Option Pricing

Lecture note

1 Some definitions

The normal distribution with mean μ and variance σ^2 is given by

$$N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{1}$$

For example, $N(0,1) = 1/\sqrt{2\pi}e^{-x^2/2}$.

2 Wiener process

If a time variate $Z(t)$ is described by the Wiener process, its small variation ΔZ over Δt is distributed as $N(0, (\sqrt{\Delta t})^2)$.

The generalized Wiener process is characterized by the variation of a time variate $X(t)$:

$$\Delta X = a\Delta t + b\Delta Z, \tag{2}$$

which follows the normal distribution $N(a\Delta t, (b\sqrt{\Delta t})^2)$.

3 Ito process

As a further generalization, one can think of a and b both as a function of X and t. It is called the Ito process, which is characterized by

$$\Delta X = a(X,t)\Delta t + b(X,t)\Delta Z. \tag{3}$$

Ito's lemma:
For the Ito process (3), the increment of a scalar function $f(X,t)$ is given by

$$\mathrm{d}f = \left(\frac{\partial f}{\partial X}a(X,t) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}b^2(X,t)\right)\mathrm{d}t + \frac{\partial f}{\partial X}\,b(X,t)\,\mathrm{d}Z. \tag{4}$$

The rigorous proof is a little involved, but one can get the essence by recalling the Taylor expansion of f:

$$\begin{aligned}\Delta f &= \frac{\partial f}{\partial X}\Delta X + \frac{\partial f}{\partial t}\Delta t \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(\Delta X)^2 + \frac{\partial^2 f}{\partial X \partial t}\Delta X\,\Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\Delta t)^2 + ...\end{aligned} \tag{5}$$

Let us retain only first-order terms. Then we might think that all the terms in the second line in equation (5) are to be dropped. This is not the case, however, because ΔX follows the Ito process (3). By noting that ΔZ follows $N(0, (\sqrt{\Delta t})^2)$, we obtain [1] $(\Delta Z)^2 = \Delta t$ in the limit of $\Delta t \to 0$, i.e.,

$$\begin{aligned}(\Delta X)^2 &= a^2(X,t)(\Delta t)^2 + a(X,t)b(X,t)\Delta Z\Delta t + b^2(X,t)(\Delta Z)^2 \\ &= a^2(X,t)(\Delta t)^2 + a(X,t)b(X,t)\Delta Z\Delta t + b^2(X,t)\Delta t. \qquad (6)\end{aligned}$$

The last term is of first-order, which should be retained. We have then equation (4).

4 The Black-Scholes model

Think about the time variation of a stock price. It'd show probably stochastic variation around mean, which drifts over a long period. Such a process may be described by

$$\mathrm{d}S = S\mu\mathrm{d}t + S\sigma\mathrm{d}Z. \qquad (7)$$

Namely $\ln S$ follows the Ito process. Ito's lemma yields, for a scalar function $f(t,S)$,

$$\mathrm{d}f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\mathrm{d}t + \frac{\partial f}{\partial S}\,\sigma\, S\mathrm{d}Z. \qquad (8)$$

The following might seem purely technical, which however is thought to be a basic type of port-folio in finance. Compare the last terms of equation (7) and (8). Notice that, by multiplying equation (7) by $\partial f/\partial S$, we obtain

$$\frac{\partial f}{\partial S}\,\sigma\, S\mathrm{d}Z = \frac{\partial f}{\partial S}\mathrm{d}S - \frac{\partial f}{\partial S}\,\mu\, S\mathrm{d}t. \qquad (9)$$

Substituting this into equation (8), we obtain

$$\frac{\partial f}{\partial S}\mathrm{d}S - \mathrm{d}f = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\mathrm{d}t. \qquad (10)$$

Apparently the uncertainty (risk) associated with $\mathrm{d}Z$ is not included in this expression. Now, if the left-hand side is not stochastic, we have

$$\frac{\partial f}{\partial S}\mathrm{d}S - \mathrm{d}f = r\left(\frac{\partial f}{\partial S}S - f\right)\mathrm{d}t. \qquad (11)$$

This is because, in the absence of risk $\mathrm{d}Z$, both the stock price S and the value f should increase as $\propto rS\mathrm{d}t$ and $\propto rf\mathrm{d}t$ with r being the risk-free interest rate.

[1] To be more precise, this is because $\mathrm{d}f$ is a stochastic integral quantity.

5 Port-folio

The argument in the last section can be explained as a practical mean of risk-hedge with a port-folio. Suppose you buy, at any time, $\partial f/\partial S$ units of a stock with price S and sell one unit of a derivative of value f. Then the value of this port-folio is

$$\frac{\partial f}{\partial S} S - f. \tag{12}$$

Clearly the value is not affected by the stochastic process $\mathrm{d}Z$. The small variation of this over Δt is then

$$\frac{\partial f}{\partial S} rS\Delta t - rf\Delta t. \tag{13}$$

From equation (10), we finally obtain the Black-Scholes equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} = rf, \tag{14}$$

which governs the evolution of the option value f.