1 The Friedmann equation for a flat universe

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\Omega_\mathrm{m}}{a^3} + \Omega_\Lambda\right] \tag{1}$$

where $\Omega_\mathrm{m}, \Omega_\Lambda$ are the present-day matter density and the density of dark energy (cosmological constant). Formal intergration yields

$$H_0 t = \int_0^a \frac{\sqrt{a}\,\mathrm{d}a}{\sqrt{\Omega_\mathrm{m} + \Omega_\Lambda a^3}}. \tag{2}$$

By changing the variable to $\chi^2 = a^3 + c$ where $c = \sqrt{\Omega_\mathrm{m}/\Omega_\Lambda}$, i.e., $2\chi d\chi = 3a^2 da$, we obtain

$$\int_0^a \frac{\sqrt{a}\,\mathrm{d}a}{\sqrt{\Omega_\mathrm{m} + \Omega_\Lambda a^3}} = \frac{1}{\sqrt{\Omega_\Lambda}}\int_{\sqrt{c}}^{\sqrt{a^3+c}} \frac{\sqrt{a}}{\chi}\frac{2\chi}{3a^2}d\chi = \frac{1}{\sqrt{\Omega_\Lambda}}\int_{\sqrt{c}}^{\sqrt{a^3+c}} \frac{2}{3}a^{-3/2}d\chi \tag{3}$$

From $a^3 = \chi^2 - c$, the integrand $a^{-3/2}$ is simply $1/\sqrt{\chi^2 - c}$. Then the above integral becomes

$$\frac{2}{3\sqrt{\Omega_\Lambda}}\int_{\sqrt{c}}^{\sqrt{a^3+c}} \frac{1}{\sqrt{\chi^2 - c}}\,d\chi. \tag{4}$$

Recall the integral of $1/\sqrt{x^2-1}$ is some combination of hyperbolic functions. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}. \tag{5}$$

With slight rescaling of $\sqrt{c}\,x = \chi$, the integration (4) is now explicity done to be

$$\frac{2}{3\sqrt{\Omega_\Lambda}}\int_1^{\sqrt{a^3/c+1}} \frac{1}{\sqrt{x^2-1}}\,dx = \frac{2}{3\sqrt{\Omega_\Lambda}}\left[\cosh^{-1} x\right]_1^{\sqrt{a^3/c+1}} \tag{6}$$

Equation (2) then reads

$$\frac{3\sqrt{\Omega_\Lambda}}{2}H_0 t = \cosh^{-1}\sqrt{a^3/c+1}, \tag{7}$$

which can be inverted to be

$$a^3/c + 1 = \cosh^2\frac{3\sqrt{\Omega_\Lambda}}{2}H_0 t. \tag{8}$$

Using $\cosh^2 x - \sinh^2 x = 1$, we finally obtain

$$a = \left(\frac{\Omega_\mathrm{m}}{\Omega_\Lambda}\right)^{1/3}\sinh^{2/3}\frac{3\sqrt{\Omega_\Lambda}}{2}H_0 t, \tag{9}$$

which has assymptotic behaviors of $a \propto t^{2/3}$ for small a and $a \propto \exp(H_0 t)$ for large a.