Holomorphic maps between generalized complex manifolds

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based on joint work with Radu Pantilie

January 4-9, 2009, IPMU, Tokyo
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The linear picture

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Linear CR-structure on $V$

$C \subset V^\mathbb{C}$ s.t. $C \cap \overline{C} = \{0\}$. 

Linear co-CR-structure on $V$

$D \subset V^\mathbb{C}$ s.t. $D + D = V^\mathbb{C}$.

Co-CR is the dual notion to CR: $D$ is co-CR $\iff$ $\text{Ann}(D)$ is CR in $(V^\mathbb{C})^\ast$.

If $J$ is a complex structure on $V$, then the corresponding $V_{1,0}$ and $V_{0,1}$ are both CR and co-CR.

Linear CR and co-CR maps $t: (V, C_V) \to (W, C_W)$ linear s.t. $t(C_V) \subseteq C_W$.

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Linear generalized complex structures

\[ L = L(E, \varepsilon) \subset V^C \oplus (V^C)^* \] which is maximally isotropic of 0 real index:

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Call $L(E \cap \overline{E}, \text{Im}(\varepsilon|_{E \cap \overline{E}}))$ the associated linear Poisson structure.
Generalized complex structure in normal form

Compatible $f$ structure and 2-form

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Normal form in tensorial language

The corresponding $\mathcal{J}$ is written as

$$\mathcal{J} = \begin{pmatrix} F & \eta \\ -\omega & -F^* \end{pmatrix}.$$
Theorem

Given a generalized complex structure $L$ and a $f$-structure $F$ s.t. $E = \pi(L)$ is the co-CR structure associated to $F$,

$$B =\begin{cases} -\text{Re}(\varepsilon) & \text{on } V_0 \\ -\varepsilon & \text{on } V_1 \end{cases}$$

and $V_0 \otimes V_1$, $0$.

The only freedom is in the choice of $F$ and, in fact, only the splitting of $V_0 \oplus V_1$ is to be chosen.
Given a generalized complex structure $L$ and a $f$-structure $F$ s.t. $E = \pi(L)$ is the co-CR structure associated to $F$, there exists a unique $B \in \Lambda^2 V^*$ s.t. $e^B(L)$ is in normal form, with associated $f$-structure $F$. 
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- Up to $B$-transforms, $t = t_1 \oplus t_2$, with $t_1$ (resp. $t_2$) a Poisson (resp. complex) map between symplectic (resp. complex) vector spaces.
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$\varphi : M \rightarrow N$ is holomorphic if $\forall x \in M$ regular, $\exists U \ni x$ s.t., up to $B$-transforms of $M$ and $N$, $\varphi\big|_U$ is a Poisson, $f$-holomorphic map between generalized complex structures in normal form.
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- If $\varphi$ is diffeo, then holomorphicity is equivalent to $\varphi_*(L_M) = L_N$ up to $B$ transforms.
- Composition preserves holomorphicity.
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Local structure of Dirac manifolds

Locally, around regular points, there exist submersions \( \varphi : M \to P \) s.t. \( \varphi^*(L) \) is a Poisson structure and \( L = \varphi^*(\varphi^*(L)) \).
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one obtains:

**Theorem**

Let \((M, L_M)\) and \((N, L_N)\) be regular real analytic generalized complex manifolds and let \( \varphi : M \to N \) be a real analytic map. If \( \varphi \) is holomorphic then, locally, up to the complexification of a real analytic \( B \)-field tranformation, the complexification of \( \varphi \) descends to a complex analytic Poisson morphism between the canonical Poisson quotients.
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Let $(M, g)$ be generalized complex in normal form with $L = L(E, i\varepsilon)$. Choose a compatible $f$-structure $F$. Quotient to $T^0M$ (integrable) to obtain local submersions $\varphi : M \to (N, J)$. On $N$ consider the canonical generalized complex structure $J$. It is in normal form. As $\varphi$ is $f$-holomorphic between g.c. structures in normal form, $\varphi$ is holomorphic.
Let \((M, g)\) be generalized complex in normal form with 
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**Proposition**

\(E\) is co-isotropic w.r.t. \(g^C\) \(\iff\) \(\varphi|_{E\cap\overline{E}}\) is pseudo-horizontally conformal (p.h.c.) \(i.e.\) it pulls back \((1, 0)\)-forms into isotropic forms.
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The notion comes from harmonic morphisms.
Inverse construction

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$\omega|\mathcal{V}$ is non-degenerate, $\text{Ker}\omega = \mathcal{H} \Rightarrow F, \omega$ are compatible.
Let $L = L(E, i\omega)$ (in normal form), with $E = \mathcal{V} \oplus \mathcal{H}^{1,0}$.

