Some applications of Finsler geometry to topological lensing

Erasmo Caponio, Politecnico di Bari, Italy
Gravitational lens effect
Gravitational lens effect

Bending of light due to the presence of a massive object between the source and the observer
Gravitational lens effect

Bending of light due to the presence of a massive object between the source and the observer.

Image: NASA/ESA
Gravitational lens effect

Bending of light due to the presence of a massive object between the source and the observer

Possibility to see multiple images of the same source at the same instant of time

Image: NASA/ESA
Gravitational lens effect

Bending of light due to the presence of a massive object between the source and the observer

Possibility to see multiple images of the same source at the same instant of time

Image: NASA/ESA

observed for the first time by D. Walsh, B. Carswell, and R. Weymann in 1979
Topological lens
Topological lens

multiple images due to a non-trivial topology of the spacetime
Topological lens

multiple images due to a non-trivial topology of the spacetime

a possible mathematical description for a stationary spacetime in terms of Finsler geometry
Overview

Stationary spacetimes
Finsler metrics associated to a stationary spacetime
Fermat principle
Convexity
Asymptotic flatness
(Standard) Stationary spacetimes

\[ M = S \times \mathbb{R} \]
Stationary spacetimes

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\[ g = g_0 + 2\omega_0 dt - \Lambda dt^2 \]
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- \( g_0 \) Riemannian metric on \( S \)
- \( \omega_0 \) a one-form on \( S \)
$M = S \times \mathbb{R}$

$g = g_0 + 2\omega_0 dt - \Lambda dt^2$

- $g_0$ Riemannian metric on $S$
- $\omega_0$ a one-form on $S$
- $\Lambda > 0$ a function on $S$
$M = S \times \mathbb{R}$

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- $g_0$ Riemannian metric on $S$
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The metric $g$ is independent from the $t$ coordinate:
(Standard) Stationary spacetimes

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- \( \omega_0 \) a one-form on \( S \)
- \( \Lambda > 0 \) a function on \( S \)

The metric \( g \) is independent from the \( t \) coordinate:

\( \partial_t \) is a timelike Killing field for \( g \)
Randers metrics from light-cone bundle

\[ g = g_0 + 2\omega_0 dt - \Lambda dt^2 \]
Randers metrics from light-cone bundle

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- Randers metrics from light-cone bundle
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\((v, 1) \in T_{x_0} S \times \mathbb{R}\) is lightlike (and future pointing)
Randers metrics from light-cone bundle

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\[ g_0(v, v) + 2\omega_0(v) - \Lambda(x_0) = 0 \]
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\(\Sigma\quad 0.\quad \text{smooth, convex}\)

\(\text{NON symmetric!}\)
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\[ g_0(v, v) + 2\omega_0(v) - \Lambda(x_0) = 0 \]

\[ F = \frac{1}{\Lambda}(\omega_0 + \sqrt{\Lambda g_0 + \omega_0^2}) \]

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Randers metrics from light-cone bundle

$g = g_0 + 2\omega_0 dt - \Lambda dt^2$

$(v, 1) \in T_{x_0} S \times \mathbb{R}$ is lightlike (and future pointing)

$g_0(v, v) + 2\omega_0(v) - \Lambda(x_0) = 0$

$\Sigma$

smooth, convex
NON symmetric!

$F = \frac{1}{\Lambda} (\omega_0 + \sqrt{\Lambda g_0 + \omega_0^2})$
Randers metrics from light-cone bundle

\[ g = g_0 + 2\omega_0 dt - \Lambda dt^2 \]

\[(v, -1) \in T_{x_0} S \times \mathbb{R} \text{ is lightlike (and past pointing)}\]

\[ g_0(v, v) - 2\omega_0(v) - \Lambda(x_0) = 0 \]

\[ \tilde{F} = \frac{1}{\Lambda}(-\omega_0 + \sqrt{\Lambda g_0 + \omega_0^2}) \]

smooth, convex
NON symmetric!
Randers metrics from light-cone bundle

\[ g = g_0 + 2\omega_0 dt - \Lambda dt^2 \]

\[ h = \frac{g_0}{\Lambda} + \left(\frac{\omega_0}{\Lambda}\right)^2 \]
Randers metrics from light-cone bundle

\[ g = g_0 + 2\omega_0 dt - \Lambda dt^2 \]

\[ h = \frac{g_0}{\Lambda} + \left(\frac{\omega_0}{\Lambda}\right)^2 \]

notice that \( \|\omega_0\|_h < 1 \) whatever \( \omega_0 \) is
These metrics where called in


