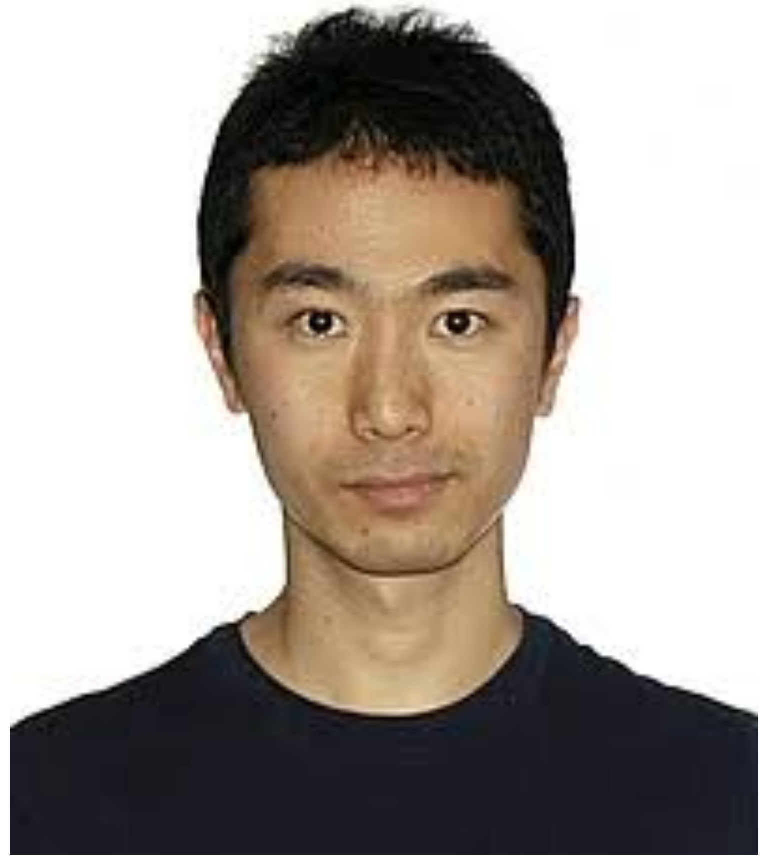
Time-Dependent Integrability from Gauge Theory