$L$ is integrable $\iff$ $J$ is integrable, $\mathcal{V}$ has minimal leaves and
$A(X, Y) := [X, Y]^\mathcal{V}$ is $(1, 1)$ w.r.t. $F$.  

Inverse construction

Let $\varphi : (M^{n+2}, g) \rightarrow (N^n, J)$ be a p.h.c. submersion.
Let $\mathcal{V} = \text{Ker} \varphi_*$, $\mathcal{H} = \mathcal{V}^\perp$, $\omega$ volume form on $\mathcal{V}$.
Let $F$ be the unique $g$-skew-symmetric $f$-structure on $M$ s.t.
$\text{Ker} F = \mathcal{V}$, $T^0 M \oplus T^{1,0} M = \varphi_*^{-1}(T^{1,0} N)$, $T^{0,1} M = \varphi_*^{-1}(T^{0,1} N)$,
$\omega|_{\mathcal{V}}$ is non-degenerate, $\text{Ker} \omega = \mathcal{H} \Rightarrow F, \omega$ are compatible.
Let $L = L(E, i\omega)$ (in normal form), with $E = \mathcal{V} \oplus \mathcal{H}^{1,0}$.

$L$ is integrable $\iff$ $J$ is integrable, $\mathcal{V}$ has minimal leaves and
$A(X, Y) := [X, Y]^\mathcal{V}$ is (1, 1) w.r.t. $F$.
For $n = 2$, $L$ integrable $\iff \varphi$ is a harmonic morphism.
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A generalized complex structure in normal form on a $(M, g)$ s.t.:
Inverse construction

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A generalized complex structure in normal form on a \((M, g)\) s.t.:
the associated \( f \)-structure is \( g \)-skew,
the induced Poisson structure has rank 2,
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A generalized complex structure in normal form on a $(M, g)$ s.t.:
the associated $f$-structure is $g$-skew,
the induced Poisson structure has rank 2,
$\|\omega\| = 1$,
is locally of this form.
Generalized Kähler manifolds

\((M, L_1, L_2)\) s.t. \(\mathcal{J}_1 \mathcal{J}_2 = \mathcal{J}_2 \mathcal{J}_1\) and \(\mathcal{J}_1 \mathcal{J}_2 < 0\).
Generalized Kähler manifolds

\((M, L_1, L_2)\) s.t. \(\mathcal{I}_1 \mathcal{I}_2 = \mathcal{I}_2 \mathcal{I}_1\) and \(\mathcal{I}_1 \mathcal{I}_2 < 0\).
By projection on \(TM\) this gives \((g, b, J_+, J_-)\).
(M, L₁, L₂) s.t. J₁J₂ = J₂J₁ and J₁J₂ < 0. By projection on TM this gives (g, b, J⁺, J⁻). Let V± be the i-eigenbundle of J±.
Generalized Kähler manifolds

\((M, L_1, L_2)\) s.t. \(J_1 J_2 = J_2 J_1\) and \(J_1 J_2 < 0\). By projection on \(TM\) this gives \((g, b, J_+, J_-)\).

Let \(V^\pm\) be the \(i\)-eigenbundle of \(J^\pm\).

Let \(L^\pm = \{X + (b \pm g)(X) \mid X \in V^\pm\}\).
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Then \(L_1 = L^+ \oplus L^-\), \hspace{1cm} L_2 = L^+ \oplus L^-\).
Generalized Kähler manifolds

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**Theorem (Gualtieri)**

$L_1$ and $L_2$ Courant-integrable $\iff$ $L^\pm$ integrable $\iff$ $J^\pm$ integrable and parallel w.r.t. $\nabla^\pm := \nabla^g \pm \frac{1}{2} g^{-1} h$, $h = db$.
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\((M, L_1, L_2)\) s.t. \(\mathcal{I}_1\mathcal{I}_2 = \mathcal{I}_2\mathcal{I}_1\) and \(\mathcal{I}_1\mathcal{I}_2 < 0\). By projection on \(TM\) this gives \((g, b, J_+, J_-)\).

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Canonical \(f\)-structures

Let \(E_j = \pi(L_j)\) \((j = 1, 2)\). Then \(E_1 = V^+ + V^-\), \(E_2 = V^+ + \overline{V^-}\).
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**Canonical \(f\)-structures**

Let \(E_j = \pi(L_j)\) \((j = 1, 2)\). Then \(E_1 = V^+ + V^-, \ E_2 = V^+ + \overline{V^-}\).

