Quandles and Topological Quantum Field Theories

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some refs

- Kashaev, Reshetikhin ('01)
- Blanchet, Geer, Patureau-Mirond, Reshetikhin ('18)
- Creutzig, Dimofte, Garner, Geer ('21)

⋮

Z

"in progress"

non-semisimple TQFT

3d CS : TQFT

$\mathcal{M}_{flat}$

$\mathfrak{g}, k$

"bulk-edge"

2d chiral CFT

KM VOA $V^k(\mathfrak{g})$

logarithmic VOA

Kazhdan-Lusztig CFT

3d TQFT

A-twist of

3d $\mathcal{N}=4$ theory

quiver gauge

Rep of $U_q(\mathfrak{g})$

mod-$U_q(\mathfrak{g})$

non-semisimple

root of unity

$\mathfrak{g} = sl_2$

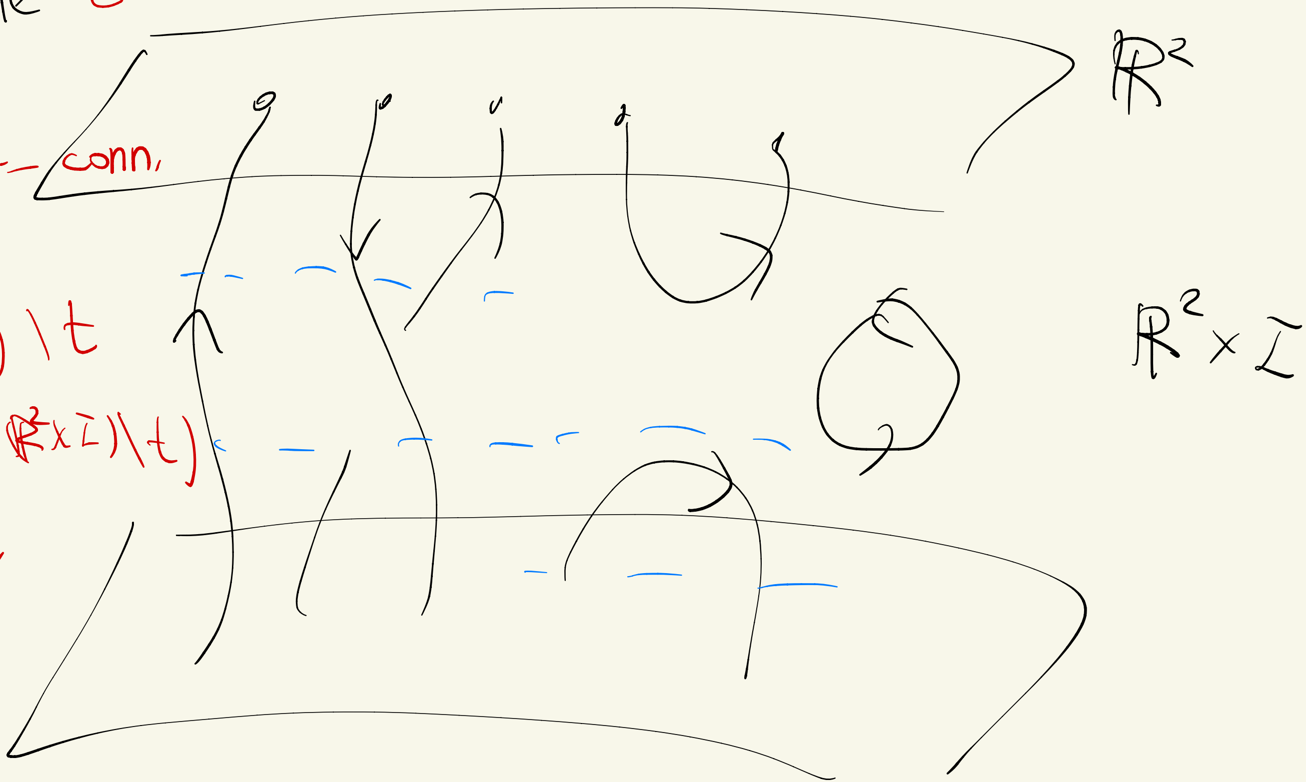
"Q: can we enrich this by (bi)quandle?"