Masahito Yamazaki

UTokyo | KAVLI IPMU

Jun 9, 2026 @ IGST 2026

Work to appear in collaboration with

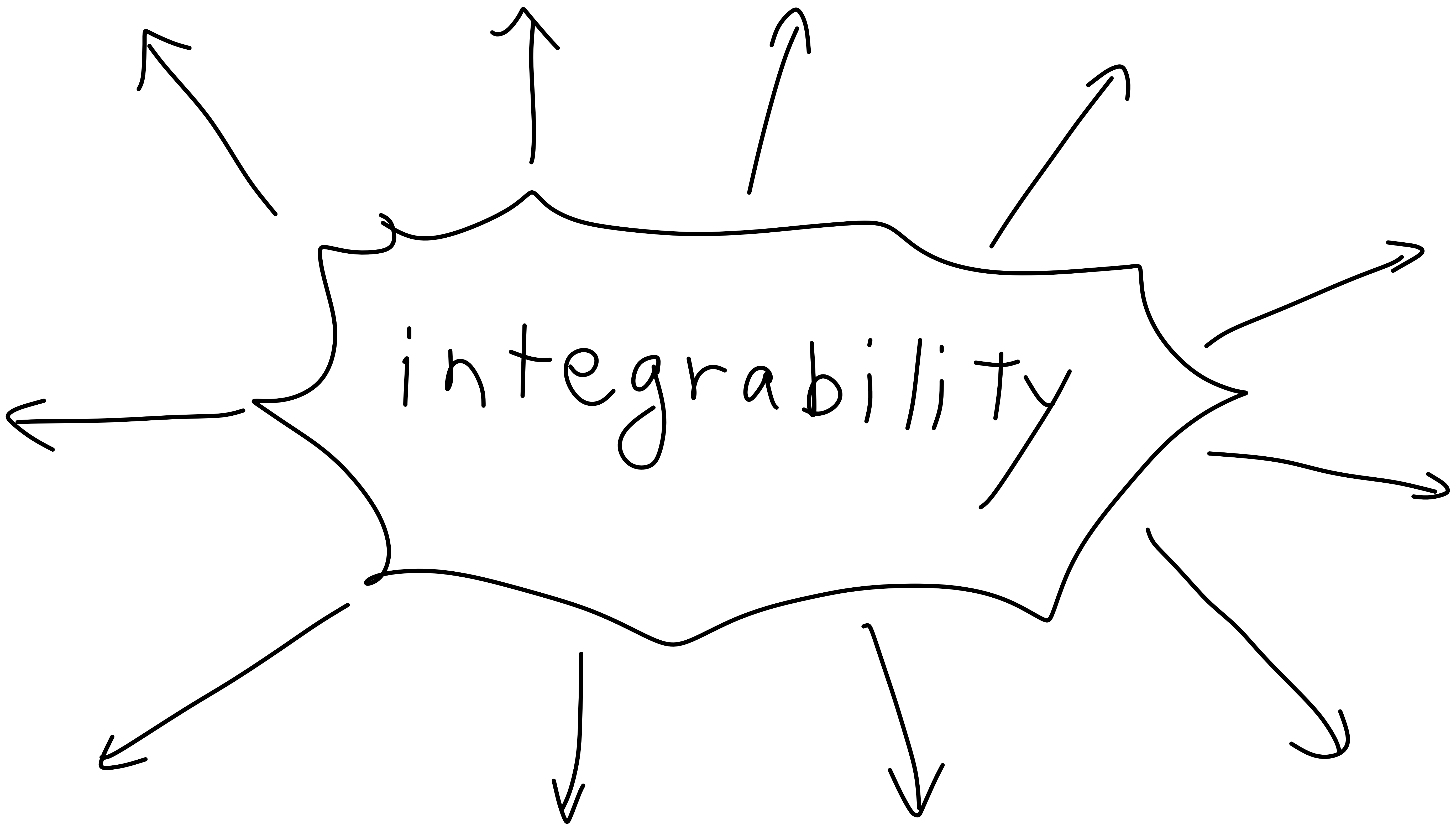


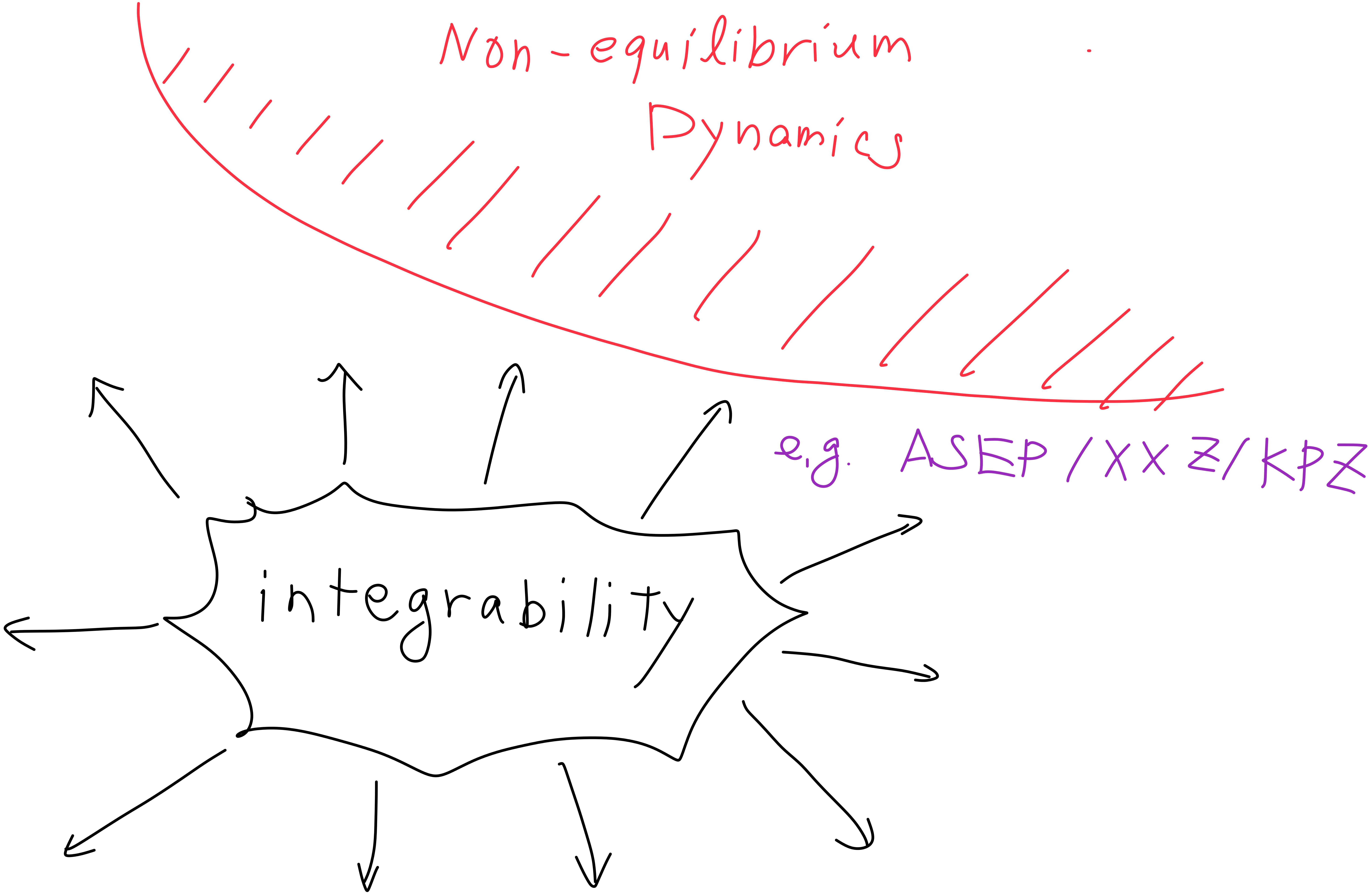
Shota Komatsu

Anders Wallberg

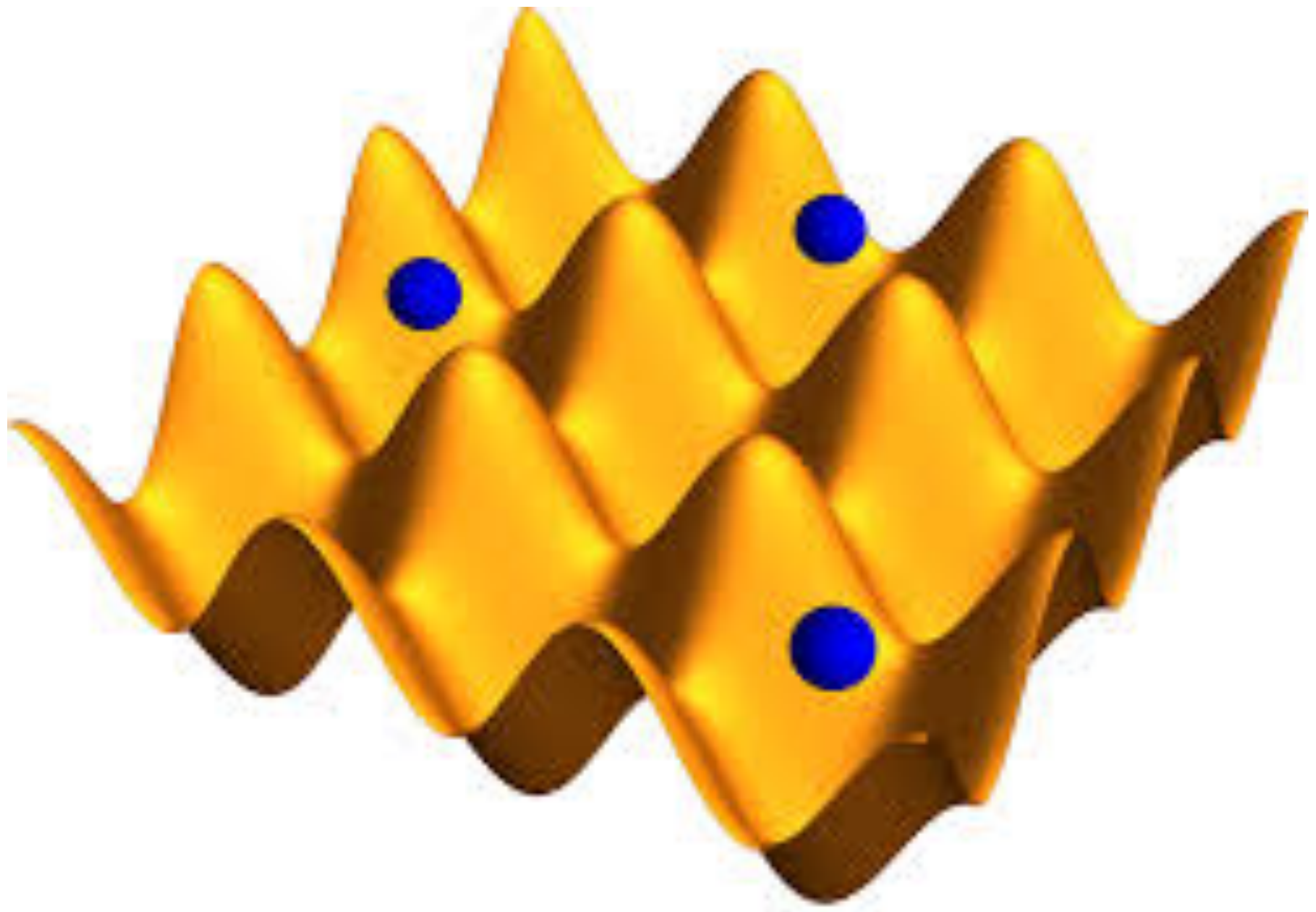
Jun-ichi Sakamoto

integrability


integrability


Non-equilibrium
Dynamics
e.g. ASEP / XXZ / KPZ
integrability

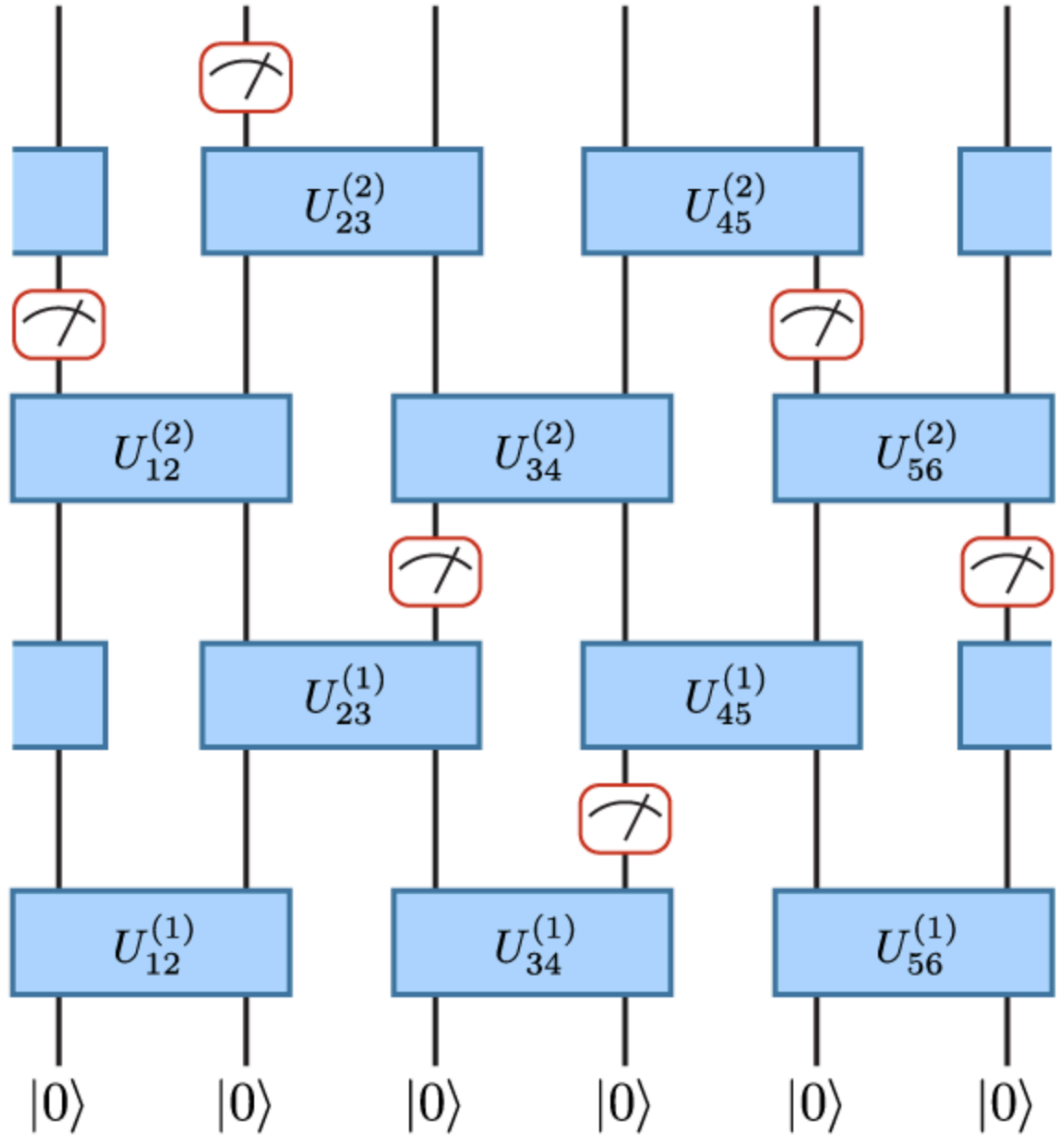
spatial modulations



e.g. optical lattices

many-body localization

temporal modulations



e.g. Floquet dynamics

quantum circuits

Q: Integrability for Dynamics = Time-Dependence?

We need a systematic framework!

- Non-autonomous integrability has a looong history
- Lax integrability since 70's (in gravity)

  [Belinsky - Zakharov ('78, '79)]

  [Burtsev - Zakharov - Mikhailov ('87)]

  [Breitenlohner - Maison ('87)]

- More recently

  [Hoare - Levine - Tseytlin ('20)]

  [Penna ('20)] [Cole - Weck ('24)]

  [Cesàro - Osten ('25)] [Ashwinkumar - Blau ('26)]

  [many papers by P. R. Pasnoori]

General Definition

This talk: 2d classical IFT

S. Komatsu on quantum aspects on Thursday

This talk: 2d classical IFT

S. Komatsu on quantum aspects on Thursday

@ Lax connection $L(z) = \sum L_\mu(z)\, dx^\mu$ ← depends on fields

satisfying flat connection condition

$$[D_\mu, D_\nu] = 0 \quad , \quad D_\mu = \partial_\mu + L_\mu$$

holds on-shell

$$L(z)[\underbrace{\phi}_{=\phi(t,x)}, \partial\phi, \cdots]$$

fields in the theory

This talk: 2d classical IFT

S. Komatsu on quantum aspects on Thursday

@ Lax connection $L(z) = \sum L_\mu(z)\, dx^\mu$

satisfying flat connection condition

$$[D_\mu, D_\nu] = 0 \quad , \quad D_\mu = \partial_\mu + L_\mu$$