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Generalized Kähler manifolds

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By projection on \(TM\) this gives \((g, b, J_+, J_-)\).
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\(L_1\) and \(L_2\) Courant-integrable \(\iff\) \(L^\pm\) integrable \(\iff\) \(J^\pm\) integrable
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**Canonical f-structures**

Let \(E_j = \pi(L_j)\) \((j = 1, 2)\). Then \(E_1 = V^+ + V^-\), \(E_2 = V^+ + \overline{V^-}\).
Then \(E_1^\perp = V^+ \cap V^-\), \(E_2^\perp = V^+ \cap \overline{V^-}\) and hence:
\(E_1, E_2\) are coisotropic w.r.t. \(g^\mathbb{C}\).
Generalized Kähler manifolds

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**Canonical \(f\)-structures**

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The skew-symmetric \(F_j\) determined by \(E_j\) and \(E_j^\perp\) are integrable \(f\)-structures.
Generalized Kähler manifolds

$$(M, L_1, L_2) \text{ s.t. } \mathcal{J}_1 \mathcal{J}_2 = \mathcal{J}_2 \mathcal{J}_1 \text{ and } \mathcal{J}_1 \mathcal{J}_2 < 0.$$ By projection on $TM$ this gives $(g, b, J_+, J_-)$.

Let $V^\pm$ be the $i$-eigenbundle of $J^\pm$.
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Theorem (Gualtieri)

$L_1$ and $L_2$ Courant-integrable $\iff$ $L^\pm$ integrable $\iff$ $J^\pm$ integrable and parallel w.r.t. $\nabla^\pm := \nabla^g \pm \frac{1}{2} g^{-1} h$, $h = db$.

Canonical $f$-structures

Let $E_j = \pi(L_j)$ ($j = 1, 2$). Then $E_1 = V^+ + V^-$, $E_2 = V^+ + \overline{V^-}$.
Then $E^\perp_1 = V^+ \cap V^-$, $E^\perp_2 = V^+ \cap \overline{V^-}$ and hence:

$E_1, E_2$ are coisotropic w.r.t. $g^\mathbb{C}$.
The skew-symmetric $F_j$ determined by $E_j$ and $E^\perp_j$ are integrable $f$-structures.

Holomorphic functions on $(M, L_1)$ resp. $(M, L_2)$ are bi-holomorphic functions on $(M, J_+, J_-)$ resp. $(M, J_+, -J_-)$.
Let $\mathcal{H}^\pm = \text{Ker}(J_+ \mp J_-)$, $\mathcal{V} = (\mathcal{H}^+ \oplus \mathcal{H}^-)^\perp$. 
Generalized Kähler manifolds with integrable $\mathcal{H}^+$

Let $\mathcal{H}^\pm = \text{Ker}(J_+ \mp J_-)$, $\mathcal{V} = (\mathcal{H}^+ \oplus \mathcal{H}^-)^\perp$.

$\mathcal{H}^\pm$, $\mathcal{V}$ invariant under $J_\pm$; $J_+ \mp J_-$ invertible on $\mathcal{V}$.
Let $\mathcal{H}^\pm = \text{Ker}(J_+ \mp J_-)$, $\mathcal{V} = (\mathcal{H}^+ \oplus \mathcal{H}^-)\perp$.
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$\mathcal{H}^\pm$, $\mathcal{V}$ distributions $\iff L_1$, $L_2$ regular.
Let $\mathcal{H}^\pm = \text{Ker}(J_+ \mp J_-)$, $\mathcal{V} = (\mathcal{H}^+ \oplus \mathcal{H}^-)^\perp$. 
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**Theorem: geometric properties of the distributions**

If $L_1$ is regular, then: $\mathcal{H}^+$ integrable $\iff \mathcal{H}^+$ geodesic $\iff \mathcal{H}^+$ holomorphic on $(M, J_+)$ $\iff \mathcal{H}^+$ holomorphic on $(M, J_-)$. 

Note that $\mathcal{H}^+$ geodesic $\iff (\mathcal{H}^+)^\perp = \mathcal{V} \oplus \mathcal{H}^-$ is a Riemannian foliation, hence defines local submersions. 