**Fermat metrics**
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**Fermat metrics**

\((v, F(v))\) is future-pointing and lightlike

\((v, -\tilde{F}(v))\) is past-pointing and lightlike
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**Fermat metrics**

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Fermat principle in General Relativity

Stationary spacetimes

Finsler metrics associated to a stationary spacetime

Fermat principle

Fermat principle in General Relativity

Fermat principle in stationary spacetimes

Fermat metrics for massive particles

Convexity

Asymptotic flatness

Light rays are the critical points of the arrival time functional
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Light rays are the critical points of the arrival time functional


\[ \gamma: (\alpha, \beta) \rightarrow M \]

\( \text{timelike, injective} \)

\[ \sigma: [a, b] \rightarrow M, \text{ lightlike,} \]

\[ \sigma(a) = p_0, \sigma(b) \in \gamma(\alpha, \beta) \]
Fermat principle in General Relativity

Light rays are the critical points of the arrival time functional


\[ \gamma: (\alpha, \beta) \rightarrow M \]
\[ \sigma: [a, b] \rightarrow M, \text{ lightlike}, \]
\[ \sigma(a) = p_0, \sigma(b) \in \gamma(\alpha, \beta) \]

\[ AT(\sigma) = \gamma^{-1}(\sigma(b)) \]
Let \( \gamma \) be an integral curve of \( \partial_t \) and

\[
\sigma(s) = (x(s), t(s))
\]

\[
AT(\sigma) = t(a) + \int_a^b \dot{t} ds
\]

\[
= t(a) + \int_x F(\dot{x}) = t(a) + \ell_F(x)
\]
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\begin{equation*}
= t(a) + \int_x F(\dot{x}) = t(a) + \ell_F(x)
\end{equation*}
\begin{equation*}
\sigma \text{ is a future-pointing (affinely parametrized) lightlike geodesics iff } x \text{ is a pre-geodesic of } (S, F)
\end{equation*}
\begin{equation*}
(\text{parametrized with } h(\dot{x}, \dot{x}) = \text{const.})
\end{equation*}
Fermat metrics for massive particles

\[(M, g)\]
Fermat metrics for massive particles

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\[(M, g) \rightarrow (M \times \mathbb{R}, g + ds^2)\]
Fermat metrics for massive particles

\[(M, g) \quad \longrightarrow \quad (M \times \mathbb{R}, g + ds^2)\]

\[\gamma, \quad g(\dot{\gamma}, \dot{\gamma}) = -1 \quad \iff \quad (\gamma, s), \text{ lightlike and } \dot{s} = 1\]
Fermat metrics for massive particles

(M, g) → (M × R, g + ds^2)

γ, g(γ, ˙γ) = −1 ⇔ (γ, s), lightlike and ˙s = 1

For a stationary space time

M = S × R
Fermat metrics for massive particles

\[(M, g) \quad \longrightarrow \quad (M \times \mathbb{R}, g + ds^2)\]

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For a stationary space time
\[M = S \times \mathbb{R} \quad \longrightarrow \quad (S \times \mathbb{R}) \times \mathbb{R}\]
Fermat metrics for massive particles

\((M, g) \rightarrow (M \times \mathbb{R}, g + ds^2)\)

\[\gamma, \quad g(\dot{\gamma}, \dot{\gamma}) = -1 \iff (\gamma, s), \text{ lightlike and } \dot{s} = 1\]