inverse scattering linear problem

$$D_\mu \Phi = 0$$

conserved charges from monodromy

$$P\exp \int L(z)$$

Time-dependent Lax connection

$$\underbrace{\mathcal{L}_\mu(z)}_{\text{space-time dependence through fields}} \rightsquigarrow \underbrace{\hat{\mathcal{L}}_\mu(z)}_{\text{now with explicit space-time dependence}}$$

$$\hat{\mathcal{L}}_\mu(z, \underbrace{t, x}_{\text{explicit time/space dependence}})[\phi, \partial\phi, \dots]$$

$$\mathcal{L}_\mu(z)[\underbrace{\phi, \partial\phi, \dots}_{\phi = \phi(t,x)}]$$

only time/space dependence

Q Time-dependent Lax connection

$$\underbrace{\mathcal{L}_\mu(z)}_{\text{space-time dependence through fields}} \rightsquigarrow \underbrace{\hat{\mathcal{L}}_\mu(z)}_{\text{now with explicit space-time dependence}}$$

$$\partial_\mu \rightsquigarrow \hat{\partial}_\mu(z) = \partial_\mu + \underbrace{f_\pm(z)}_{f_\pm(t,x,z)}\,\partial_z$$

impose

$$[\hat{\partial}_\mu(z), \hat{\partial}_\nu(z)] = 0$$

Q Time-dependent Lax connection

$$\hat{\partial}_\mu(z)\hat{L}_\nu(z) - \hat{\partial}_\nu(z)\hat{L}_\mu(z) + [\hat{L}_\mu(z), \hat{L}_\nu(z)] = 0$$

$$[\underbrace{\hat{D}_\mu(z)}_{\hat{\partial}_\mu(z)+\hat{L}_\mu(z)}, \hat{D}_\nu(z)] = 0$$

$$[\hat{\partial}_\mu(z), \hat{\partial}_\nu(z)] = 0$$

$\cancel{\rightarrow}$ conserved charges not guaranteed

$$\hat{D}_\mu(z)\Phi = 0$$

linear problem ; inverse scattering method

Bottom-up Example

Gross – Neveu model

- Action N fermions $\psi_\pm^{i=1\sim N}$

$$S = \int \sum_{i=1}^{N} i\left(\psi_+^i \partial \psi_+^i + \psi_-^i \partial_+ \psi_-^i\right) + h \underbrace{\mathrm{Tr}\left(J_+^a J_-^a\right)}_{\text{four-Fermi}}$$

