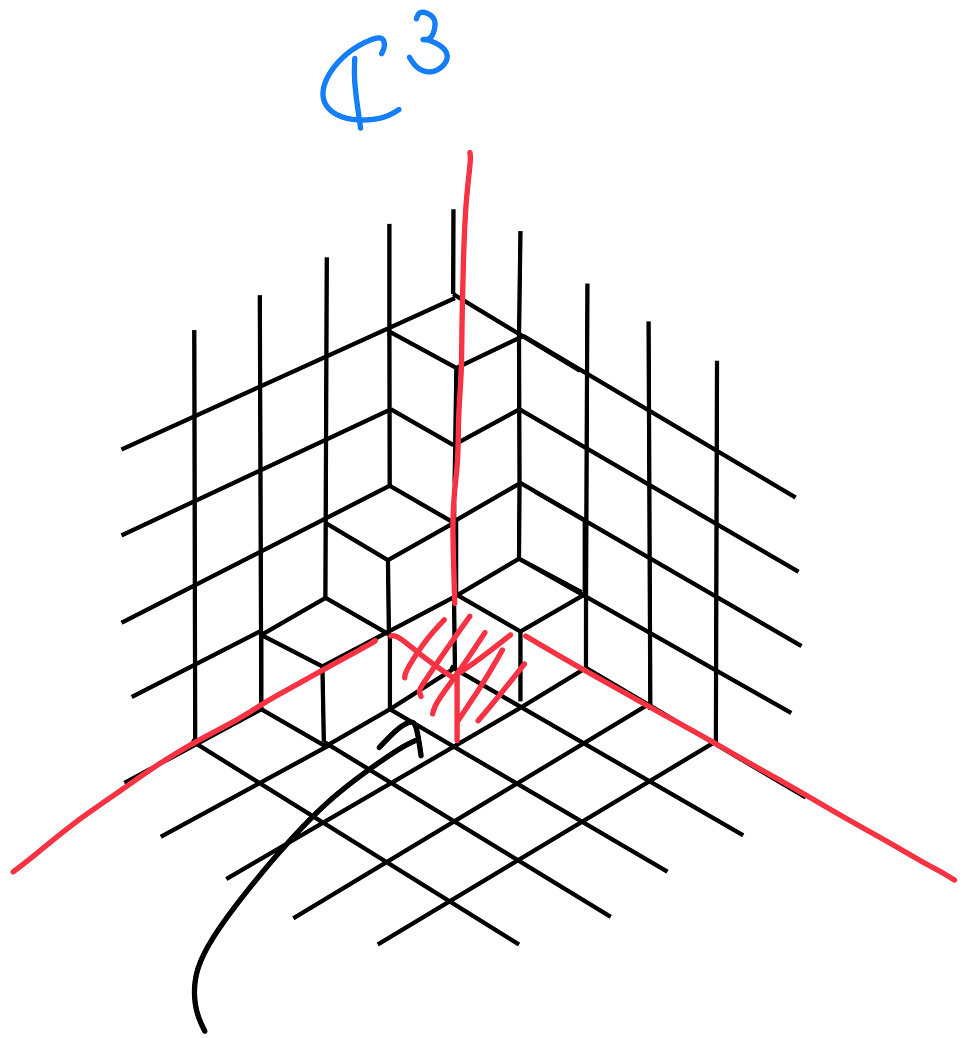


$$\psi_0 \sim \frac{z-z_c}{(z-z_a)(z-z_b)}$$



$$\psi_0 \sim \frac{z-z_d}{z}$$

example of truncation

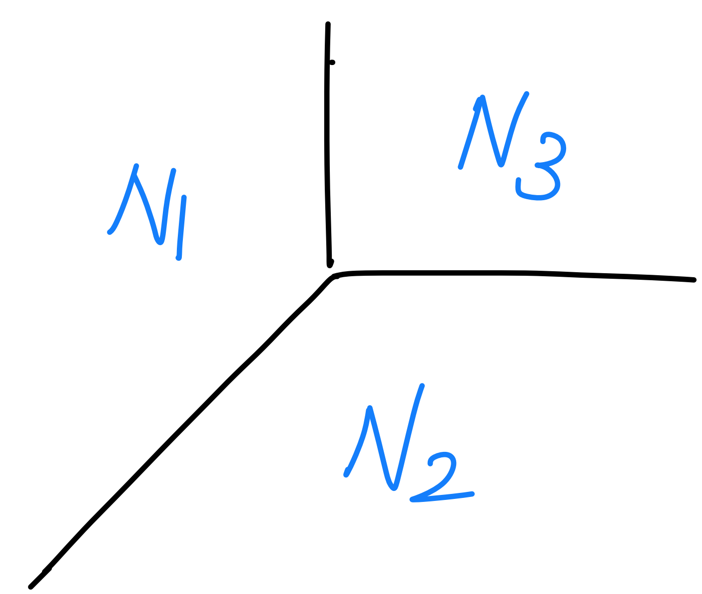


pit: location of null state

@ (N_1, N_2, N_3)

There is a corresponding truncation of the algebra studied by [Gaiotto-Rapcak]