**Corollary**

Let $(M, L^M_1, L^M_2)$ and $(N, L^N_1, L^N_2)$ with $\mathcal{H}^+=M$ and $\mathcal{H}^+=N$ integrable. Then any holomorphic $\phi: (M, L^M_1) \to (N, L^N_1)$ descends, locally, w.r.t. the above Riemannian submersions, to a holomorphic map between the Kähler quotients.
Let $\mathcal{H}^\pm = \text{Ker}(J_+ \mp J_-)$, $\mathcal{V} = (\mathcal{H}^+ \oplus \mathcal{H}^-)^\perp$.
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Corollary

Let $(M, L^M_1, L^M_2)$ and $(N, L^N_1, L^N_2)$ with $\mathcal{H}^+_M$ and $\mathcal{H}^+_N$ integrable.
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**Corollary**

Let $(M, L^M_1, L^M_2)$ and $(N, L^N_1, L^N_2)$ with $\mathcal{H}^+_M$ and $\mathcal{H}^+_N$ integrable. Then any holomorphic $\varphi : (M, L^M_1) \to (N, L^N_1)$ descends, locally, w.r.t. the above Riemannian submersions, to a holomorphic map between the Kähler quotients.
Products of Kähler manifolds

Let \((M_j, g_j, J_j)\) Kähler manifolds, \((j = 1, 2)\).
On \(M_1 \times M_2\) there are 2 non-equivalent g.K. structures:
Products of Kähler manifolds

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A generalization of a theorem of Apostolov-Gualtieri

Products of Kähler manifolds

Let \((M_j, g_j, J_j)\) Kähler manifolds, \((j = 1, 2)\). On \(M_1 \times M_2\) there are 2 non-equivalent g.K. structures:

- the **first product** is just the Kähler product structure,
- the **second product**: \(L_1 = L(T^{1,0}M_1 \times TM_2, i\omega_2)\) and \(L_2 = L(T^{1,0}M_2 \times TM_1, i\omega_1)\)
Let \((M_j, g_j, J_j)\) Kähler manifolds, \((j = 1, 2)\).

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Both \(L_1\) and \(L_2\) are in normal form; the corresponding almost \(f\)-structures are skew-adjoint (and, thus, unique with this property).
Products of Kähler manifolds

Let $(M_j, g_j, J_j)$ Kähler manifolds, $(j = 1, 2)$. On $M_1 \times M_2$ there are 2 non-equivalent g.K. structures:

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From the geometric properties of the distributions we get:
A generalization of a theorem of Apostolov-Gualtieri

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From the geometric properties of the distributions we get:

**Theorem**

Any generalized Kähler manifold with \( \nu = 0 \) (*i.e.* \( [J_+, J_-] = 0 \)) is, up to a unique \( B \)-field transformation, locally given by the second product of two Kähler manifolds.

In particular, \( h = db = 0 \).
Generalized Kähler manifolds with $\mathcal{H}^- = 0$. 

Tamed symplectic manifolds 

$(M, \varepsilon, J)$ s.t. $\varepsilon(JX, X) > 0$, $J$ and $\varepsilon^{-1}J^*\varepsilon$ integrable, $d\varepsilon = 0.$
Generalized Kähler manifolds with $\mathcal{H}^- = 0$.

**Tamed symplectic manifolds**

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**Structure theorem (see also Gualtieri ’07)**

Let $\varepsilon$ be non-degenerate on $M$ and $J$ almost complex structure. Let $J_+ = J$, $J_- = -\varepsilon^{-1}J^*\varepsilon$. Let $g$, $b$ be the symmetric and skew-symmetric parts of $\varepsilon J$. 

Let $L_1 = L(TM_C, 2\omega_I - i(\omega_J - \omega_K))$, $L_2 = L(TM_C, -i(\omega_J + \omega_K))$. 

**Generalized Kähler manifolds with $\mathcal{H}^- = 0$.**

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$(M, \varepsilon, J)$ is tamed symplectic $\iff (g, b, J_+, J_-)$ is g.K. with $J_+ + J_-$ invertible.
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Up to a unique $B$- transform, any g.K. structure with $J_+ + J_- \text{ invertible}$ is of this kind.

Example (cf. Hitchin '06)

Let $(M, g, I, J, K)$ be hyperkähler and $\varepsilon := - (\omega_I + \omega_J)$. Then $(M, \varepsilon, J)$ is tamed symplectic with associated g.K. structure $(g, b, J_+, J_-) = (g, \omega_I, J, K)$. Here, $L_1 = L(TM^\mathbb{C}, 2\omega_I - i(\omega_J - \omega_K))$, $L_2 = L(TM^\mathbb{C}, -i(\omega_J + \omega_K))$. 
Generalized Kähler manifolds with $\mathcal{H}^- = 0$.

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Tamed symplectic manifolds

$(M, \varepsilon, J)$ s.t. $\varepsilon(JX, X) > 0$, $J$ and $\varepsilon^{-1}J^*\varepsilon$ integrable, $d\varepsilon = 0$.