- Lax

$$L_\pm(t,x,z) = \frac{J_\pm(t,x,z)}{z - z_\pm}$$

current

$$J_\pm^a = \psi_\pm^i \, \overset{\text{SO(N) generator}}{\overset{\downarrow}{T^a}} \, \psi_\pm^i$$

light-cone coordinates

$$x^\pm = t \pm x$$

- Introduce time / space dependence

$$S = \int \sum_{i=1}^{N} i\left(\psi_+^n \partial \psi_+^n + \psi_-^n \partial_+ \psi_-^n\right) + \hat{h}(t,x)\, \mathrm{Tr}(J_+ J_-)$$

Lax

$$\hat{L}_\pm(t,x,z) = \frac{J_\pm(t,x,z)}{z - \hat{z}_\pm(t,x)}$$

• Introduce time / space dependence

$$S = \int \sum_{i=1}^{N} i\left(\psi_+^n \partial \psi_+^n + \psi_-^n \partial_+ \psi_-^n\right) + \underbrace{\hat{h}(t,x)}_{\displaystyle \| \atop \displaystyle \frac{1}{\hat{z}_+(t,x) - \hat{z}_-(t,x)}} \mathrm{Tr}\left(J_+ J_-\right)$$

Lax

$$\hat{L}_\pm(t,x,z) = \frac{J_\pm(t,x,z)}{z - \hat{z}_\pm(t,x)}$$

Ansatz <4d CS

- Introduce time / space dependence

$$S = \int \sum_{i=1}^{N} i\left(\psi_+^n \partial_- \psi_+^n + \psi_-^n \partial_+ \psi_-^n\right) + \underbrace{\hat{h}(t,x)}_{= \frac{1}{\hat{z}_+(t,x) - \hat{z}_-(t,x)}} \operatorname{Tr}(J_+ J_-)$$

Lax

$$\hat{L}_\pm(t,x,z) = \frac{J_\pm(t,x,z)}{z - \hat{z}_\pm(t,x)}$$

- Impose flatness:

$$\partial_\mp \hat{z}^\pm = 0 \rightsquigarrow \hat{h}(t,x) = 1 \Big/ \left(g_+(x_+) - g_-(x_-)\right)$$

- If time-dependence only: $\hat{h}(t) = 1 \big/ (at+b)$

What is this deformation?

$$\hat{h}(t) = \frac{1}{at+b}$$

What is this deformation?

$$\hat{h}(t) = \frac{1}{at+b}$$

$$\partial_t \hat{h}(t) = -a\,\hat{h}(t)^2$$

What is this deformation?

$$\hat{h}(t) = \frac{1}{at+b}$$

$$\underbrace{\partial_t}_{\text{" }\ln\mu\text{ "}} \hat{h}(t) = -a\,\hat{h}(t)^2 \propto \underbrace{\beta(\hat{h}(t))}_{\text{one-loop }\beta\text{-function!!}}$$

What is this deformation?

$$\hat{h}(t) = \frac{1}{at+b}$$

$$\underset{\text{" } \ln\mu \text{ "}}{\partial_t} \hat{h}(t) = -a\,\hat{h}(t)^2 \propto \underbrace{\beta(\hat{h}(t))}_{\text{one-loop } \beta\text{-function !!}}$$

Lesson

- time/space dependence : zoo of possibilities
- time dependence only : unique & RG

[Hoare – Levine – Tseytlin ('20)]

General Framework ;

4d Chern- Simons

4d Chern-Simons Theory

[Costello, Witten, MY ('17, '18); Costello, MY ('19)]

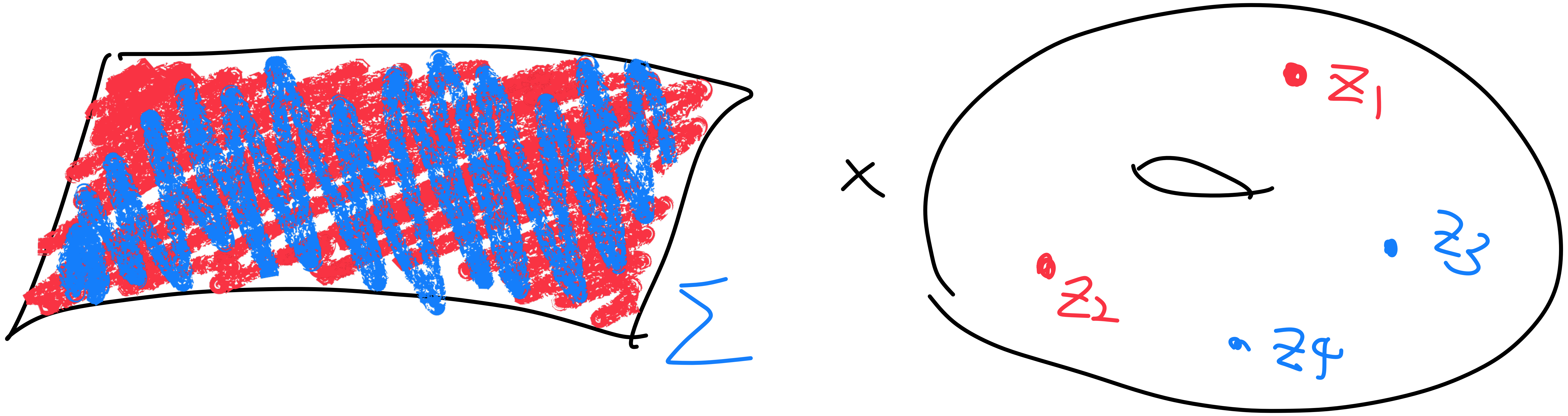
$$S = \frac{1}{2\pi\hbar} \int_{\underbrace{\Sigma_w \times C_z}} \underbrace{\omega} \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$A = A_w dw + A_{\bar{w}} d\bar{w} + A_{\bar{z}} d\bar{z}$$

