2025 / 01 / 29 Lecture 7

$\boxed{\text{Summary} \sim \text{Lec 6}}$

- fusion algebra ⊂ categorical sym

$$\alpha \cdot \beta = \sum_{\gamma} N_{\alpha\beta}{}^{\gamma} \gamma$$

- transverse-field Ising model

$$H = -g^{-1} \sum_{i=1}^{L} Z_i Z_{i+1} - g \sum_{i=1}^{L} X_i$$

- Kramers-Wannier transformation

$$\mathcal{D} = \frac{1}{\sqrt{2^L}} \sum_{\{s_i\},\{\hat{s}_{i+\frac{1}{2}}\}} (-1)^{\sum_{j=1}^{L} (s_j+s_{j+1})\hat{s}_{j+\frac{1}{2}}} \left| \{\hat{s}_{i+\frac{1}{2}}\} \right\rangle \left\langle \{s_i\} \right|$$

- towards research:

  * generalization: $\mathbb{Z}_2 \to \mathbb{Z}_N$
    qubit   qudit

Cao-Li-Yamazaki
arXiv: 2406.05454
[cond-mat.str-el]

  * composition/sandwich:

$$\hat{\mathcal{D}} = \mathcal{D} \cup \mathcal{D}$$

today

Recall. qudit $\mathbb{Z}_N$ $|s\rangle$ $s = 0, \ldots, N-1 \in \mathbb{Z}_N$

$$X = \sum_{s=0}^{N-1} |s+1\rangle\langle s| , \quad Z = \sum_{s=0}^{N-1} \omega^s |s\rangle\langle s| ; \quad \omega^N = 1$$

- L-qubit $|\{s_i\}\rangle = |s_1\rangle \otimes \cdots \otimes |s_L\rangle$

X_i , Z_i

$X_i^N = Z_i^N = I$

$Z_i X_i = \omega X_i Z_i , \quad Z_i X_j = X_j Z_i \quad (i \neq j)$

- $\mathbb{Z}_N$ version of Ising model = clock model

$$H = -g^{-1} \sum_{i=1}^{L} \left(Z_i^{+} Z_{i+1} + Z_i Z_{i+1}^{+} \right) - g \sum_{i=1}^{L} \left(X_i + X_i^{+} \right)$$

- symmetry

$$\eta = \prod_{i=1}^{L} X_i \quad , \quad C = \sum_{\{s_i\}} |\{-s_i\}\rangle\langle\{s_i\}|$$

Kramers–Wannier transformation D

$$D : Z_{j-1}^{\dagger} Z_j \longrightarrow X_j , \quad X_j \longrightarrow Z_{j-1}^{\dagger} Z_j$$

$$\left(D \cdot (Z_{j-1}^{\dagger} Z_j) = X_j D , \quad D X_j = Z_{j-1}^{\dagger} Z_j D \right)$$

fusion algebra

$$D^{\dagger} \cdot D = \sum_{k=1}^{N} \eta^{k} = 1 + \eta + \cdots + \eta^{N-1} \qquad (\eta^{N} = 1)$$

$\mathbb{Z}_N$ - sym.

Hadamard matrix

- $N=2$ case

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\Big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\Big)$$

- general N

$$U^{H} = \frac{1}{\sqrt{N}}\sum_{\alpha,\beta=1}^{N} \omega^{-\alpha\beta}|\beta\rangle\langle\alpha|$$

U^H exchanges Z and X

$$U^H : X \to Z^\dagger, \quad Z \to X.$$

$C = (U^H)^2$: charge conjugation

$$C : X \to X^\dagger, \quad Z \to Z^\dagger$$

on L-qubit Hilbert space

$$U^H_{total} = \prod_{i=1}^{L} U^H_i, \quad C_{total} = \prod_{i=1}^{L} C_i$$

check

$$U^H X = \left(\frac{1}{\sqrt{N}} \sum_{\alpha,\beta=1}^{N} \omega^{-\alpha\beta} |\alpha\rangle\langle\beta| \right) \left(\sum_{\gamma} |\gamma+1\rangle\langle\gamma| \right)$$

$$= \frac{1}{\sqrt{N}} \sum_{\alpha,\beta,\gamma=1}^{N} \omega^{-\alpha\beta} |\alpha\rangle \underbrace{\langle\beta|\gamma+1\rangle}_{\delta_{\beta,\gamma+1}} \langle\gamma|$$

$$= \frac{1}{\sqrt{N}} \sum_{\alpha,\gamma=1}^{N} \underbrace{\omega^{-\alpha(\gamma+1)} |\alpha\rangle}_{\omega^{-\alpha\gamma} \underbrace{\omega^{-\alpha}|\alpha\rangle}_{Z_i^{\dagger}|\alpha\rangle}} \langle\gamma|$$

$$= Z_i^{\dagger} \left(\frac{1}{\sqrt{N}} \sum_{\alpha,\gamma=1}^{N} \omega^{-\alpha\gamma} |\alpha\rangle\langle\gamma| \right) = Z_i^{\dagger} U^H$$

