Basic Numerical Analysis

October 30, 2012

Problem Set 4

1. A warm-up.
Write a simple program that solves the following equations for x_i:

$$\begin{bmatrix} 5 & 2 & 2 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 8 & 0 \\ 0 & 0 & 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}. \tag{1}$$

Use Jacobi's method and Gauss-Seidel method. The true solutions are $(-3/23, 6/23, 13/23, 25/92, 21/115)$. Report how many times of iterations your program needed to get sufficiently converged solutions.

2. Apply Crank-Nicholson scheme to solve the 1D diffusion equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{2}$$

subject to the boundary conditions

$$\frac{\partial T}{\partial x} = 0 \text{ at } x = \pm L, \tag{3}$$

where the system is evolved on the domain $-L < x < L$. Set L to be sufficiently large. As a simplest case, set the initial condition $T = \exp(-x^2/4)$ at $t = 1$. First write down a matric equation for $T(i, t = 1 + \Delta t)$, where i runs the spatial coordinate. Try to use Gauss-Seidel to solve the matrix equation.

3. Burgers equation. Write a C program that advances the Burgers equation

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = 0, \tag{4}$$

for the initial conditions

$$f(t=0) = \begin{cases} 1 & (x \leq -1) \\ -x & (-1 < x < 0) \\ 0 & (x \geq 0) \end{cases} \tag{5}$$

Try a forward, a backward, and a central difference scheme.
The analytic solutions is

$$\begin{aligned} f(x,t) &= 1 \;\; (x \leq -1+t) \\ & \quad -\frac{x}{1-t} \;\; (-1+t < x < 0) \\ & \quad 0 \;\; (x \geq 0). \end{aligned} \tag{6}$$

Does your program give an accurate numerical solution ?