$$\underbrace{C = \mathbb{CP}^1}_{\text{rational case}}$$

$$\omega = \underbrace{\varphi}_{\text{twist function.}} dz$$

4d CS + 2d surface defects



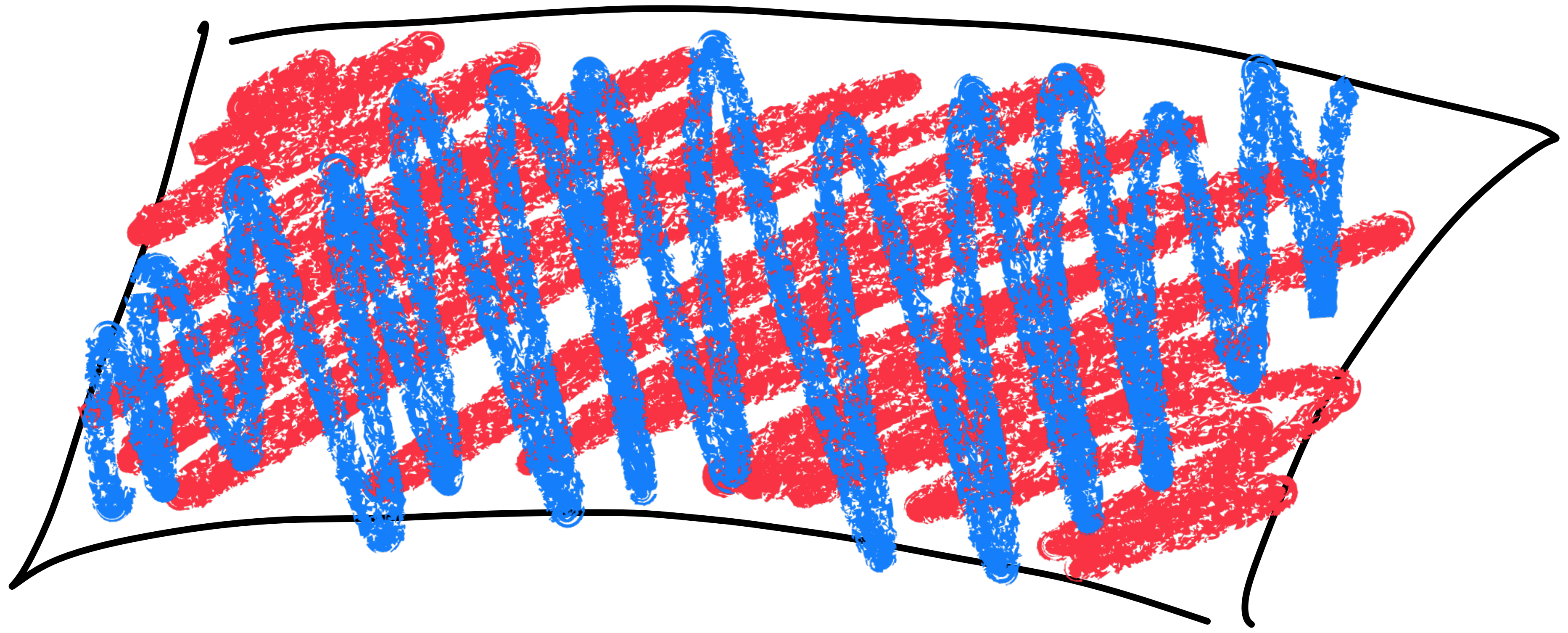
$C = \mathbb{CP}^1$

integrate along C

$S_{4d\,CS} + S_{2d\ defect}$

$S_{2d\ IFT}$

2d Integrable QFTs



Lax integrability automatic

4d

$S_{4d\,CS} \longrightarrow$ e.o.m. in gauge $A_{\bar{z}} = 0$

e.o.m. (for $A_{\bar{z}} = 0$)

$\partial_{\bar{z}} A_+ = \partial_{\bar{z}} A_- = 0$

$F_{+-} = 0$

$\rightsquigarrow$

2d

Lax connection

$\mathcal{L} = A_+(z)\,dx^+ + A_-(z)\,dx^-$

We have

$S_{4d\ CS} + \boxed{S_{2d\ \text{defect}}}$

- order defect (ultralocal)

  coupling to 2d defect Lagrangian

  e.g. Gross-Neveu model

- disorder defect (non-ultralocal)