For a stationary space time

\(M = S \times \mathbb{R} \rightarrow (S \times \mathbb{R}) \times \mathbb{R}\)

\[F_\Lambda = \frac{1}{\Lambda} (\omega_0 + \sqrt{\Lambda (g_0 + ds^2) + \omega_0^2})\]

on \(S \times \mathbb{R}\)
Fermat metrics for massive particles

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For a stationary space time
\[ M = S \times \mathbb{R} \quad \longrightarrow \quad (S \times \mathbb{R}) \times \mathbb{R} \]

\[
\tilde{F}_\Lambda = \frac{1}{\Lambda}(-\omega_0 + \sqrt{\Lambda(g_0 + ds^2) + \omega_0^2})
\]
on \[ S \times \mathbb{R} \]
$D \subset S$, open subset with smooth ($C^2$) boundary $\partial D$
Convexity of a domain in a Finsler manifold

$D \subset S$, open subset with smooth ($C^2$) boundary $\partial D$

$D$ is said convex (w.r.t. $F$) if any two points $x_0, x_1 \in D$ are connected by at least one geodesic of $(S, F)$, $\gamma \subset D$, $\gamma$ having minimum length among all the curves in $D$.
Convexity of a domain in a Finsler manifold

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Hessian of a function on a Finsler manifold $(S, F)$: let $(x, y) \in TS$ then for any $v \in T_x S$
Convexity of a domain in a Finsler manifold

\( D \subset S \), open subset with smooth \((C^2)\) boundary \( \partial D \)

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Hessian of a function on a Finsler manifold \((S, F)\): let \((x, y) \in TS\) then for any \( v \in T_xS \)

\[
\text{Hess}_F^\varphi(x, y)[v, v] = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x)v^j v^i - \frac{\partial \varphi}{\partial x^k}(x) \Gamma^k_{ij}(x, y)v^i v^j
\]
Convexity of a domain in a Finsler manifold

\( D \subset S \), open subset with smooth \( (C^2) \) boundary \( \partial D \)

\( D \) is said convex (w.r.t. \( F \)) if any two point \( x_0, x_1 \in D \) are connected by at least one geodesic of \( (S, F) \), \( \gamma \subset D \), \( \gamma \) having minimum length among all the curves in \( D \)

\[ \varphi > 0 \quad \partial D \]

\[ \varphi < 0 \]

Hessian of a function on a Finsler manifold \( (S, F) \): let \( (x, y) \in TS \) then for any \( v \in T_xS \)

\[ \text{Hess}_\varphi^F(x, y)[v, v] = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x)v^jv^j - \frac{\partial \varphi}{\partial x^k}(x)\Gamma^k_{ij}(x, y)v^iv^j \]

component of the Chern connection of \( F \)
Convexity of a domain in a Finsler manifold

$D \subset S$, open subset with smooth ($C^2$) boundary $\partial D$

$D$ is said convex (w.r.t. $F$) if any two points $x_0, x_1 \in D$ are connected by at least one geodesic of $(S, F)$, $\gamma \subset D$, $\gamma$ having minimum length among all the curves in $D$

$\partial D$ is $F$-convex if

\[
\text{Hess}^F_{\varphi}(x, v)[v, v] = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x)v^j v^j - \frac{\partial \varphi}{\partial x^k}(x)\Gamma^k_{ij}(x, v)v^i v^j \leq 0
\]

for all $x \in \partial D$ and $v \in T_x(\partial D)$
When $F$ is the Fermat metric of a standard stationary spacetime.
When $F$ is the Fermat metric of a standard stationary spacetime

\[ \downarrow \]

light-convexity of $(\partial D) \times \mathbb{R}$
When $F$ is the Fermat metric of a standard stationary spacetime

\[\text{light-convexity of } (\partial D) \times \mathbb{R}\]