Tangle t

$+$

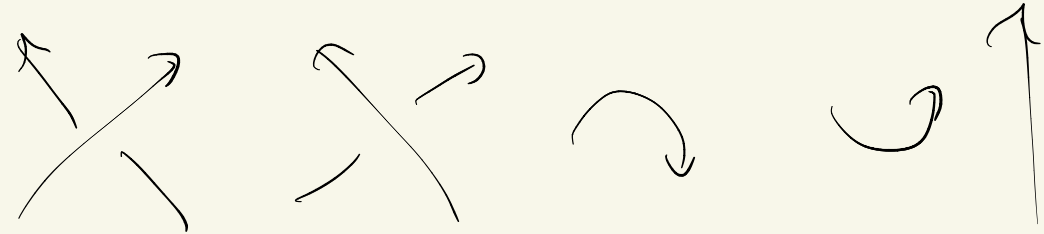
flat G-conn. in $(\mathbb{R}^2 \times I) \setminus t$

$\rho : \pi_1((\mathbb{R}^2 \times I) \setminus t) \to G$

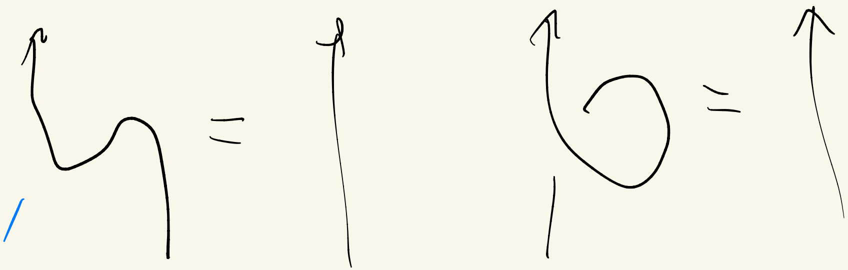


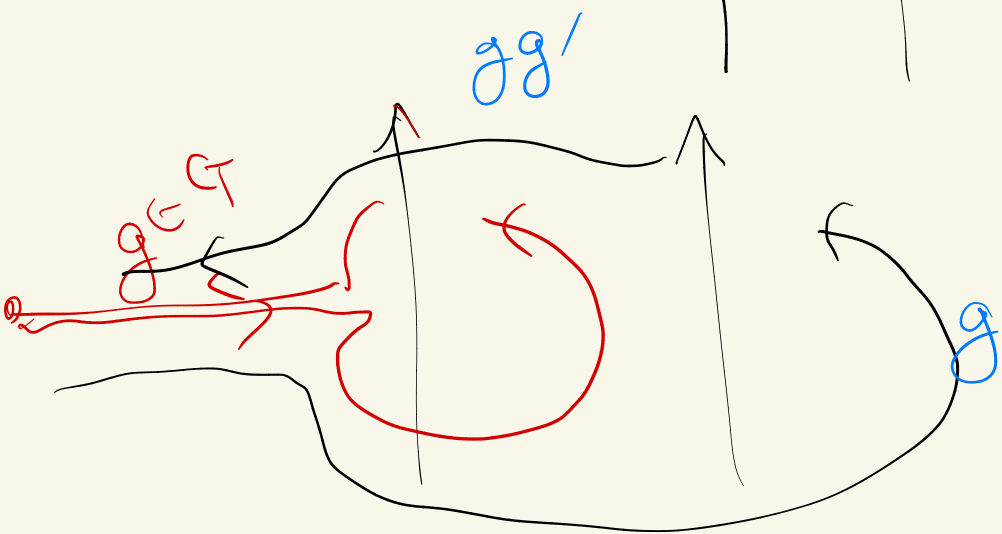
tangle diagram

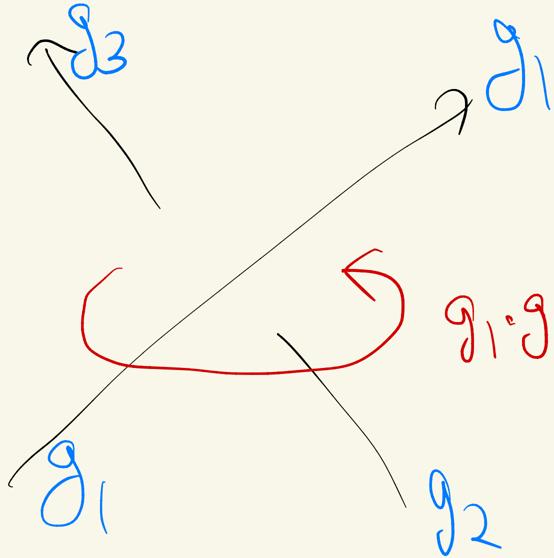
elementary moves



moves



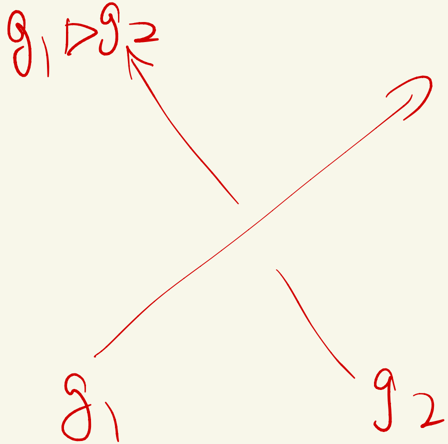