$$Y(\hat{gl}_1) \to Y_{N_1, N_2, N_3}$$

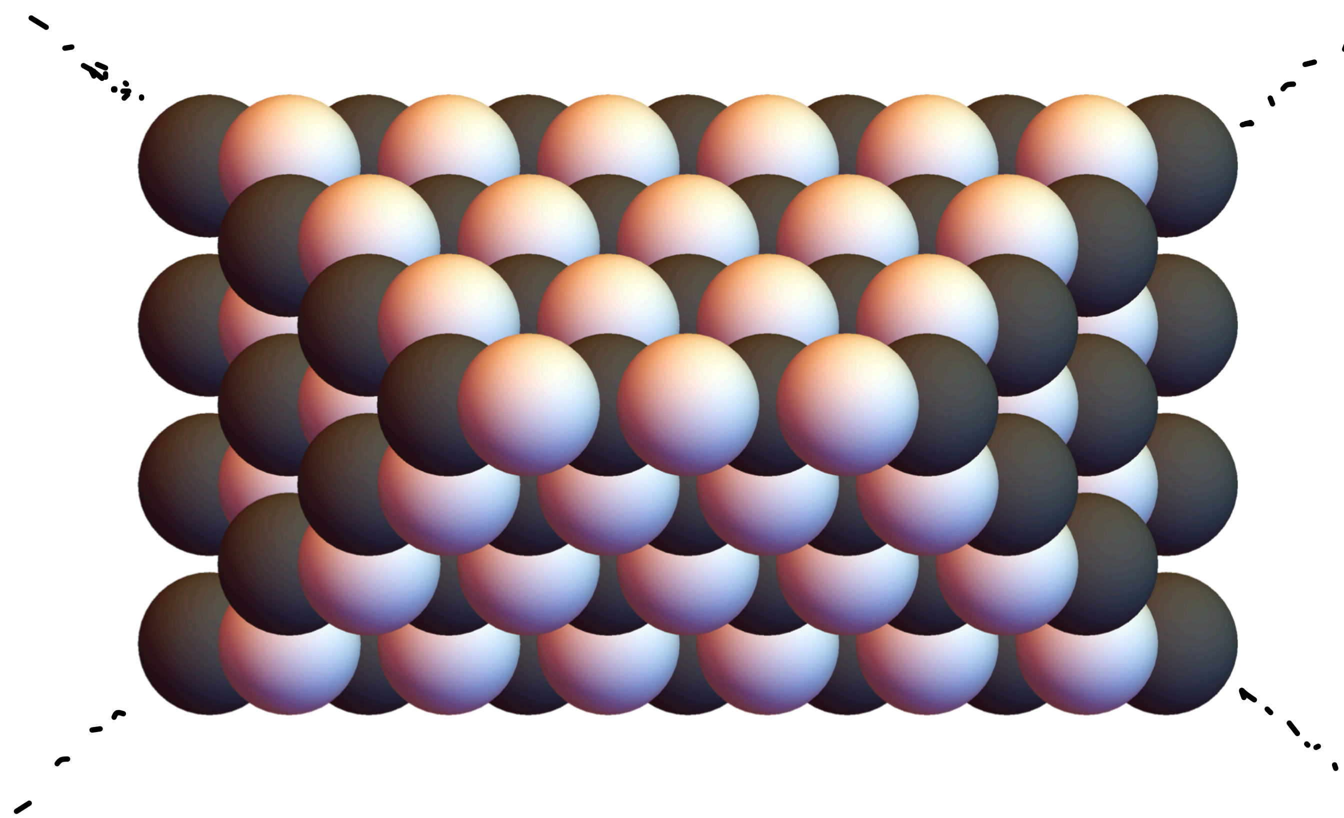


D-branes wrapping divisors

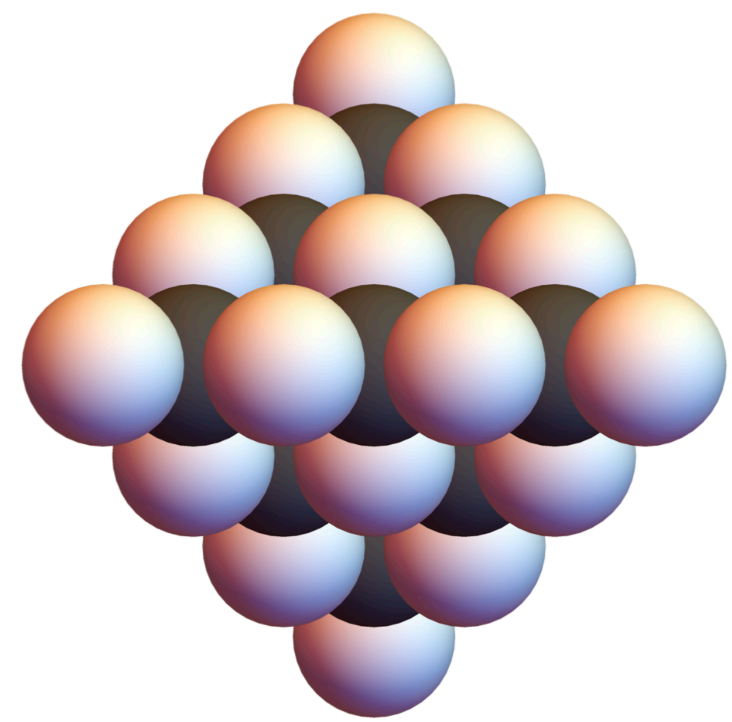
(framing of quiver)