When $F$ is the metric $F_\Lambda$ of a standard stationary spacetime

\[\text{time-convexity of } (\partial D) \times \mathbb{R}\]
$\partial D$ is $F$-convex if

$$\text{Hess}^F_\varphi(x, v)[v, v] = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x) v^j v^j - \frac{\partial \varphi}{\partial x^k}(x) \Gamma^k_{ij}(x, v) v^i v^j \leq 0$$

for all $x \in \partial D$ and $v \in T_x(\partial D)$
\[ \partial D \text{ is } F\text{-convex if} \]

\[
\text{Hess}^F_\varphi(x, v)[v, v] = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x)v^i v^j - \frac{\partial \varphi}{\partial x^k}(x)\Gamma^k_{ij}(x, v)v^i v^j \leq 0
\]

for all \( x \in \partial D \text{ and } v \in T_x(\partial D) \)

\[
\Updownarrow
\]

\[
\text{Hess}^h_\varphi(x, v)[v, v] + \sqrt{h(v, v)}|d\omega(v, \nabla^h\varphi)| \leq 0
\]

for all \( x \in \partial D \text{ and } v \in T_x(\partial D) \)

\[\omega : = \omega_0/\Lambda\]
The boundary $\partial D$ is $F$-convex if

$$\text{Hess}_F^F(x, v)[v, v] = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x) v^j v^j - \frac{\partial \varphi}{\partial x^k}(x) \Gamma^k_{ij}(x, v) v^i v^j \leq 0$$

for all $x \in \partial D$ and $v \in T_x(\partial D)$.

\[\uparrow\]

$$\text{Hess}^h_\varphi(x, v)[v, v] + \sqrt{h(v, v)}|d\omega(v, \nabla^h \varphi)| \leq 0$$

for all $x \in \partial D$ and $v \in T_x(\partial D)$.

$$\omega : = \omega_0 / \Lambda$$

(In particular $\partial D$ $F$-convex $\Rightarrow$ $\partial D$ $h$-convex.)
∂D is $F$-convex if

$$\text{Hess}_\varphi^F(x, v)[v, v] = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x)v^j v^j - \frac{\partial \varphi}{\partial x^k}(x) \Gamma^k_{ij}(x, v)v^i v^j \leq 0$$

for all $x \in \partial D$ and $v \in T_x(\partial D)$

$$\upuparrows$$

$$\text{Hess}_\varphi^h(x, v)[v, v] + \sqrt{h(v, v)}|d\omega(v, \nabla^h \varphi)| \leq 0$$

for all $x \in \partial D$ and $v \in T_x(\partial D)$

$$\omega: = \omega_0/\Lambda$$

(the same for $\tilde{F}$: there is no forward or backward convexity!)
\[ \partial D \] is \( F \)-convex if

\[
\text{Hess}_F^\varphi(x, v)[v, v] = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x)v^j v^j - \frac{\partial \varphi}{\partial x^k}(x)\Gamma^k_{ij}(x, v)v^i v^j \leq 0
\]

for all \( x \in \partial D \) and \( v \in T_x(\partial D) \)

\[
\iff
\]

\[
\text{Hess}_h^\varphi(x, v)[v, v] + \sqrt{h(v, v)}|d\omega(v, \nabla_h^\varphi)| \leq 0
\]

for all \( x \in \partial D \) and \( v \in T_x(\partial D) \)

\[
\omega \colon = \omega_0/\Lambda
\]

...and for a $F_\Lambda$ (or $\tilde{F}_\Lambda$) metric

$$(\partial D) \times \mathbb{R} \text{ is } F_\Lambda\text{-convex}$$
...and for a $F_\Lambda$ (or $\tilde{F}_\Lambda$) metric

$$(\partial D) \times \mathbb{R} \text{ is } F_\Lambda\text{-convex}$$

$\uparrow$

$\partial D$ is $F$-convex, $\nabla^h \Lambda$ does not point outside $D$, i.e. $h(\nabla^h \varphi, \nabla^h \Lambda) \geq 0$ and
...and for a $F_\Lambda$ (or $\tilde{F}_\Lambda$) metric

$$(\partial D) \times \mathbb{R} \text{ is } F_\Lambda\text{-convex}$$

$\uparrow$

$\partial D$ is $F$-convex, $\nabla^h \Lambda$ does not point outside $D$, i.e.

