Linear Velocity Field

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1 Linear potential evolution

In comoving coordinate, the Poisson equation is written as

$$\nabla^2_{\mathbf{x}}\Phi(\mathbf{x},t) = 4\pi G\,\bar{\rho}\,a^2\,\delta(\mathbf{x},t). \tag{1}$$

The equation holds (of course!) at an initial epoch $t = t_i$:

$$\nabla^2_{\mathbf{x}}\Phi_i(\mathbf{x}) = 4\pi G\,\bar{\rho}_i\,a_i^2\,\delta_i(\mathbf{x}). \tag{2}$$

Note that we have explicitly denoted $\nabla^2_{\mathbf{x}}$, differentiation with respect to the comoving coordinate x. By noting $\bar{\rho}a^3 = \bar{\rho}_i a_i^3$ for matter, and that the linear evolution of density perturbations is given by

$$\delta(\mathbf{x},a) = D(a)\,\delta_i(\mathbf{x}), \tag{3}$$

where $D(a)$ is normalized such that $D(a_i) = 1$, we rewrite equation (1) as

$$\nabla^2_{\mathbf{x}}\Phi = 4\pi G\bar{\rho}_i\,\frac{a_i^3}{a^3}\,a^2\,\delta = 4\pi G\bar{\rho}_i a_i^2\,\frac{a_i}{a}\,D\,\delta_i = \frac{a_i}{a}\,D\,\nabla^2_{\mathbf{x}}\Phi_i. \tag{4}$$

The potential is simply related to the initial one:

$$a\Phi(\mathbf{x},t) = D\,a_i\Phi_i(\mathbf{x}). \tag{5}$$

2 The Zeldovich approximation

We use the linearized Euler equation to obtain the corresponding velocity field for a given density distribution in the early universe:

$$\dot{\mathbf{v}} + H\mathbf{v} + \frac{1}{a}\nabla\Phi = 0. \tag{6}$$

You can easily check that the solution is given by

$$\mathbf{v} = -\frac{a_i\nabla\Phi_i}{a}\int\frac{D}{a}\mathrm{d}t. \tag{7}$$

It is important to notice that the above solution can be written as

$$\mathbf{v} = f(t)\nabla\Phi_i. \tag{8}$$

That is that the velocity as a later time $a(t)$ is simply determined by the local gradient of the *initial* potential. In other words, in linear theory, a particle moves on a straight trajectory extrapolated from the initial velocity vector.