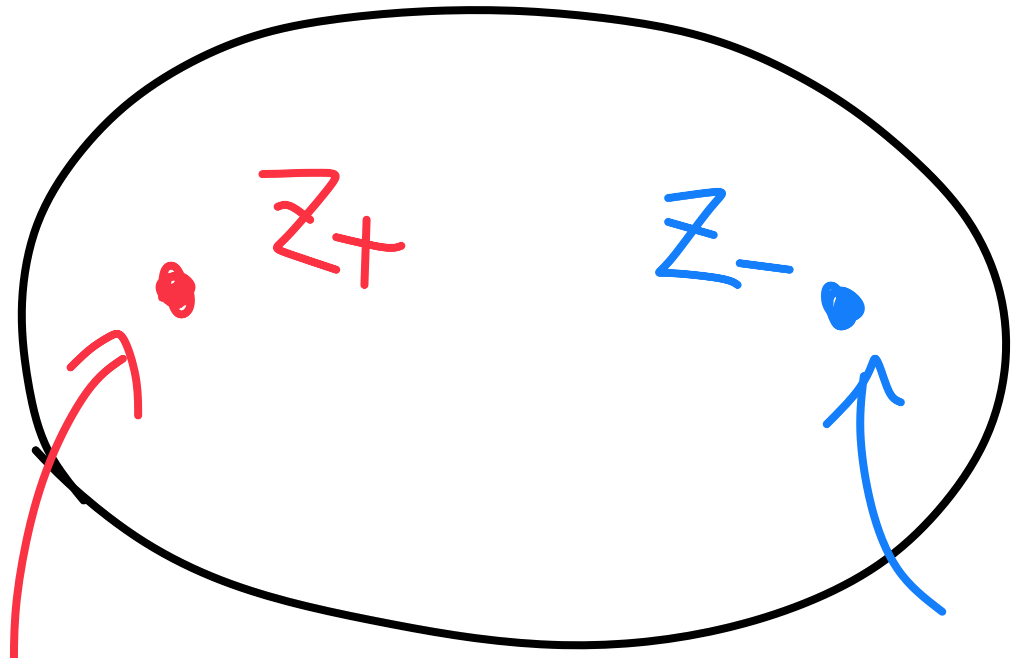
  modification of one-form ω

  e.g. principal chiral model

  $S_{4d\ CS} \sim \int \boxed{\omega} \wedge CS[A]$

Order Defect

e.g. chiral free fermion defects



anti-chiral fermion

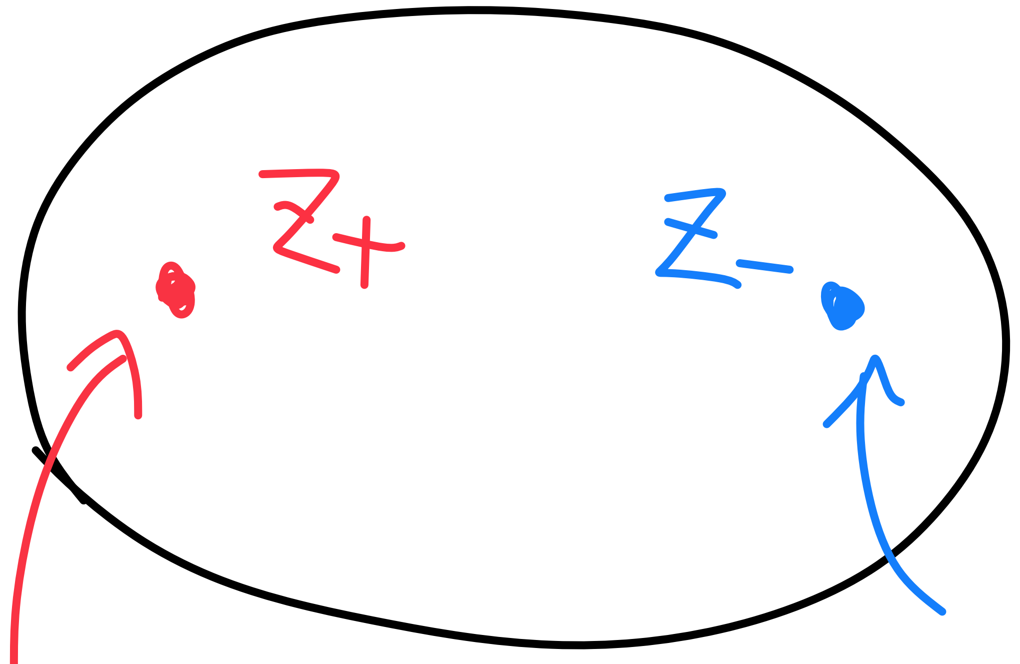
chiral fermion

$$\mathcal{L}_1 = \bar{\Psi}_+ (\not{\partial} + \not{A}_+) \Psi_+$$

$$\mathcal{L}_2 = \bar{\Psi}_- (\not{\partial} + \not{A}_-) \Psi_-$$

Order Defect

e.g. chiral free fermion defects

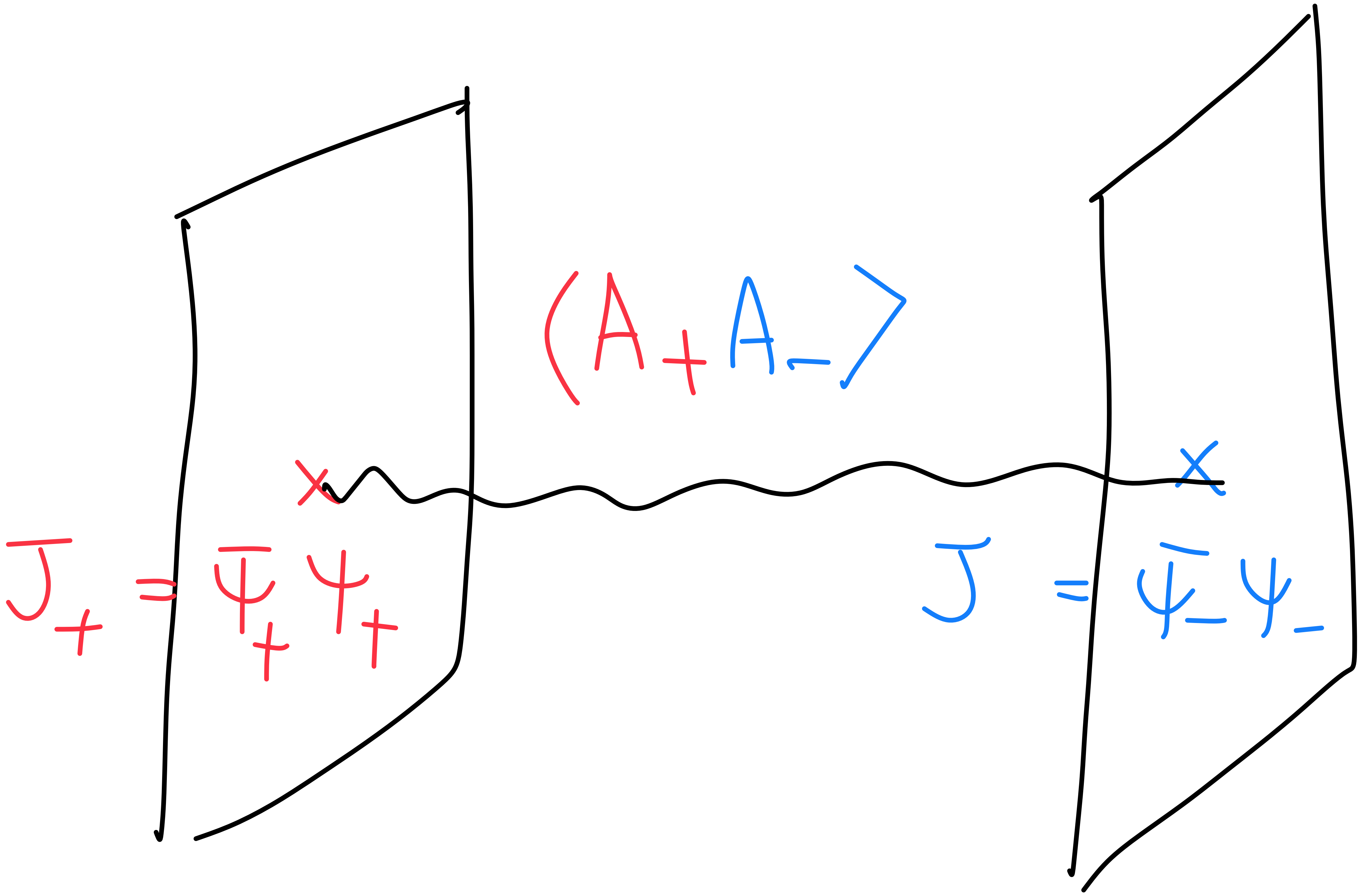


chiral fermion

anti-chiral fermion

$$\mathcal{L}_1 = \bar{\psi}_+ (\not{\partial} + \not{A}_+)\psi_+ \qquad \mathcal{L}_2 = \bar{\psi}_- (\not{\partial} + \not{A}_-)\psi_-$$

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \frac{1}{z_1 - z_2}\underbrace{(\bar{\psi}_+\psi_+)(\bar{\psi}_-\psi_-)}_{\text{4-fermi interaction}}$$



