von Neumann's Stability Analysis

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1 Hyperbolic Systems

Let us consider a general second order partial differential equation of hyperbolic type:

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0. \tag{1}$$

Formal solutions for the above equation are given by

$$f = f_1(x - ct) + f_2(x + ct). \tag{2}$$

It is easy to understand the solution by noting that

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) f = 0. \tag{3}$$

Namely, the evolution of a physical system governed by equation (1) is also discribed by the solution for the following two *first-order* equations

$$\frac{\partial f}{\partial t} - c\frac{\partial f}{\partial x} = 0, \tag{4}$$

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = 0. \tag{5}$$

2 Difference schemes and the stability analysis

We'd like to solve the above hyperbolic equations by means of a difference method, i.e., by discretizing the equations. One can easily notice that equation (2) represent two propagating waves; one in forward direction with velocity c, the other in backward direction with velocity $-c$. Essentially, it is sufficient to study the stability for the forward case. The other case can be treated in exactly the same manner by changing c to $-c$[1] .

We have learned that there are three simple ways to discretize the equation

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = 0. \tag{6}$$

[1] This fact should be kept in mind. In this section, we use forward and backward difference schemes, but a forward scheme can be a *backward* one when the sign of c is changed.

Because we consider an evolving system, let us adopt a forward scheme $[f(x, t+\Delta t) - f(x, t)]/\Delta t$ for the time derivative. For the spatial derivative, the forward scheme is

$$\frac{f(x, t+\Delta t) - f(x, t)}{\Delta t} + c\,\frac{f(x+\Delta x, t) - f(x, t)}{\Delta x} = 0. \tag{7}$$

The backward scheme is, simply,

$$\frac{f(x, t+\Delta t) - f(x, t)}{\Delta t} + c\,\frac{f(x, t) - f(x-\Delta x, t)}{\Delta x} = 0. \tag{8}$$

One can come up with yet another way of evaluating the spatial derivative as the mean of the above two. Namely, the central difference gives

$$\frac{f(x, t+\Delta t) - f(x, t)}{\Delta t} + c\,\frac{f(x+\Delta x, t) - f(x-\Delta x, t)}{2\Delta x} = 0. \tag{9}$$

It would be a useful exercise for you to derive that the forward and backward schemes are of the first order, that is that the truncation error associated with the spatial derivative is $O(\Delta x^2)$, whereas the central scheme is of the second-order. Apparently, the central scheme is a better method. Nevertheless... see below.

As usual, we'd like to know how the solution develops if we set the initial condition to be a plane parallel wave:

$$f_{\rm init} = \exp(\frac{ikx}{\Delta x}). \tag{10}$$

Let us set our computational domain and the grid points such that the j-th grid has the x-coordinate of $j\Delta x$, where j runs from 1 to N. Then the initial condition is

$$f_j^0 = \exp(ikj). \tag{11}$$

Here after we use the notation $f_j^n = f(j\Delta x, n\Delta t)$.

It'd be best (if not ideal) if numerical integration yields the solution

$$f_j^n = \exp\left[ik(j - \frac{cn\Delta t}{\Delta x})\right], \tag{12}$$

because we know the formal analytic solution is $f(x - ct)$. You can see this easily by setting the time step width as $\Delta t = \Delta x/c$. Then the value of f at $j-1$-th grid is simply transferred to the next j-th grid over a period of Δt, at the correct velocity c.

Now, let us examine if a good solution is obtained by any of the three difference methods. The solution for the forward scheme is obtained to be

$$f_j^n = \left\{1 - \frac{c\Delta t}{\Delta x}[\exp(ik) - 1]\right\}^n \exp(ikj). \tag{13}$$

It is non-trivial to derive this solution, but it is indeed straightforward to check whether this solution satisfies equation (7). Similarly, the solution for the backward and the central schemes are, respectively,

$$f_j^n = \left\{1 - \frac{c\Delta t}{\Delta x}[1 - \exp(-ik)]\right\}^n \exp(ikj), \tag{14}$$

and

$$f_j^n = \left\{1 - i\frac{c\Delta t}{\Delta x}\sin(k)\right\}^n \exp(ikj). \quad (15)$$

The first thing to notice is that all the three solutions asymptotically become $\sim \exp[ik(j - nc\Delta t/\Delta x)]$ for small k when $\Delta t \to 0$, because the front factors $\{...\}$ then approach $\{1 - ik\frac{c\Delta t}{\Delta x}\}$. Long-wavelength modes are accurately represented at least initially.

