Option Pricing

Due date: January 31, 2013

European Call option

A call option is a basic type of financial derivatives. One can buy the right (or a contract) to buy a stock at a fixed price at a certain time in the future. Suppose that you want to buy a stock A for $X = 100{,}000$ JPY on January 31, 2013. The actual price of the stock on the day is not known yet, and would indeed be highly variable. But you'd perhaps 'guess' the expected price and then you may want to buy A at a lower price if possible. If the price $S_{1/31}$ turns out to be higher than what you actually pay $(=X)$, then you will get a profit of $S_{1/31} - X$ JPY. Now the question is, what is the *value* of this option *today*, say, on December 11th, 2012 ?

Black-Scholes model

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel (often called Nobel Prize in Economics) in 1997 was awarded to Myron Scholes, who, together with late Fischer Black, developed a pioneering formula for the valuation of stock options.

The Black-Scholes model posits that the value of an option (called "premium") is determined by a second order stochastic partial differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} = rf \quad \text{for } t < T. \tag{1}$$

Here, $f = f(S,t)$ is the value of the option, S is the price of the stock at time t. An European Call option sets the boundary condition

$$f(S_{\mathrm{T}}, t = T) = \max\{S_{\mathrm{T}} - X, 0\}. \tag{2}$$

The condition can be easily understood by noting that, at time T in the future, you'll have the right to buy the stock at a pre-fixed price X, say, 100,000 JPY. If the actual price S_{T} is 120,000 JPY, higher than X, then effectively you get a profit of $S_{\mathrm{T}} - X = 20{,}000$ JPY per stock.

The above equation is specified with two constants; σ is called volatility and r is called risk-free interest rate. The former is perhaps easier for physics student if it were called "dispersion". Namely, σ is the overall dispersion of the (uncertain) stock price S if it is distributed like a Gaussian. The overall trend of S is characterized by the rate r (see Problem 2 next page).

Problem 1

Write a C program that solves equation (1), subject to boundary condition (2). You might have noticed that equation (1) consists of a time derivative, an advection term (first derivative w.r.t. S), a diffusion term (second derivative), and a source term rf. Note that you need to treat equation (1) essentially in a time-backward manner, i.e., f evolves starting from the future time T back to the present t.

Consider the following example: the exercize price X = 95,000 JPY, volatility $\sigma = 0.1$, the risk-free interest rate $r = 0.03$, and set the time $T =$ 0.5 years from now (time is in units of year, by convention).
What is the value of $f(S = 100,000\mathrm{JPY}, t - T = -0.5)$?

Problem 2

It'd simplify equation (1) if we adopt the following transformation $(S, t) \rightarrow (u, x)$ such that

$$\begin{aligned} u &= \ln\frac{S}{X} + (r - \frac{\sigma^2}{2})(T - t) && (3) \\ x &= T - t. && (4) \end{aligned}$$

Show that, by assuming $f = e^{-rx}y(u, x)$, equation (1) reduces to

$$\frac{\partial y}{\partial x} = \frac{\sigma^2}{2}\frac{\partial^2 y}{\partial u^2}, \tag{5}$$

which is nothing but an ordinary diffusion equation that you are familiar with! Derive the corresponding boundary conditions for y and solve eq. (5).

Problem 3

The true *value* of the work by Black and Scholes is that they derived a formal analytic solution for equation (1). The solution is given by

$$f(S, t) = SN(d_1) - Xe^{-r(T-t)}N(d_2), \tag{6}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,\mathrm{d}y, \tag{7}$$

$$d_1 = \frac{\ln\frac{S}{X} + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}, \tag{8}$$

and

$$d_2 = \frac{\ln\frac{S}{X} + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}. \tag{9}$$

Surely you can evaluate $N(d_1)$ and $N(d_2)$ using numerical integration. Compare the solution (6) with the result of your calculation in Problem 1.