$$g_1 g_2 = g_3 g_1 \to \underbrace{g_3 = g_1^{-1} g_2 g_1}_{\parallel}$$

$$g_1 \triangleright g_2 \quad \text{"quandle"}$$

$\mathrm{Conj}(G)$



RI $\quad g \triangleright g = g \quad \forall g$

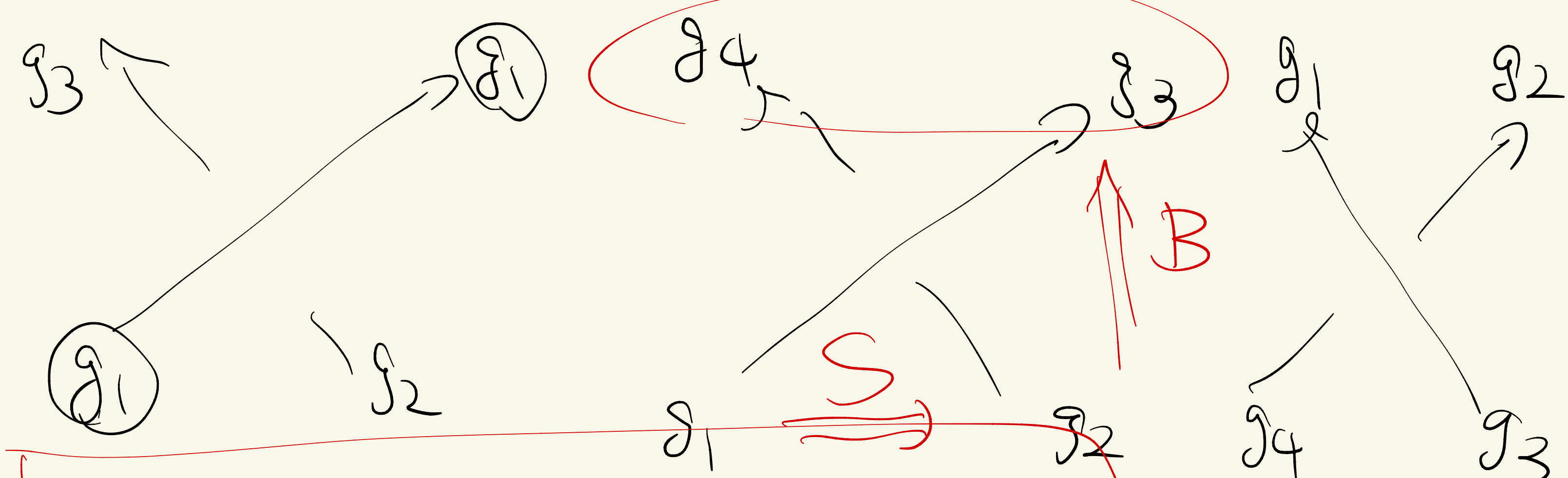
RII $\quad \forall g_1, g_2 \in G \quad \exists! \, g_3 \quad$ s.t. $\quad g_1 \triangleright g_3 = g_2$