We can obtain rather general reps by using starter / pauser / stoppers

e.g. open / closed BPS state counting and their wall crossings



conifold : ∞-chamber

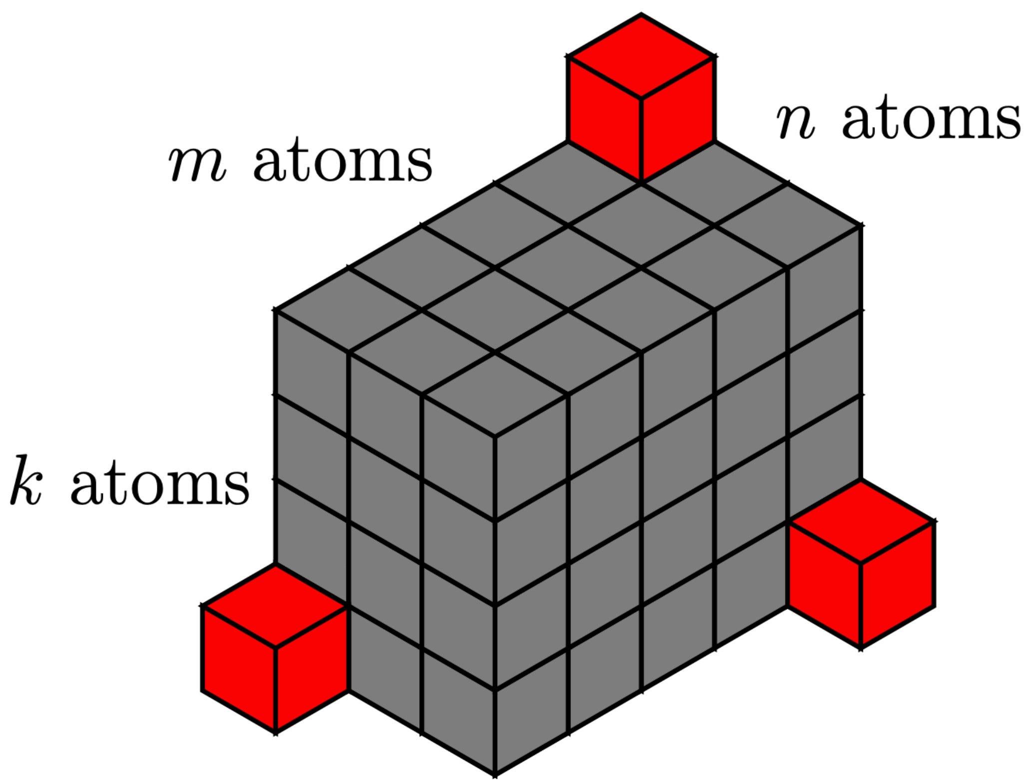


conifold : finite chamber

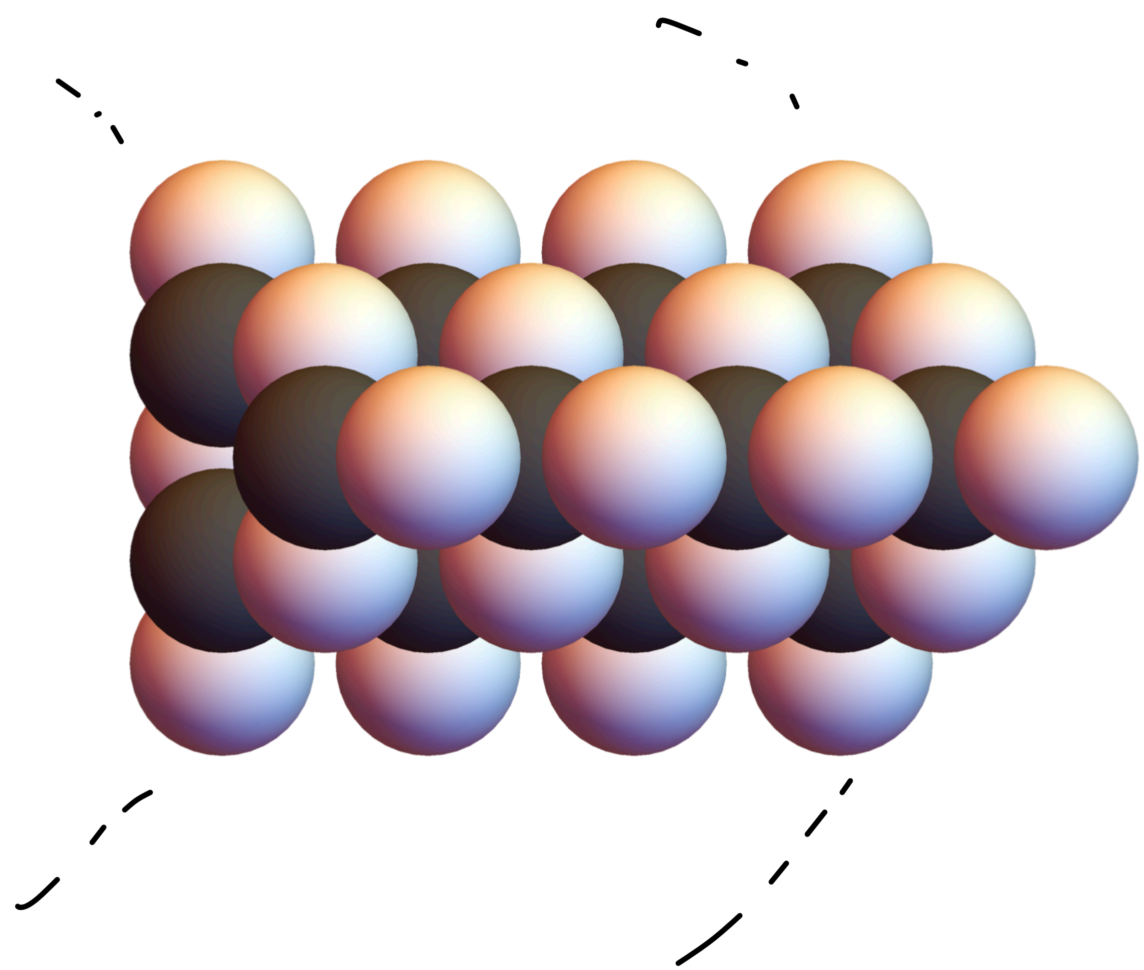
[Nagao-Nakajima; Jafferis-Moore; Chuang-Jafferis,$\cdots$ '08]

