Spatial Modulation and Topological Current in Holographic QCD Matter

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Outline

$p$-wave pion condensation in nuclear matter
An old but vital idea of inhomogeneity in nuclear physics, but...

Landau-Migdal interaction in a Fermi liquid
Almost abandoned... and revived recently in quark matter

Interaction controlled by topological current in $B$
– Chiral Magnetic Effect –
Effect of the topological current induced by the magnetic field

Analysis in the Sakai-Sugimoto model
Inhomogeneous states favored or disfavored by $B$???
Results consistent with what happened to the pion condensation
Historical Phase Diagram of QCD

Baym (1983)
Historical Phase Diagram of QCD

Baym (1983)

What is this?
**p-wave Pion Condensation**

Sawyer-Scalapino, Migdal (1972)

\[
\Pi(\omega, k) \rightarrow D^{-1}(\omega = 0, k = k_c) = 0 \text{ at } \rho = \rho_c
\]

One Pion Exchange Potential (OPEP)

\[
V = \frac{m_\pi^2 g^2}{3} \left[ \frac{4\pi}{3} \left( \sigma_1 \cdot \sigma_2 \frac{e^{-m_\pi r}}{r} + S_{12} \left( 1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2} \right) \frac{e^{-m_\pi r}}{r} \right) - \frac{g^2}{3} (\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2) \delta(r) \right]
\]

**Landau-Migdal (short-ranged) Interactions**

\[
f + g \sigma_1 \cdot \sigma_2 + f' \tau_1 \cdot \tau_2 + g' (\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2)
\]
**p-wave Pion Condensation**

**Why p-wave?**

Relative angular-momentum must be 1 (p-wave)

\[
N | \quad \pi \quad \text{(negative parity)}
\]

\[
\langle \sigma \rangle \sim \chi \cos (2qz) \quad \langle \pi^0 \rangle \sim \chi \sin (2qz)
\]

**Landau-Migdal Parameter**

\(g'\) is sensitive to the spin-isospin collective excitation

Gamow-Teller resonance → large \(g'\) → **No pion condensation**?

Most of nuclear physicists consider no pion condensation up to a few times normal nuclear density


$p$-wave Pion Condensation

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History repeats itself (Nuclear $\rightarrow$ Quark Matter)
One Possible QCD Phase Diagram

Fukushima-Sasaki (2013)

Inhomogeneous Chiral Condensate

Quark-Gluon Plasma

Hadronic Phase

Quarkyonic Regime

Color Superconductors

Temperature $T$

Chemical Potential $\mu_B$

Nuclear Superfluid

Liquid-Gas

Crystalline States

sQGP

2SC

uSC
dSC

CFL
**Energy Gain by Spiral**

**Chiral spiral (DGR Ansatz)**

\[ \psi(x) = e^{i \gamma_5 \tau_3 q z} \psi'(x) \text{ with } \chi = \langle \bar{\psi}' \psi' \rangle \]

\[ \langle \sigma \rangle \sim \langle \bar{\psi} \psi \rangle = \chi \cos(2qz) \]

\[ \langle \pi^0 \rangle \sim \langle \bar{\psi} \gamma_5 \tau_3 \psi \rangle = \chi \sin(2qz) \]

**Quasi-particle dispersion relation**

\[ \omega = \sqrt{p_{\perp}^2 + \left( \sqrt{p_z^2 + (M^2 \pm q)^2} \right)^2} \]

- **Vacuum** favors large \( M \)
- **Competitive**
- **High Density** favors small \( M \)

**Vacuum + High Density** favors large \( q \sim M \) most!
Energy Gain by Spiral

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Counterpart of the Landau-Migdal int?  
Can be the true ground state with strong \( B \)
Landau Quantization

Energy dispersion relation in $B$

$$\omega^2 = p_z^2 + 2|gB|(n + 1/2) + m^2 - 2s gB$$

Transverse motion = Harmonic Oscillator

- Light fermions ($s=1/2$) have zero mode.
  Pseudo-(1+1) dimensional system of fermions
- Light vector bosons have (Nielesen-Olesen) instability.
  Gluons in the chromo-$B$ / $\rho$ in a superstrong $B$ (Chernodub)
- Charged scalar bosons are all massive.
  $\pi^+$, $\pi^-$, ... Explicit breaking of isospin symmetry
  Etc, etc...