Let us now compare the three solutions with the true solution (equation [12]) and re-write them as

$$f_j^n = (g_{\rm F})^n f_{\rm true}, \quad (16)$$
$$f_j^n = (g_{\rm B})^n f_{\rm true}, \quad (17)$$
$$f_j^n = (g_{\rm C})^n f_{\rm true}. \quad (18)$$

The error produced per one time step is now given (estimated) by the norm of $g_{\rm F}, g_{\rm B}, g_{\rm C}$, respectively. By now you'd have guessed that the norm should be less than unity for the solution to be stable. You are right. Why don't you check it explicitly:

$$g_{\rm F} = \{1 - \alpha[\exp(ik) - 1]\}\exp(ik\alpha), \quad (19)$$
$$g_{\rm B} = \{1 - \alpha[1 - \exp(-ik)]\}\exp(ik\alpha), \quad (20)$$
$$g_{\rm C} = \{1 - \alpha[1 - i\alpha\sin(k)]\}\exp(ik\alpha), \quad (21)$$

where we have denoted the so-called Courant factor as $\alpha = c\Delta t/\Delta x$. It is nice that the error per timestep is given by a function of the dimensionless factor α for an arbitrary wavenumber k. The remaining task is straightforward, although somewhat tedious. The actual calculations are left as your exercise:

$$\begin{aligned} |g_{\rm F}| = & |(1 + \alpha - \alpha\cos(k))\cos(\alpha k) + \alpha\sin(k)\sin(\alpha k) \\ & + i\{(1 + \alpha - \alpha\cos(k))\sin(\alpha k) - \alpha\sin(k)\cos(\alpha k)\}|. \end{aligned} \quad (22)$$

See Figure 1, which shows $|g_{\rm F}|$ for three ks as a function of α. It'd be nice to get a fresh surprise by seeing that $|g_{\rm F}|$ is always larger than unity! Such a scheme produces unstable solutions. (You might have seen this already when you solved the Burgers equation with the forward difference scheme.) What about the backward scheme ? After similar algebra, we obtain Figure 2, which shows that $|g_{\rm B}|$ is less than unity for $0 < \alpha < 1$. This is the necessary condition (not sufficient though) for the backward scheme to yield a stable solution, with respect to the true solution.

3 The case for diffusion equations

We can easily extend the discussion in the previous section to the stability of the central difference scheme for the diffusion equation. Consider forward in time, and central in space:

$$\frac{f(x,t+\Delta t)-f(x,t)}{\Delta t}=D\,\frac{f(x+\Delta x,t)-2f(x,t)+f(x-\Delta x,t)}{\Delta x^2}, \quad (23)$$

or, with our simpler notations,

$$f_j^{n+1}=f_j^n+\alpha(f_{j-1}^n-2f_j^n+f_{j+1}^n), \quad (24)$$

where $\alpha = D\Delta t/\Delta x^2$.

Let us examine the stability for the initial condition

$$f_j^0=\exp(ikj). \quad (25)$$

One time step advancement yields

$$\begin{aligned} f_j^1 &= f_j^0+\alpha[\exp(ik(j-1))-2\exp(ikj)+\exp(ik(j+1))] \\ &= \exp(ikj)+\alpha[\exp(-ik)-2+\exp(ik)]\exp(ikj) \\ &= [1+\alpha(2\cos(k)-2)]\exp(ikj) \\ &= \left[1-4\alpha\sin^2\frac{k}{2}\right]\exp(ikj) \end{aligned} \quad (26)$$

Notice that $\sin^2$ is an oscillating function in [0,1], and thus the solution is stable only if $\alpha < 1/2$. It is now clear for you why, in the last week's problem set, you saw a catastrophic failure for large Δt.