Some representations have no known CY$_3$/geometry counterparts

$Y(\hat{gl}_1)$ $\mathbb{C}^3$-like



$Y(\hat{gl}_{1|1})$ conifold-like



semi-infinite

[Galakhov-Li-MY '21]

R - matrix

Bethe Ansatz

[Galakhov-Li-Y ('22)]

We now have an algebra $\mathcal{Y}$
and a representation $\mathcal{C}$ from crystal

$\rightsquigarrow$

Natural to consider
"crystal chains" $\mathcal{C}^{\otimes n}$

&

R-matrix, BAE, ...

Resolution of the Puzzle?

2d $\mathcal{N}=(2,2)$

Q, W

$\rightsquigarrow$

Vacuum equation

$$\exp\left(\frac{\partial W}{\partial \sigma}\right) = 1$$

[Nekrasov-Shatashvili ('08)] $\parallel$

Gauge/Bethe

BAE

$Y(\hat{g})$

$\neq$

integrable model $\mathcal{Y}_{Q,W}$

What is $\mathcal{Y}_{Q,W}$?

quiver Yangian $Y_{Q,W}$?

should be some ∞-dim. algebra

twisted superpotential FI/stability param.

$$\mathcal{W}(\vec{\sigma}) = \mathrm{i} \sum_{a\in Q_0} \sum_{i=1}^{N_a} t_a \sigma_i^{(a)} + \sum_{(I:a\to b)\in Q_1} \sum_{i=1}^{N_a} \sum_{j=1}^{N_b} \mathbf{w}\left(\sigma_j^{(b)} - \sigma_i^{(a)} - h_I\right),$$

$$\mathbf{w}(\sigma) = \sigma(\log\sigma - 1).$$

Vacuum equation = "Would-be BAE"

$$1 = \mathbf{BAE}_i^{(a)}(\vec{\sigma}, \vec{u}, \vec{\mathfrak{q}}) := \mathfrak{q}_a^{-1} \prod_{\substack{1\le j\le N_a \\ j\neq i}} \varphi^{a\Leftarrow a}\left(\sigma_i^{(a)} - \sigma_j^{(a)}\right) \times$$