$$h(\nabla^h \varphi, \nabla^h \Lambda) \geq 0$$

and for each $y \in T\partial D$, either

$$d\omega(y, \nabla^h \varphi)^2 + \frac{h(\nabla^h \varphi, \nabla^h \Lambda)}{\Lambda} H^h_\varphi(y, y) \leq 0$$

or $h(\nabla^h \varphi, \nabla^h \Lambda) > 0$ and

$$2H^h_\varphi(y, y) + \frac{\Lambda}{h(\nabla^h \varphi, \nabla^h \Lambda)} d\omega(y, \nabla^h \varphi)^2 + \frac{h(\nabla^h \varphi, \nabla^h \Lambda)}{\Lambda} h(y, y) \leq 0$$
If $D$ satisfies some completeness assumptions then

\[ D \text{ is convex } \iff \partial D \text{ is convex} \]
If $D$ satisfies some completeness assumptions then

$$D \text{ is convex} \iff \partial D \text{ is convex}$$

Convexity of a domain in a Finsler manifold

If $D$ satisfies some completeness assumptions
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If $D$ satisfies some completeness assumptions

$$
\bar{B}^D_s(x, r) \text{ compact}
$$
If $D$ satisfies some completeness assumptions

$$\bar{B}_s^D(x, r) \text{ compact}$$

$$d_s^D(x_0, x_1) = 1/2(d_F^D(x_0, x_1) + d_F^D(x_1, x_0))$$

$$C_{x_0x_1}(\bar{D}) = \left\{ \gamma: [0,1] \to \bar{D} : \gamma \text{ continuous, piecewise smooth} \right\}$$

$$d_F^D(x_0, x_1) = \inf_{\gamma \in C_{x_0x_1}(\bar{D})} \ell_F(\gamma)$$
Convexity of a domain in a Finsler manifold

If \( D \) satisfies some completeness assumptions

**Role of the symmetrized balls**

In Lorentzian manifolds, global hyperbolicity \( \sim \) completeness of Riemannian manifolds

\[
\bar{B}_s^D(x, r) \text{ compact } \iff (D \times \mathbb{R}, g) \text{ globally hyperbolic}
\]

E. Caponio, M. A. Javaloyes, and M. Sánchez. On the interplay between Lorentzian causality and Finsler metrics of Randers type.

If $D$ satisfies some completeness assumptions then

$$D \text{ is convex} \iff \partial D \text{ is convex}$$

If $D$ satisfies some completeness assumptions then

$D$ is convex $\iff \partial D$ is convex


Moreover, if $D$ is not contractible
If $D$ satisfies some completeness assumptions then

\[ D \text{ is convex } \iff \partial D \text{ is convex} \]


Moreover, if $D$ is not contractible

then a sequence $\{\gamma_n\}$ of geodesics of $(D, F)$ exists such that $\ell_F(\gamma_n) \to \infty$
Assume that $\bar{B}^D_s(x, r)$ are compact for any $x \in D$ and $r > 0$

Any point $(x_0, t_0) \in D$ and any integral line of $\partial_t$ passing through $x_1 \in D$ can be joined by a future-pointing (and a past-pointing) lightlike geodesic remaining in $D \times \mathbb{R}$ and minimizing the arrival time

$(\partial D) \times \mathbb{R}$ is light-convex
Assume that $\bar{B}^D_s(x, r)$ are compact for any $x \in D$ and $r > 0$

For any point $(x_0, t_0) \in D$, any integral line of $\partial_t$ passing through $x_1 \in D$ and any a-priori fixed interval $[a, b]$, there exists a future-pointing (and a past-pointing) timelike geodesic $\gamma : [a, b] \rightarrow D \times \mathbb{R}$ parametrized with respect to the proper time and minimizing the arrival time