We can have multiple order defects

$$S_{2d} = \int \sum_{i=1}^{N_+} \mathcal{L}_+^i[\phi_+^i] + \sum_{j=1}^{N_-} \mathcal{L}_-^j[\phi_-^j]$$

$$+ \sum_{i=1}^{N_+} \sum_{j=1}^{N_-} J_+^{i,a} \underbrace{r_{ab}(z_i^+ - z_j^-)}_{\text{classical } r\text{-matrix}} J_-^{j,b}$$

$$h_{ij} \propto \frac{1}{z_i^+ - z_j^-} \quad \text{rational case}$$

We can have multiple order defects

$$S \ni \sum_{i=1}^{N_+} \sum_{j=1}^{N_-} h_{ij} \, J_+^i \, J_-^j$$

$$\hat{h}_{ij} = \frac{1}{\hat{z}_i^+(x^+) - \hat{z}_i^-(x^-)}$$ time/space dependance

$$= \frac{1}{at + b_i^+ + b_j^-}$$ time-only dependence

matches RG

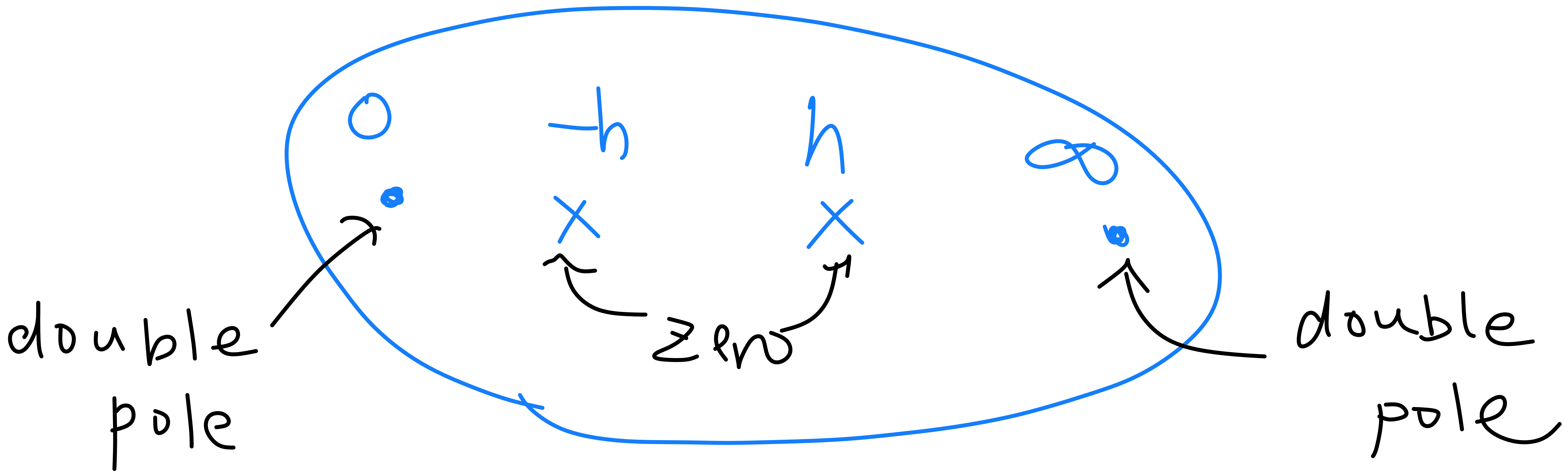
Disorder defect

$$S = \int_{\Sigma \times C} \omega \wedge CS[A]$$

$$\omega = \varphi(z)\, dz$$

e.g. principal chiral model $S = \frac{\hbar}{2} \int \mathrm{Tr}(g^{-1}\partial_- g, g^{-1}\partial_+ g)$

$$\omega = \frac{\hbar^2 - z^2}{2z^2}\, dz$$



Disorder defect

$$S = \int_{\Sigma \times C} \omega \wedge CS[A]$$

$$\omega = \varphi(z)\, dz$$

$$\Big\Downarrow$$

$$\hat{\omega} = \hat{\varphi}(t, x, z)\, dz + \hat{\Psi}_+(t, x, z)\, dx^+ + \hat{\Psi}_-(t, x, z)\, dx^-$$

time/space dependence introduced

$$\delta S_{\substack{4d \\ CS}} \sim \int_{\Sigma \times \mathbb{CP}^1} \hat{\omega} \wedge F \wedge \delta A - \int_{\Sigma \times \mathbb{CP}^1} d\hat{\omega} \wedge \mathrm{Tr}(A \wedge \delta A)$$

$$\delta S_{4d\,CS} \sim \int_{\Sigma\times\mathbb{CP}^1} \underbrace{\hat{\omega}\wedge F}\wedge \delta A - \int_{\Sigma\times\mathbb{CP}^1} d\hat{\omega}\wedge \mathrm{Tr}(A\wedge\delta A)$$

We have

$\hat{\omega}\wedge F = 0$

$\downarrow$

$F=0$ away from $\underbrace{\text{zeros of } \omega}_{\text{singularities of Lax} = A}$

We need boundary condition

$$\delta S_{4d\,CS} \sim \int_{\Sigma\times\mathbb{CP}^1} \hat{\omega}\wedge F \wedge \delta A - \int_{\Sigma\times\mathbb{CP}^1} \underbrace{d\hat{\omega}} \wedge \mathrm{Tr}(A\wedge \delta A)$$

$$\hat{\omega} = \hat{\varphi}\, dt + \hat{\Phi}_+ dx^+ + \hat{\Phi}_- dx^-$$

bulk contribution

$$d\hat{\omega} = (\partial_+\hat{\varphi} - \partial_t \hat{\Phi}_+)\, dx^+\wedge dt$$

$$+ (\partial_-\hat{\varphi} - \partial_t \hat{\Phi}_-)\, dx^-\wedge dt$$

$$+ (\partial_-\hat{\Phi}_+ - \partial_+ \hat{\Phi}_-)\, dx^-\wedge dx^+$$

$$\delta S_{4d\,CS} \sim \int_{\Sigma\times\mathbb{CP}^1} \hat{\omega}\wedge F \wedge \delta A - \int_{\Sigma\times\mathbb{CP}^1} \underbrace{d\hat{\omega}} \wedge \mathrm{Tr}(A\wedge\delta A)$$

$$\hat{\omega} = \hat{\varphi}\, dt + \hat{\Phi}_+ dx^+ + \hat{\Phi}_- dx^-$$

bulk contribution

$$\begin{aligned} d\hat{\omega} = {} & \underbrace{\boxed{(\partial_+\hat{\varphi} - \partial_t \hat{\Phi}_+)}}_{=0} dx^+\wedge dt \\ & + \underbrace{\boxed{(\partial_-\hat{\varphi} - \partial_t \hat{\Phi}_-)}}_{=0} dx^-\wedge dt \\ & + \underbrace{\boxed{(\partial_-\hat{\Phi}_+ - \partial_+ \hat{\Phi}_-)}}_{=0} dx^-\wedge dx^+ \end{aligned}$$

$$\delta S_{\substack{4d\\CS}} \sim \int_{\Sigma\times\mathbb{CP}^1} \hat{\omega}\wedge F \wedge \delta A - \int_{\Sigma\times\mathbb{CP}^1} \underbrace{d\hat{\omega}} \wedge \mathrm{Tr}(A\wedge\delta A)$$

we need to cancel remaining localized contributions

$$d\hat{\omega} \ni \delta^{i}(z-p_i)$$

$$\text{if} \quad \hat{\omega} \sim \frac{1}{z-p_i}$$

$$\left(\partial_{\bar{z}} \frac{1}{z} \sim \delta(z)\right)$$

For time-only-dependence,

$$\omega = \hat{\varphi}(t,z)\,dz + \hat{\Psi}(t,z)\,dt + a\,dx$$

Similar logic gives

$$\underbrace{\partial_t \hat{\varphi} = \partial_z \hat{\Psi}(z,t)}_{\text{matches one-loop RG}}$$