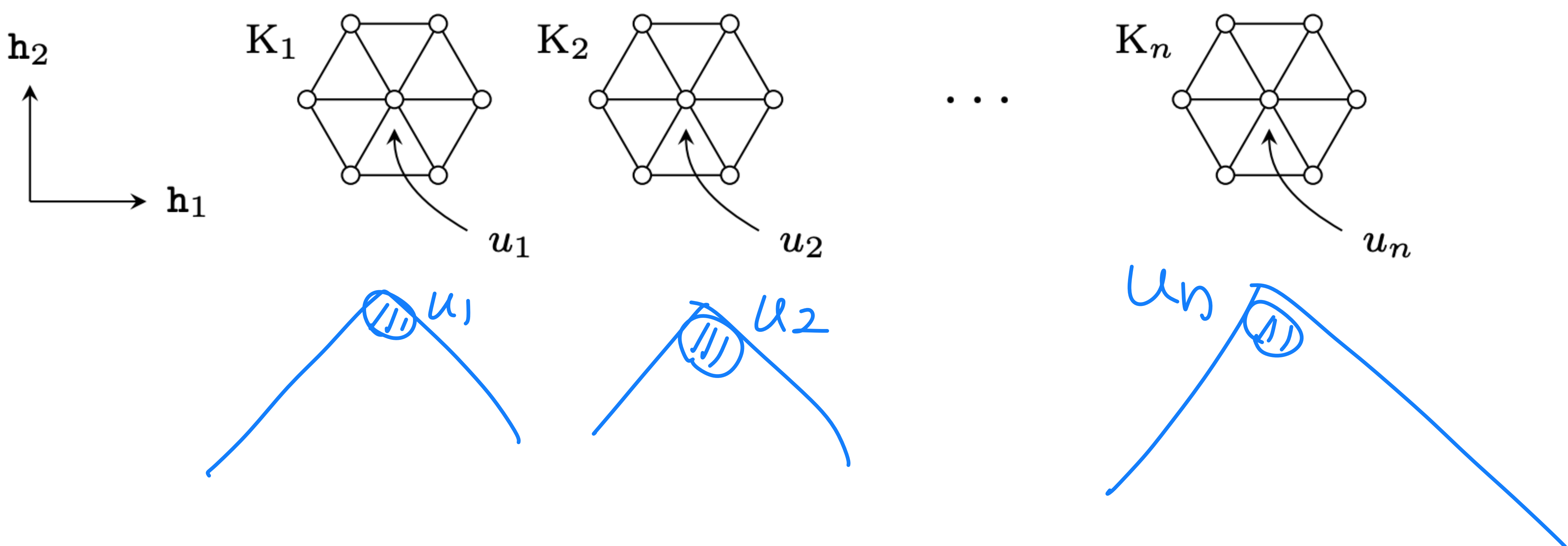
$$\times \prod_{\substack{b\in Q_0 \\ b\neq a}} \prod_{k=1}^{N_b} \varphi^{a\Leftarrow b}\left(\sigma_i^{(a)} - \sigma_k^{(b)}\right) \prod_{\mathfrak{f}} \varphi^{a\Leftarrow \mathfrak{f}}\left(\sigma_i^{(a)} - u_{\mathfrak{f}}\right)$$

$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u + h_I)}{\prod_{I\in\{a\to b\}}(u - h_I)}$$

We can make "crystal chains" by bringing together crystals in

spectral-parameter plane

[Galakhov-Y, Galakhov-Li-Y ('21)]



$$|\mathrm{K}_1, {}^\sharp\mathcal{C}_1\rangle_{u_1} \otimes |\mathrm{K}_2, {}^\sharp\mathcal{C}_2\rangle_{u_2} \otimes \ldots \otimes |\mathrm{K}_n, {}^\sharp\mathcal{C}_n\rangle_{u_n} .$$

We can derive representations

[Galakhov-Y, Galakhov-Li-Y ('21)]

$$\mathbf{\Delta}_0^{(n)}(\psi(z)) \bigotimes_{i=1}^{n} |\mathrm{K}_i\rangle_{u_i} = \prod_i \Psi_{\mathrm{K}_i}(z-u_i) \times \bigotimes_i |\mathrm{K}_i\rangle_{u_i} \,,$$