//

composite KW transformation:

$$\hat{D} = D^{\dagger} U^H_{tot} D = \underbrace{T}_{\text{shift index by } 1} D\, U^H_{tot} D$$

Recall:

$D \;:\; Z^{\dagger}_{j-1} Z_j \to X_{j-1}, \quad X_j \to Z^{\dagger}_{j-1} Z_j$

$U^H \;:\; X_i \to Z_i^{\dagger}, \quad Z_i \to X_i$

$T \;:\; X_i \to X_{i+1}, \quad Z_i \to Z_{i+1}$

$$\begin{tikzcd}
X_i \arrow[r, "D"'] \arrow[dr, bend right, "\hat{D}"'] & Z_{i-1}^\dagger Z_i \arrow[r, "U^{-1}"'] & X_{i-1}^\dagger X_i \arrow[r, "D"'] & (Z_{i-2}^\dagger Z_{i-1})^\dagger Z_{i-1}^\dagger Z_i \arrow[d, equal] \\
& Z_{i-1}(Z_i^\dagger)^2 Z_{i+1} & & Z_{i-2}(Z_{i-1}^\dagger)^2 Z_i \arrow[ll, bend left, "T"]
\end{tikzcd}$$

$$\hat{D}: X_i \to Z_{i-1}(Z_i^\dagger)^2 Z_{i+1}$$

$$
\begin{tikzcd}
Z_{i-1}(Z_i^\dagger)^2 Z_{i+1} = (Z_{i-1}^\dagger Z_i)^\dagger (Z_i^\dagger Z_{i+1}) \arrow[rr, "D"] \arrow[d, "D"'] & & X_i^\dagger X_{i+1} \arrow[d, "U^H"] \\
X_i^\dagger & X_{i+1}^\dagger \arrow[l, "T"] & Z_i^\dagger Z_{i+1} \arrow[l, "D"']
\end{tikzcd}
$$

$$\hat{D}: \; Z_{i-1}(Z_i^\dagger)^2 Z_{i+1} \to X_i^\dagger$$

Ising/Clock model

$$H = -g^{-1} \sum \left(Z_i^\dagger Z_{i+1} + Z_i Z_{i+1}^\dagger \right) - g \sum \left(X_i + X_i^\dagger \right)$$

KW D

$$\eta = \prod X_i \,, \quad C$$

dipole Ising model

$$H = -g^{-1} \sum \left(Z_{i-1} (Z_i^\dagger)^2 Z_{i+1} + Z_{i-1}^\dagger (Z_i)^2 Z_{i+1}^\dagger \right) - g \sum \left(X_i + X_i^\dagger \right)$$

(generalized) KW $\not{D}$

$$\eta_Q = \prod_{i=1}^{L} X_i \,, \quad \eta_L = \prod_{i=1}^{L} (X_i)^i \quad C$$

$$H \supset \underbrace{Z_i^\dagger}_{\omega} \underbrace{Z_{i+1}}_{\omega^{-1}}$$

$$\left(\begin{array}{ll} Z_i X_i = \omega X_i Z_i, & Z_i X_j = \omega X_j Z_i \\ Z_i^\dagger X_i = \omega^{-1} X_i Z_i^\dagger & (i \neq j) \end{array} \right)$$

$$\begin{cases} \eta Z_i^\dagger Z_{i+1} = \omega Z_i^\dagger \eta Z_{i+1} \\ \qquad = \omega \omega^{-1} Z_i^\dagger Z_{i+1} \eta = Z_i^\dagger Z_{i+1} \eta \end{cases}$$

$$H \supset Z_{i-1} (Z_i^\dagger)^2 Z_{i+1} \qquad \left(\eta_Q = \prod X_i , \quad \eta_D = \prod (X_i)^i \right)$$

$$Z_{i-1} (Z_i^\dagger)^2 Z_{i+1}$$

$$\begin{cases} \eta_Q : \omega^{-1} \omega^2 \cdot \omega^{-1} = 1 \\ \eta_D : \omega^{-(i-1)} (\omega^i)^2 \omega^{-(i+1)} = 1 \end{cases}$$