RIII $\quad g_1 \triangleright (g_2 \triangleright g_3) = (g_1 \triangleright g_2) \triangleright (g_1 \triangleright g_3)$

✓ tangle colored by conj(G)

tangle colored by X : biquandle

Rep $U_q(g)$ @ root of unity



$$B(g_1, g_2) = \big(\underbrace{B_1(g_1, g_2)}_{g_3}, \underbrace{B_2(g_1, g_2)}_{g_4}\big)$$

$$S(g_1, g_4) = \big(\underset{S_1(g_1, g_4)}{\underset{\parallel}{g_2}}, \underset{S_2(g_1, g_4)}{\underset{\parallel}{g_3}}\big)$$

RI $\quad S(x, x) = (\alpha(x), \alpha(x)) \quad \alpha$: bij

RII $\quad S(B_1(x,y), x) = (B_2(x,y), y)$

RIII $\quad B$: satisfies YBE

$$(Id \otimes B) \circ (B \otimes Id) \circ (Id \otimes B)$$
$$= (B \otimes Id) \; (Id \; B) \; (B \otimes Id)$$

$y \triangleright x \qquad y \triangleright x$

biquandle X $\longrightarrow$ quandle Q

$S_1(x,y) \qquad B_1(x, S_1(x,y))$

Lebed Vandromin

$U_q(sl_2)$ $\quad K, K^{-1}, E, F$ $\quad q = e^{\pi i/k}$ $\quad q^{2k} = 1$

$KEK^{-1} = q^2 E$

$KFK^{-1} = q^{-2} F$

$[E, F] = \dfrac{K - K^{-1}}{q - q^{-1}}$

central elements

$\{E^k, F^k, K^{2k}\}$

$E^k = e \in \mathbb{C}$, $F^k = f \in \mathbb{C}$, $K^{2k} = \kappa \in \mathbb{C}^\times$

Δ coproduct

ε: counit

S: antipode

$\left.\begin{matrix} \Delta \\ \varepsilon \\ S \end{matrix}\right\}$ Hopf alg.

R

rep.

$V_{(e, f, \kappa)}$

$KE^kK^{-1} = q^{2k}E^k = E^k$

encode (e, f, k) into

$$G^* = \left\{ \left(\underbrace{\begin{pmatrix} k & 0 \\ f & 1 \end{pmatrix}}_{g_+}, \underbrace{\begin{pmatrix} 1 & e \\ 0 & k \end{pmatrix}}_{g_-} \right) \in GL_2 \times GL_2 \right\}$$