$$\mathbf{\Delta}_0^{(n)}(e(z)) \bigotimes_{i=1}^{n} |\mathrm{K}_i\rangle_{u_i} = \sum_i \sum_{\square\in\mathrm{Add}(\mathrm{K}_i)} \prod_{j<i} \Psi_{\mathrm{K}_j}(u_i+h_\square-u_j) \times \frac{[\mathrm{K}_i \to \mathrm{K}_i+\square]}{z-(u_i+h_\square)} \times \bigotimes_{j<i} |\mathrm{K}_j\rangle_{u_j} \otimes |\mathrm{K}_i+\square\rangle_{u_i} \otimes \bigotimes_{k>i} |\mathrm{K}_k\rangle_{u_k} \,,$$

$$\mathbf{\Delta}_0^{(n)}(f(z)) \bigotimes_{i=1}^{n} |\mathrm{K}_i\rangle_{u_i} = \sum_i \sum_{\square\in\mathrm{Rem}(\mathrm{K}_i)} \prod_{k>i} \Psi_{\mathrm{K}_k}(u_i+h_\square-u_k) \times \frac{[\mathrm{K}_i \to \mathrm{K}_i-\square]}{z-(u_i+h_\square)} \times \bigotimes_{j<i} |\mathrm{K}_j\rangle_{u_j} \otimes |\mathrm{K}_i-\square\rangle_{u_i} \otimes \bigotimes_{k>i} |\mathrm{K}_k\rangle_{u_k} \,,$$

※ not inv. under permutations

and "standard coproduct"

$$\mathbf{\Delta}_0 e = e\otimes 1 + \psi \overset{\rightarrow}{\otimes} e \,,$$

$$\mathbf{\Delta}_0 f = 1\otimes f + f \overset{\leftarrow}{\otimes} \psi \,,$$

$$\mathbf{\Delta}_0 \psi = \psi\otimes\psi \,.$$

However,

$$\mathbf{\Delta}_0 e = e \otimes 1 + \psi \overset{\rightarrow}{\otimes} e\,,$$
$$\mathbf{\Delta}_0 f = 1 \otimes f + f \overset{\leftarrow}{\otimes} \psi\,,$$
$$\mathbf{\Delta}_0 \psi = \psi \otimes \psi\,.$$

- Δ_0 does NOT reproduce R-matrix needed for (BAE) = (vacuum equation)
- For rational/Yangian case does NOT come from a coproduct $\Delta_0 : Y \to Y \otimes Y$

[Prochazka ('15)] [Galakhov-Li-Y ('22)]

We need to search "correct" Δ:

cf. stable envelope of [Maulik-Okounov]

$$\Delta = \mathcal{U}^{-1} \Delta_0 \mathcal{U}$$

"Yes-Go"

[Galakhov-Li-Y ('22)]
See also [Feigin-Jimbo-Miwa-Mukhin ('15)]
[Litvinov-Vilkovisky ('20)] [Chistyakova-Litvinov-Orlov ('21)]
[Kolyaskin, A. Litvinov, and A. Zhukov ('22)] [Bao ('22)]

We can choose shift $=0$

* For 2d-crystal repr. (Fock module) of $Y(\hat{\mathfrak{g}})$ w/ $\mathfrak{g} = \mathfrak{gl}_{m|n}, D(2,1;\alpha)$

We can derive BAE

and verify Gauge/Bethe!

"No-Go" [Galakhov-Li-Y ('22)]