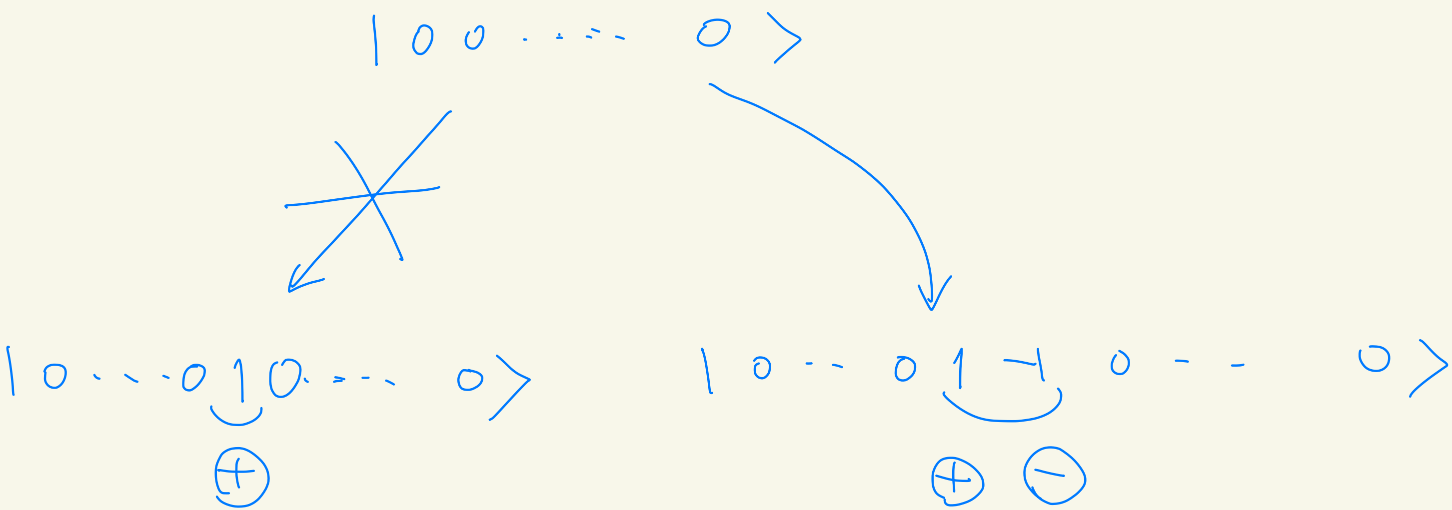
meaning? (easier if $X_i \to Z_i$)

First, $\eta_Q = \prod_{i=1}^{L} Z_i$

$Z_i |s_i\rangle = \omega^{s_i} |s_i\rangle$

$\eta_Q |\{s_i\}\rangle = \omega^{\underbrace{\sum s_i}} |\{s_i\}\rangle$

"total charge"



$$\eta_D = \prod (Z_i)^i \qquad \eta_D |\{s_i\}\rangle = \omega^{\sum i s_i} |\{s_i\}\rangle$$

$$|0 \cdots 0\rangle$$

$$|0 \cdots 1\ 0 \cdots 0\rangle \qquad |0 \cdots \underbrace{1\ {-1}}_{i\ \ i+1}\ 0 \cdots 0\rangle \qquad |0 \cdots 0\ \underbrace{1\ {-2}\ 1}_{\text{dipole}} \cdots\rangle$$

$+$

$+ \rightarrow -$

$+ \rightarrow -$

$- \leftarrow +$

dipole (2重極)は保存

quadruple (4重極)は変わってもよい

fusion algebra

$$\eta_Q = \prod_{i=1}^{L} X_i \,, \quad \eta_D = \prod_{i=1}^{L} (X_i)^i \,, \quad \hat{D} \,, \quad C + T$$

$$\begin{cases} \eta_Q \hat{D} = \eta_D \hat{D} = \hat{D}\eta_Q = \hat{D}\eta_D = \hat{D} \\ \hat{D} \times \hat{D} = \left(\sum_{k=1}^{N} \eta_Q^k\right)\left(\sum_{k=1}^{N} \eta_D^k\right) C \\ C\eta_Q = \eta_Q^\dagger C \,, \quad C\eta_D = \eta_D^\dagger C \\ T\eta_Q = \eta_Q T \\ T\eta_D = \eta_Q^\dagger \eta_D T \,, \quad C^2 = 1 \,, \quad T^L = 1 \end{cases}$$

$$\hat{D}^\dagger = C\hat{D} = \hat{D}C$$

(not Tambara-Yamagami)

N=2 case $\qquad c = 1$

$$\begin{cases} \eta_Q \hat{D} = \eta_D \hat{D} = \hat{D}\eta_Q = \hat{D}\eta_D = \hat{D} = \hat{D}^\dagger \\ \hat{D} \times \hat{D} = (1+\eta_Q)(1+\eta_D) \end{cases}$$

$$\mathrm{Rep}(D_8)$$