Fano threefolds and mirror duality

Lecture 1

Program (Coates - Corti - Galkin - Golyshev - van Straten - ...) :

Mirror : Fanos $\to$ Objects of special nature, the so called Landau-Ginzburg models.

LGs: actual pencils of motives or their realization.

Slogan: Classify LGs $\longrightarrow$ classify Fanos

Therefore : Start where classification is known (dimension 3), study LGs, make guess at what can be LG, go 1 or 2 dimension higher.

Remark : Dimensions 1 and 2 too specific to infer anything definitive.

Definition A Fano: smooth, complex, projective, $-K$ ample.

Known Classification $\mathbb{P}^1$, del Pezzos, Mori-Mukai.

Mori-Mukai : Classify Fano according to their Picard

rank $\rho = h^2(F, \mathbb{Z})$, index$d =$ index of $-K \cdot \mathbb{Z}$ in $-K \cdot \mathbb{Q} \cap NS$.

worth noticing The Hodge diamond of a 3-Fano:

$$
\begin{matrix}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & \rho & & 0 & \\
0 & & \frac{b_3}{2} & & \frac{b_3}{2} & & 0 \\
 & 0 & & \rho & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{matrix}
$$

Comment on each zero of the Hodge diamond (if enough time, otherwise exercise). Hint: vanishing theorems.

Fano threefolds 105 deformation families (Fano, Iskovskikh, Mori - Mukai) In particular, 17 deformation families of $\rho = 1$ Fanos (Iskovskikh).

Landau-Ginzburg models No definition or construction of the LG model of a Fano is known, which makes the zest of it.

However [most deeply] by HMS, the Fukaya - Seidel category of the/an LG is equivalent to $D^b_{coh}F$

[most practicably] by the original mirror conjecture

The regularised quantum D-module of a Fano isisomorphic to the Gauss-Manin conjection in its LG.

For the most of the remainder of Lecture 1 and part of Lecture 2 we make the Original Mirror Conjecture (or, VHS Mirror Conjecture) as specific and concrete as possible.

To this end must

(I) define regularised quantum D-module

(II) review Gauss-Manin connections and pin down the candidate pencils that can aspire to be LGs.

Quantum Cohomology We review samll quantum cohomology for Fano manifolds.

i. Gromov -Witten invariants : Glossary p.6

ii. Quantum multiplication, first connection: Glossary 24-33

iii. The original mirror conjecture : Glossary 34-36

remark. The quantum Lefschetz principle of Givental, Coates, Lee can be interpreted as saying that the regularised quantum D-module of F holds information on the GW–count on a generic CY anticanonical selection X in F in a [n arguably] more natural way than it does for the GW–count on F itself.

Round-up: Now what?

How to complete regularized quantum D-modules: known methods:

We will mostly concentrate on

A Naive Gromov-Witten count. Generally good for high index.

B Homogeneous varieties: known by
Peterson-Fulton-Woodward-Bertram-Buch–Ciocan-F . . . .

C Toric: Batyrev, Givental, ...

D Complete intersections in the above.

E Toric degenerations.

We will mostly concentrate on E in these lectures.

What will we see: It is clear that the world of Fanos-like objects is much smaller than that of pencils. One then expects that one is going to end up with pencils with very special properties. The basics insight here is this. Imagine a class of functions which is the opposite of Morse, i.e. such where you want the critical values come as much together as possible. There are known extremes on this way. It is these animals that we call ”extremal pencils”.

Our organising principle in search for Fanos is the start with the study of the extremal ones and understand how mirror duality works for these.

Next lecture:

i. Extremal local systems : definition;

ii. Local system from a Laurent poly;

iii. Mirror symmetry in the Laurent poly setup;

iv. Computing quantum D-modules of $\mathrm{rk}\,1$ Fano threefolds;

Lecture 2

Reminder We consider a Fano threefold F,

the quantum D – module $D\zeta = \zeta(-K\cdot)$ on a torus given by the quantum multiplication by $-K$

the regularized quantum DE $L\Phi = 0$, which is basically Fourier transform of the above.