$\uparrow$

$(\partial D) \times \mathbb{R}$ is time-convex
Consequences for stationary spacetimes III

$$\partial D \ F\text{-convex and } \bar{B}^D_s(x, r) \text{ compact, for all } x \in D \text{ and } r > 0$$

\[\Downarrow\]

$$(D \times \mathbb{R}, g) \text{ is } \textit{causally simple}$$
Consequences for stationary spacetimes IV

\[ \partial D \text{ F-convex and } \bar{B}_s^D(x, r) \text{ compact, for all } x \in D \text{ and } r > 0, \]
\[ + \]
\[ D \text{ non-contractible} \]

\[ \Downarrow \]

in \( D \times \mathbb{R} \), there exist infinitely many future-pointing (and past-pointing) light rays, between any point \( (x_0, t_0), x_0 \in D \), and any integral line of \( \partial_t \) passing through \( x_1, x_1 \in D, x_0 \neq x_1 \), having arrival time

\[ AT((\gamma_n, t_{\gamma_n}) \rightarrow \infty (AT((\gamma_n^-, t_{\gamma_n^-}) \rightarrow -\infty) \]
Asymptotic flatness

Stationary spacetimes
Finsler metrics associated to a stationary spacetime
Fermat principle
Convexity

Asymptotic flatness
Geodesic asymptotic flatness for Randers spaces
Large cylinders in geodesic asymptotically flat $F_\Lambda$
Kerr
Light-convex shell in Kerr
Geodesic asymptotic flatness for Randers spaces

\[ K \subset S, \text{ compact} \]
Geodesic asymptotic flatness for Randers spaces

\[ K \subset S, \text{ compact} \quad S \setminus K = \bigcup_{k=1}^{m} E^k, \quad E^i \cap E^j = \emptyset \]
Geodesic asymptotic flatness for Randers spaces

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Geodesic asymptotic flatness for Randers spaces

Large cylinders in geodesic asymptotically flat $F^\Lambda_{\Lambda}$

Kerr

Light-convex shell in Kerr

\[
K \subset S, \text{ compact } \quad S \setminus K = \bigcup_{k=1}^{m} E^k, \quad E^i \cap E^j = \emptyset
\]

\[
E_k \text{ diffeomorphic to } \mathbb{R}^n \setminus \{0\}
\]
Geodesic asymptotic flatness for Randers spaces

\[ K \subset S, \text{ compact} \quad S \setminus K = \bigcup_{k=1}^{m} E^k, \quad E^i \cap E^j = \emptyset \]

\[ E_k \text{ diffeomorphic to } \mathbb{R}^n \setminus \{0\} \]

there exist a coordinate chart in \( E^k \), \( x = (x^1, \ldots, x^n) \), and \( p, q > 0 \) such that:

\[ h_{ij} = \delta_{ij} + O\left(\frac{1}{|x|^p}\right) \]

\[ \partial_k h_{ij} = O\left(\frac{1}{|x|^{p+1}}\right) \quad \text{as } |x| \to +\infty \]

\[ \Omega_{ij} := \partial_i(\omega_j) - \partial_j(\omega_i) = O\left(\frac{1}{|x|^{q+1}}\right) \]
Geodesic asymptotic flatness for Randers spaces

$
K \subset S$, compact \\
S \setminus K = \bigcup_{k=1}^{m} E^k, \quad E^i \cap E^j = \emptyset
$

$E_k$ diffeomorphic to $\mathbb{R}^n \setminus \{0\}$

there exist a coordinate chart in $E^k$, $x = (x^1, \ldots, x^n)$, and $p, q > 0$ such that:

$h_{ij} = \delta_{ij} + O(1/|x|^p)$

$\partial_k h_{ij} = O(1/|x|^{p+1})$ as $|x| \to +\infty$

$\Omega_{ij} := \partial_i(\omega_j) - \partial_j(\omega_i) = O(1/|x|^{q+1})$

Large spheres $S^{n-1}(r) = \{x \in E^k : |x|^2 = r^2\}$, $r$ big enough, are (strongly) convex
Large cylinders in geodesic asymptotically flat $F_\Lambda$

\[ S^{n-1}(r) \times \mathbb{R} \subset E^k \times \mathbb{R} \]
Large cylinders in geodesic asymptotically flat $F_\Lambda$

\[ S^{n-1}(r) \times \mathbb{R} \subset E^k \times \mathbb{R} \]

is $F_\Lambda$-(strongly) convex for large $r$ if
Large cylinders in geodesic asymptotically flat $F_\Lambda$

\[ S^{n-1}(r) \times \mathbb{R} \subset E^k \times \mathbb{R} \]

is $F_\Lambda$-(strongly) convex for large $r$ if

\[ \Lambda = C_1 + O(1/|x|^{q'}) \]

\[ \partial_i \Lambda = O(1/|x|^{q'+1}) \quad \text{as} \quad |x| \to \infty \]

\[ \partial_r \Lambda \sim -\frac{C_2}{|x|^{q'+1}} \]

for some $C_1, C_2 > 0$, and $q' \in [0, 2q)$

\[ \downarrow \]

$S^{n-1}(r) \times \mathbb{R}$, for $r$ big enough, are (strongly) convex
Large cylinders in geodesic asymptotically flat $F_{\Lambda}$

Stationary spacetimes
Finsler metrics associated to a stationary spacetime
Fermat principle
Convexity
Asymptotic flatness
Geodesic asymptotic flatness for Randers spaces

$S^{n-1}(r) \times \mathbb{R} \subset E^k \times \mathbb{R}$

$\partial_r \Lambda \sim \frac{C}{|x|^{q'+1}}$, as $|x| \to \infty$

for $C > 0$ and $q' > 0$

$S^{n-1}(r) \times \mathbb{R}$ is not $F_{\Lambda}$-convex for any $r > r_0$ large enough
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Kerr
Light-convex shell in Kerr

\[ S^{n-1}(r) \times \mathbb{R} \subset E^k \times \mathbb{R} \]

\[ \partial_r \Lambda \sim \frac{C}{|x|^{q' + 1}}, \quad \text{as } |x| \to \infty \]

for $C > 0$ and $q' > 0$

\[ S^{n-1}(r) \times \mathbb{R} \] is not $F_\Lambda$-convex for any $r > r_0$ large enough

In fact $h(\nabla^h \varphi, \nabla^h \Lambda)$ is definitively negative
Large cylinders in geodesic asymptotically flat $F_\Lambda$

For physically reasonable asymptotically flat stationary spacetimes
Large cylinders in geodesic asymptotically flat $F_{\Lambda}$

For physically reasonable asymptotically flat stationary spacetimes

\[
| (g_0)_{ij} - \delta_{ij} | + |x||\partial_l(g_0)_{ij}| = O(1/|x|^{p_0}), \quad \text{as } |x| \to \infty
\]
\[
| (\omega_0)_j | + |x||\partial_l(\omega_0)_j| = O(1/|x|^{q_0}),
\]
\[
|\Lambda - 1| + |x||\partial_l\Lambda| = O(1/|x|^{q'}),
\]
for some $p_0, q_0, q' > 0$. 

Kavli IPMU, September 30 – October 3, 2013
Large cylinders in geodesic asymptotically flat $F_\Lambda$

For physically reasonable asymptotically flat stationary spacetimes

\[
\begin{align*}
| (g_0)_{ij} - \delta_{ij} | + |x| | \partial_l (g_0)_{ij} | &= O(1/|x|^{p_0}), \\
| (\omega_0)_i | + |x| | \partial_l (\omega_0)_i | &= O(1/|x|^{q_0}), \quad \text{as } |x| \to \infty \\
| \Lambda - 1 | + |x| | \partial_l \Lambda | &= O(1/|x|^{q'})
\end{align*}
\]

for some $p_0, q_0, q' > 0$.

\[\downarrow\]

generalized asymptotic flatness for the Fermat metric $F$

with $p = \min\{p_0, 2q_0, q'\}$ and $q = q_0$
Large cylinders in geodesic asymptotically flat $F_\Lambda$