Sep. 27, 2013 @ IPMU
**Pseudo-(1+1) dimensional System**

Dirac Lagrangian in (1+1) dimensions

\[
L = \bar{\psi} \left[ (\partial_4 + \mu) \gamma^4 + \partial_3 \gamma^3 \right] \psi
\]

\[
= \bar{\psi}' \left[ \partial_4 \gamma^4 + \partial_3 \gamma^3 \right] \psi'
\]

Thermodynamic potential

\[
\Omega / V = - \int_{-\Lambda - \mu}^{\Lambda - \mu} \frac{dp}{2\pi} \frac{|\varepsilon(p)|}{2} - \int_{-\Lambda + \mu}^{\Lambda + \mu} \frac{dp}{2\pi} \frac{|\varepsilon(p)|}{2} + \ldots
\]

\[
= \Omega (\mu = 0) / V \left( - \frac{\mu^2}{2\pi} \right)
\]

Surface integral: Anomaly origin
c.f. CS term in Sakai-Sugimoto

No mass suppression as compared to the homogeneous case
Energy gain by spiral maximized in (1+1) dimensions!
Pseudo-(1+1) dimensional System

If the zero-density system has a condensate:

$$\langle \bar{\psi} \psi \rangle = \text{(homogeneous chiral condensate)}$$

Rotated system has the same condensate: $$\langle \bar{\psi}' \psi' \rangle$$

(1+1)-dimensional system forms a “spiral”

$$\langle \bar{\psi} \psi \rangle = \langle \bar{\psi}' \psi' \rangle \cos(2\mu z)$$

$$\langle \bar{\psi} \gamma^3 \gamma^4 \psi \rangle = \langle \bar{\psi}' \psi' \rangle \sin(2\mu z)$$

Chiral Magnetic Spiral  
Basar-Dunne-Kharzeev (2010)
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Chiral Magnetic Spiral  

Basar-Dunne-Kharzeev (2010)

\( B + \mu \) causes another interesting phenomenon!
Chiral Magnetic (Separation) Effect

Classical Picture

Right-handed Quarks
= momentum parallel to spin

Left-handed Quarks
= momentum anti-parallel to spin

\[ J_5 \neq 0 \text{ if } \mu \text{ or } N \neq 0 \]

Kharzeev-McLerran-Warringa (2007)
Fukushima-Kharzeev-Warringa (2008)
Quantum Formula

Chiral Magnetic (Separation) Effect

\[ j_5 = \frac{e^2 \mu}{2 \pi^2} B \]

\[ (j^\mu = \epsilon^{\mu \nu \rho \sigma} \partial_\nu \varphi F_{\rho \sigma}) \]

Vector = Axial-Vector (\( \gamma^5 = \gamma^0 \gamma^1 \)) in (1+1)

\[ \gamma^\mu \gamma^5 = -\epsilon^{\mu \nu} \gamma_\nu \]

\[ j_V^\mu = \bar{\psi} \gamma^\mu \psi, \quad j_A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi \]

\[ j_V^1 = j_A^0, \quad j_A^1 = j_V^0 \]
Question

At high density
Chiral spiral is expected?

At strong $B$
Chiral spiral is favored by low dimensionality...

Axial-current is strengthened leading to –
\[ g' (\sigma_1 \cdot \sigma_2) (\tau_1 \cdot \tau_2) \]
that disfavors the chiral spiral...
What we need is...