Then Mirror conjecture : L is an operator of Richard - Fuchs type i.e there is a pencil $E \xrightarrow{\pi} \mathbb{A}^1$ in which it is realized as the variation of the middle relative cohomology.

Clear Many more pencils than there are Fanos. Which functions?

Guess (at least for odd-dimensional Fanos) : Extremal: want critical values to come together as strongly as possible.

Want To define that precisely.

Extremal local systems:

Geometric ramification: Glossary 43

Extremal: Glossary 44

Further specialization of the Mirror hypothesis

Extremal Laurent polynomials: Glossary 51 – 53

Computing the PF operator from Laurent polynomial. It turns out that it is an algorithmic problem to compute the Picard - Fuchs equation given a Laurent polynomial P.

The constant term series : $\sum \mathrm{c.\ t.}\,(P^i) z^i = \sum c_i t^{-i}$.

Proposition There exists a relative cycle γ_t in E_t, t near infinity, and a fiberwise differential form ω_t such that one has an expansion around ∞:

$$\int_{\gamma_t} \omega_t = \sum c_i t^{-1}.$$

Remarks ω_t is the residue form of the form $\bigwedge \frac{dx_i}{x_i}$ on the ambient torus; γ_s: define

$$\begin{aligned} T_s &= \{(x_1, ..., x_n) \in \pi \, : \, |x_1| = |x_2| = |x_3| = s\}, \\ R_\delta &= \bigcup T_s \delta \leq s \leq 1, \end{aligned}$$

assume t small and δ small,

$$\gamma_t = Y_t \cap R_\delta \subset Y_t$$

Hence: given a Laurent polynomial P, can build the Picard-Fuchs equation by finding an L such that $L \sum c.t.\,(P^i)t^{-i} = 0$. Perform the standard Fuchsian procedure to determine the symbol and the conjugacy classes of the local monodromies $\Rightarrow$ find geometric ramification.

In other words, can effectively tell if $R = 2 * \mathrm{rk}\, L$, by looking at L

Example. Consider $\mathbb{P}^2$. The regularized quantum DE with respect to the coordinate $z = t^{-1}$ is

$$\left[D^2 - 27z^3(D+1)^2\right]\Phi(z) = 0$$

and $\Phi(z) = \sum \frac{(3n!}{(n!)^3} z^{3n}$.

One computes singularities: at $t = \infty$, $t = 3\sqrt[3]{1}^i$, $i = 0, 1, 2$

Conjugacy classes of all four local monodromies: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$R = 4 = 2 \cdot 2 \Rightarrow$ extremal.

Extremality is acyclicity:

Proposition. Let F be a constructible sheaf of $\mathbb{C}$–vector spaces on a complex analytic smooth projective curve X. Denote by U an open subset over which F is locally constant, j the open embedding. Let x be a point in U and let $X \setminus U = X_0 = \{x_i\}$. Denote by F_x the fiber of F over x. This turns F_x into a $\pi_1(U)$–module. One has the Euler–Poincare formula:

$$\chi(X,F) = \sum(-1)^r h^r(X,F) = (2-2g)\dim F_x - \sum(\dim F_{x_i} - \dim F_x).$$

If, additionally, $F = j_* j^* F$, then $\dim F_(x_i) = \dim F_x^{I_i}$.

Recall that $H^2(X, j_*(j^*(F)))$ is dual to $H_c^0(X, j_*(j^*(F)^\vee))$. Remember that we have agreed to strip away the trivial constituents from our sheaf.

Hence, in our situation, with $X = \mathbb{P}^1(\mathbb{C})$ and $H^0(X, F) = 0$ and $H^2(X, F) = 0$, the Euler-Poincare formula implies

$\boxed{\text{extremal} \Leftrightarrow H^1(X, F) = 0}$.