$$\downarrow$$

$$G' \subset GL_2$$

$$\psi$$

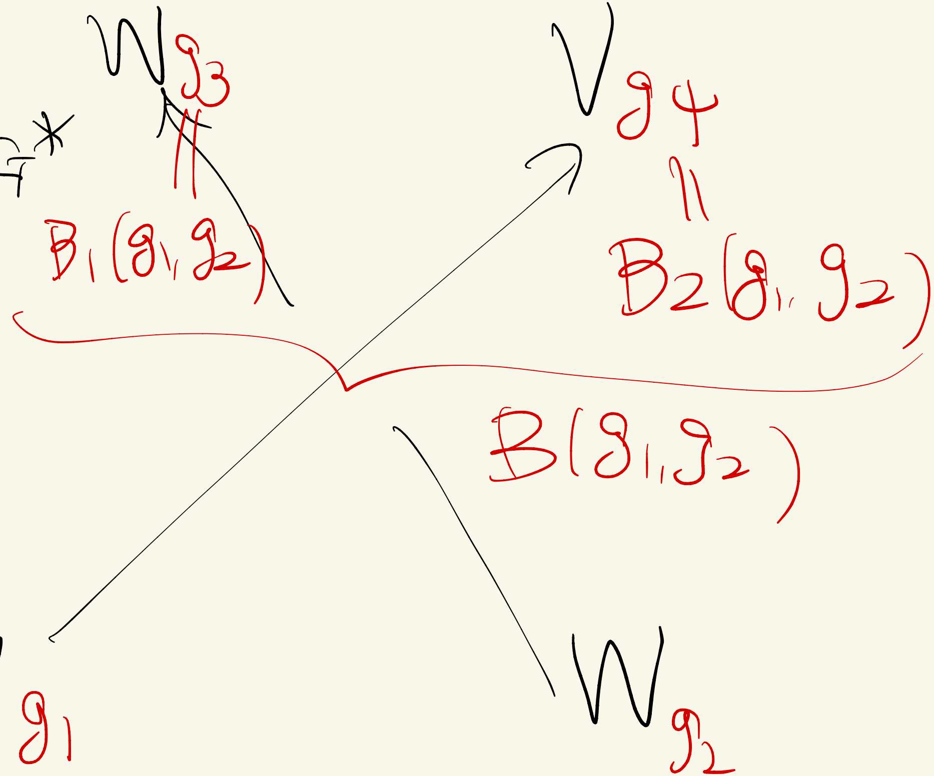
$$g = g_+ \times (g_-)^{-1} = \begin{pmatrix} k & -e \\ f & \frac{1}{k} - \frac{ef}{k} \end{pmatrix} \in GL_2$$

root-of-unity rep. of $U_q(\mathfrak{sl}_2)$

$q = e^{\pi i/k}$

then V_g, $g \in G^*$

$g = (e, f, K)$



$B : G^* \times G^* \to G^* \times G^*$

Statement

$$G^* = \left\{ \begin{pmatrix} k & 0 \\ f & 1 \end{pmatrix}, \begin{pmatrix} 1 & e \\ 0 & k \end{pmatrix} \right\}$$

$$(B_1, B_2) = B : G^* \times G^* \to G^* \times G^*$$

S

defines biquandle

tangle + flat G-conn

$\updownarrow$

(bi) quandle-colored tangle

$\updownarrow$

Rep $U_q(\mathfrak{sl}_2)$

root of unity

non-semisimple

☆ 3d TQFT
non-semi simple
local operator

$\xleftarrow{\text{A-twist}}$
H-twist

3d $\mathcal{N}=4$ theory $\quad \mathfrak{g}=\mathfrak{sl}_n$

$T[SU(n)] \curvearrowleft G$ / GTR

3d mirror

$G^\vee \curvearrowright \mathcal{M}_C \longleftrightarrow \mathcal{M}_H \curvearrowleft G$

Nakajima quiver var.

$(1) - (2) \cdots (n-1) - \boxed{n}$

L_i

local op.

L_j

$\mathrm{Hom}(L_i, L_j) = \mathbb{C}\,\delta_{ij}$

☆ $\boxed{\text{2d logarithmic VOA} \quad \text{Feigin-Typunin}}$

Zhu's alg.

→ HK

chiralization

$\mathcal{U}_{H}$