(+ some conditions, e.g. $\hat{\Psi}$ has same poles as $\hat{\varphi}$)

[Delduc - Lacroix - Sfetsos - Siampos ('21)]

[Lacroix - Levine - Wallberg ('25)]

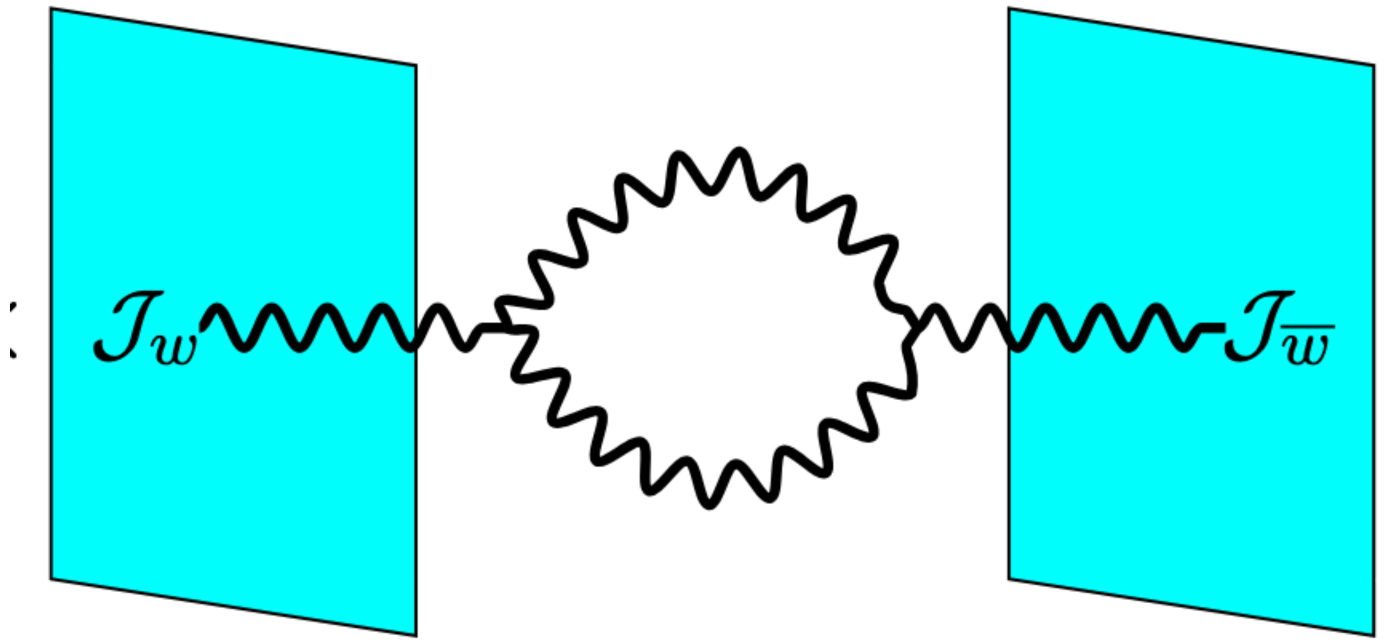
One-loop RG can be derived in 2d

[Kutasov ('89)] [Gerganov-LeClair-Moriconi ('01)] ...

This can be derived more intrinsically in 4d

To appear with

Meer Ashwinkumar



Example $\quad PCM_{\underbrace{h}_{\text{coupling}}} + WZ_{\underbrace{k}_{\text{level}}}$

- one-form

$$\omega = \varphi(z)\, dz \qquad \varphi = \frac{1}{2} \frac{h^2 - z^2}{(z-k)^2} dz$$

- Lax connection

$$L = \frac{h}{h-z}\left(1 - \frac{k}{h}\right) j_+ dx^+ + \frac{h}{h+z}\left(1 + \frac{k}{h}\right) j_- dx^-$$

- RG flow

$$\frac{d}{d\ln\mu} h = c_G \left(1 - \frac{k^2}{h^2}\right)$$

$\uparrow$ dual coxeter number

Introduce time-only dependence $h \to \hat{h}(t)$, $\omega \to \hat{\omega}$ etc.

$$\hat{\omega} = \hat{\varphi}(z,t)\left(dz - \hat{f}_+(z,t)\,dx^+ - \hat{f}_-(z,t)\,dx^-\right)$$

$$\hat{\varphi} = \frac{1}{2}\,\frac{\hat{h}(t)^2 - z^2}{(z-k)^2} \qquad \hat{f}_\pm(z,t) = \frac{a(z-k)(\hat{h}(t) \mp k)}{\hat{h}(t)\,(z \mp \hat{h}(t))}$$

Lax

$$\hat{L} = \frac{\hat{h}(t)}{\hat{h}(t) - z}\left(1 - \frac{k}{\hat{h}(t)}\right) j_+ dx^+ + \frac{\hat{h}(t)}{\hat{h}(t) + z}\left(1 + \frac{k}{\hat{h}(t)}\right) j_- dx^-$$

With $\hat{h}(t)$ defined by RG flow

$$\frac{d}{dt}\hat{h}(t) = a\left(1 - \frac{k^2}{\hat{h}(t)^2}\right)$$

The procedure works for zoo of models!

- massless non-Abelian Thirring model
- Faddeev-Reshetikhin model

order defect

- PCM with WZ term
- η-/λ- deformations of PCM
- coupled PCM + WZ

⋮

disorder defect

many examples worked out thanks to

Comment 1:

time-dependent Lax $\rightsquigarrow$ [Belinsky - Zakharov ('78, '79)]

Spectral parameter z constant

we worked out relations explicitly in many examples

[Breitenlohner-Maison ('87)]

Spectral parameter $z(t)$ time - dependent

Comment 2: Time-dependence arises by coupling to dilaton gravity

$$S = \int \sqrt{-g}\left(\Phi R + \mathcal{L}_{matter}(\psi, g_{\mu\nu} \,|\, \Phi)\right)$$

$$\downarrow \quad ds^2 = e^{\varphi}\eta_{\mu\nu}\,dx^\mu dx^\nu$$

$$S = \int \Phi\, \partial_\mu \partial^\mu \varphi + \mathcal{L}_{matter}(\psi \,|\, \Phi)$$

$$\downarrow \quad \partial_+\partial_-\Phi = 0, \quad \Phi = g_+(x_+) + g_-(x_-); \quad \text{choose } \Phi = t$$

$$S = \int \mathcal{L}_{matter}(\psi \,|\, t)$$

[cf. Belinsky - Zakharov]

4d Einstein gravity $\rightarrow$ 2d dilaton gravity

Summary

- time / space - dependent integrability

  promising direction for
  - theoretical framework
  - application to dynamics

- Realization via 4d CS + order/disorder defects

  $\Rightarrow$ Lax integrability
  - huge parameter sp for deformations
  - time-only dependence = RG flow

Many open questions

Non-equilibrium Dynamics

Quantum Time Dependence

string theory realizations

ODE/IM

Inverse Scattering

(ultra) discretization

isomonodromy problem

connection w/ GR/BH

Time-dependent Yangian