Spatial Inhomogeneity

Ooguri-Park, Chuang-Dai-Kawamoto-Lin-Yeh (2010)

Topological Current

Yee, Rebhan-Schmitt-Stricker (2009)

Non-perturbative Method → Sakai-Sugimoto Model

Holographic QCD Model with $D4 + D8 + \overline{D8}$  

Sakai-Sugimoto (2004)
Phase Diagram in SSM

Bergman-Lifschytz-Lippert (2007)

Deconfined / Chiral Symmetric
Finite Density

Deconfined
Chiral Broken
Zero Density

Confined
Chiral Broken
Zero Density

Confined
Chiral Broken
Finite Density

Fukushima-Sasaki (2013)
Inhomogeneous States in SSM

Inhomogeneous States in SSM


\[ T \]

\[ \log_{10} \mu \]

QGP

Vacuum

Nuclear Matter

Spatially Modulated

\[ + B \]
Sketch of the Calculations

Action for the flavor sector

\[ S = N \int d^4 x \, du \, u^{1/4} \sqrt{-\det \left( g_{\alpha\beta} + F_{\alpha\beta} \right)} + \frac{\alpha}{3!} N \int d^4 x \, du \, \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} A_{\mu_1} F_{\mu_2 \mu_3} F_{\mu_4 \mu_5} \]

DBI action

Density \( \bar{a}_0(u) \quad \bar{a}_0(\infty) = \mu \)

Current \( \bar{a}_z(u) \)

Magnetic field \( \bar{F}_{xy} = B(u) = B \)

\[ S \sim \int du \, u^{5/2} \sqrt{\left( 1 - a_0'^2 + f \, a_z'^2 \right) \left( 1 + B^2 \, u^{-3} \right)} + 4 \alpha \, B \int du \, \bar{a}_z \bar{a}_0' \]
Sketch of the Calculations

Equations of motion (w.r.t. $a_0$ and $a_z$)

\[ \rho = u \overline{a}_0' \sqrt{\frac{u^3 + B^2}{1 - \overline{a}_0'^2 + f \overline{a}_z'^2}} - 4 \alpha B \overline{a}_z \quad \text{and} \quad 0 = u f \overline{a}_z' \sqrt{\frac{u^3 + B^2}{1 - \overline{a}_0'^2 + f \overline{a}_3'^2}} - 4 \alpha B \overline{a}_0 \]

Asymptotic solutions:

\[ \overline{a}_z(u \to \infty) \simeq -\frac{8 \alpha}{3} B \mu u^{-3/2} \]

\[ \overline{a}_0(u \to \infty) \simeq \mu - \frac{3 \alpha}{2} \rho u^{-3/2} \]

\[ \sim j_5 \]

Introducing spatial modulations:

\[ F_{\alpha \beta} \rightarrow F_{\alpha \beta} + f_{\alpha \beta}(\omega, k) \]

\[ f_{\alpha \beta} = \partial_{\alpha} a_{\beta} - \partial_{\beta} a_{\alpha} \]

\[ D^{-1}(\omega = 0, k = k_c) = 0 \]
Sketch of the Calculations

Equations of motion (w.r.t. \( a_x, a_y, a_z \))

From the condition that they are normalizable or
\[
\begin{align*}
  a_x(u = \infty) &= a_y(u = \infty) = a_z(u = \infty) \to 0
\end{align*}
\]

\[ k_x, k_y, k_z \text{ are found for } \rho > \rho_c \]

Without \( B \)

\[ \rho_c = 3.715 u_T^{5/2} \] (Ooguri-Park 2010)

With \( B \)

The presence of \( j_5 \) completely changes the results!

(Fukushima-Morales 2013)
Phase Diagram with $B$

Smaller with increasing $B$

Fukushima-Morales (2013)

Same tendency as what happened to the $p$-wave pion condensation
Sakai-Sugimoto model is a powerful tool to investigate the *chiral sector of large-$N_c$ QCD*.

→ Phase Diagram ??

Axial-vector interaction disfavors the spatial modulations in the same way as discussed by nuclear physicists long long time ago.

Full structure of the phase diagram with an extra axis of strong $B$ ??

Earlier chiral phase transition? (Magnetic Inhibition) Deconfinement?

(Fukushima-Hidaka 2012)