Structure theorem (see also Gualtieri ’07)

Let $\varepsilon$ be non-degenerate on $M$ and $J$ almost complex structure.
Let $J_+ = J$, $J_- = -\varepsilon^{-1}J^*\varepsilon$. Let $g$, $b$ be the symmetric and skew-symmetric parts of $\varepsilon J$.
$(M, \varepsilon, J)$ is tamed symplectic $\iff (g, b, J_+, J_-)$ is g.K. with $J_+ + J_-$ invertible.
Up to a unique $B$- transform, any g.K. structure with $J_+ + J_-$ invertible is of this kind.

Example (cf. Hitchin ’06)

Let $(M, g, I, J, K)$ be hyperkähler and $\varepsilon := -(\omega_I + \omega_J)$.
Then $(M, \varepsilon, J)$ is tamed symplectic with associated g.K. structure $(g, b, J_+, J_-) = (g, \omega_I, J, K)$. 

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Here, $L_1 = L(TM^\mathbb{C}, 2\omega_I - i(\omega_J - \omega_K))$, $L_2 = L(TM^\mathbb{C}, -i(\omega_J + \omega_K))$. 
A g.K. manifold with $\mathcal{H}^+ \text{ integrable and } \mathcal{H}^- = 0$ is, up to a unique $B$-transform, locally a product $(M \times N, L_1^M \times L_1^N, L_2^M \times L_2^N)$ where $(L_1^M, L_2^M)$ comes from a Kähler structure on $M$ and $(L_1^N, L_2^N)$ is a g.K. structure on $N$ with $J_+ + J_-$ and $J_+ - J_-$ invertible.
Induced holomorphic Poisson structure (cf. Hitchin ’06)

For a g.K. \((M, L_1, L_2)\) coming from a tamed symplectic structure, let \(\rho^\pm : TM^\mathbb{C} \to T_{\pm}^{1,0} M\). Then \(\rho^\pm_*(L_2)\) is a holomorphic Poisson structure on \((M, J_{\pm})\).
Generalized Kähler manifolds with $\mathcal{H}^- = 0$.

**Induced holomorphic Poisson structure (cf. Hitchin ’06)**

For a g.K. $(M, L_1, L_2)$ coming from a tamed symplectic structure, let $\rho^\pm : TM^\mathbb{C} \to T^{1,0}_\pm M$. Then $\rho^\pm_*(L_2)$ is a holomorphic Poisson structure on $(M, J^\pm)$. The converse holds only if $J^+ - J^-$ is invertible. In this case, $\rho^\pm_*(L_2)$ are holomorphic symplectic structures.
Generalized Kähler manifolds with $\mathcal{H}^- = 0$.

**Induced holomorphic Poisson structure (cf. Hitchin ’06)**

For a g.K. $(M, L_1, L_2)$ coming from a tamed symplectic structure, let $\rho^\pm : TM^C \to T^{1,0}_{\pm}M$. Then $\rho^\pm_* (L_2)$ is a holomorphic Poisson structure on $(M, J_\pm)$.

The converse holds only if $J_+ - J_- \text{ is invertible. In this case, } \rho^\pm_* (L_2) \text{ are holomorphic symplectic structures.}$

The associated Poisson bivectors on $(M, J_\pm)$ are

$$\eta_- = -\eta_+ = \frac{1}{4} [J_+, J_-] g^{-1}.$$

The symplectic foliation associated to $\eta_+$ is precisely $\mathcal{V}$. 
Holomorphic maps between generalized Kähler manifolds with $\mathcal{H}^{-} = 0$.

**Induced holomorphic Poisson morphism**

Let $(M, L_{1}^{M}, L_{2}^{M})$ and $(N, L_{1}^{N}, L_{2}^{N})$ be generalized Kähler manifolds, with $J_{+}^{M} + J_{-}^{M}$ and $J_{+}^{N} + J_{-}^{N}$ invertible, and let $\varphi : M \rightarrow N$ be a map.
Holomorphic maps between generalized Kähler manifolds with $\mathcal{H}^- = 0$.

**Induced holomorphic Poisson morphism**

Let $(M, L^M_1, L^M_2)$ and $(N, L^N_1, L^N_2)$ be generalized Kähler manifolds, with $J^M_+ + J^M_-$ and $J^N_+ + J^N_-$ invertible, and let $\varphi : M \to N$ be a map.

If

- $\varphi : (M, L^M_2) \to (N, L^N_2)$ is holomorphic and,
- at least one of $\varphi : (M, J^M_+) \to (N, J^N_+)$ and $\varphi : (M, J^M_-) \to (N, J^N_-)$ is holomorphic,

then $\varphi$ is a holomorphic Poisson morphism between the associated holomorphic Poisson structures.