shift $\neq 0$

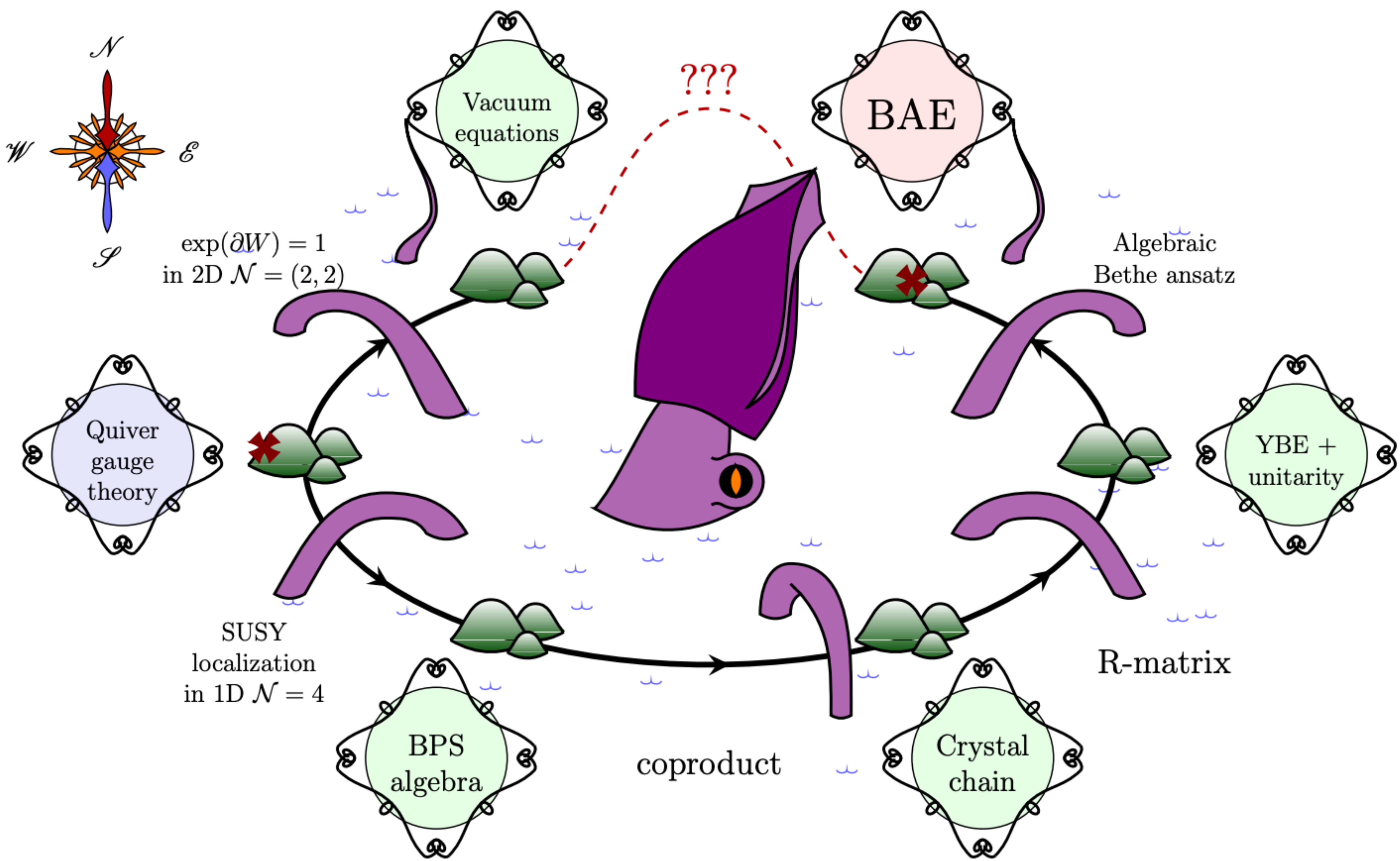
* For $Y(Q, W)$ without underlying $\mathfrak{g}$

[chiral quiver / toric CY_3 with 4-cycle]

We have obstructions (under some assumptions)

to finding consistent Δ / R

whose BAE matches vacuum eqn.



$\mathscr{N}$
$\mathscr{W}$
$\mathscr{E}$
$\mathscr{S}$
Vacuum equations
???
BAE
$\exp(\partial W) = 1$ in 2D $\mathcal{N} = (2,2)$
Algebraic Bethe ansatz
Quiver gauge theory
YBE + unitarity
SUSY localization in 1D $\mathcal{N} = 4$
R-matrix
BPS algebra
coproduct
Crystal chain

Summary

String theory

toric CY_3

Quiver Yangian

$Y(Q,W)$

new algebras

SUSY

QM

(Q,W)

new repr.

repr. in crystal melting

counts

BPS states / DT inv.