For physically reasonable asymptotically flat stationary spacetimes

$$\partial_r \Lambda \sim \frac{C}{|x|^{q'+1}}$$
Large cylinders in geodesic asymptotically flat $F_\Lambda$

For physically reasonable asymptotically flat stationary spacetimes

$$\partial_r \Lambda \sim \frac{C}{|x|^{q'+1}}$$

for $C' > 0$
For physically reasonable asymptotically flat stationary spacetimes

\[ \partial_r \Lambda \sim \frac{C}{|x|^{q'+1}} \]

For example, if \((S \times \mathbb{R}, g) \neq \mathbb{L}^4\) is an asymptotically flat stationary solution of the Einstein field equation. Assume that, at each end, there exists a constant \(C\) such that

\[ \partial_r \Lambda = C/|x|^2 + o(1/|x|^2), \quad \text{as } |x| \to +\infty. \]

and

\[ \text{Ric}(g)(\partial t, \partial t) \geq 0, \]

with \(\text{Ric}(g)(\partial t, \partial t)\) integrable on \(S\). Then \(C > 0\)
\[ ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\varphi)^2 \]
\[ + \frac{\sin^2 \theta}{\rho^2} \left((r^2 + a^2) d\varphi - a dt\right)^2 + \frac{\rho^2}{\Delta} d\varphi^2 + \rho^2 d\theta^2, \]

\[ \Delta = r^2 - 2mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \]

where \( m \) is the ADM mass and \( a = j/m \) with \( j \) the ADM angular momentum of the spacetime.
\[ ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\varphi)^2 + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2)d\varphi - adt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \]

\[
\Delta = r^2 - 2mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \]

where \( m \) is the ADM mass and \( a = j/m \) with \( j \) the ADM angular momentum of the spacetime.

For \( a^2 < m^2 \)

standard stationary outside the stationary limit hypersurface

\[
r = m + \sqrt{m^2 - a^2 \cos^2 \theta} \]
Its Randers structure was studied in

\[ \Lambda = \frac{r^2 - 2mr + a^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} = 1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta}, \]
\[ \Lambda = \frac{r^2 - 2mr + a^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} = 1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta}, \]

\[ \partial_r \Lambda \sim \frac{2m}{r^2}. \]
Kerr

\[ \Lambda = \frac{r^2 - 2mr + a^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} = 1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta}, \]

\[ \partial_r \Lambda \sim \frac{2m}{r^2} \]

\[ C = 2m > 0 \]

and \( S^2(r) \times \mathbb{R} \) are definitively not time-convex.
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\[ \Lambda = \frac{r^2 - 2mr + a^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} = 1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta}, \]

\[ \partial_r \Lambda \sim \frac{2m}{r^2} \]

\[ C = 2m > 0 \]

and $S^2(r) \times \mathbb{R}$ are definitively not time-convex

as Kerr is asymptotically flat they are (strongly) light-convex
For any $\varepsilon \in (0, m)$ and $r_0 \in (0, \infty)$

$$D_{\varepsilon,r_0}^a = \{(x, y, z) \in \mathbb{R}^3 : m + \sqrt{m^2 + \varepsilon - a^2 \cos^2 \theta} < r < r_0\}.$$ 

Then there exists $a_0 > 0$, depending on $\varepsilon$, such that $D_{\varepsilon,r_0}^a \times \mathbb{R}$ has strongly light-convex boundary, for each $|a| < a_0$, provided that $r_0$ is large enough.
The light-convexity of the hypersurface $H_\varepsilon \times \mathbb{R}$ ($H_\varepsilon$ is the hypersurface $r = m + \sqrt{m^2 + \varepsilon - a^2 \cos^2 \theta}$) was proved in Antonio Masiello. *Variational Methods in Lorentzian Geometry*. Pitman Research Notes in Mathematics Series. Longman Scientific & Technical, New York, 1994