Lecture 7 | Global theory for meromorphic connections with regular singular points.

Let F be holom. v.b. with merom. conn. ∇ over $\mathbb{C}P^1$,
$a_1, \dots, a_n$ – singular points of ∇. Assume they all regular,
and ∞ is not one of them.

Let O_i be small neighborhoods of a_i, choose trivializations for $F|_{O_i}$.
∇ defines system of linear ODE's $\frac{dy}{dz} = B(z)\,y$ (7.1) with reg. sing. point a_i.
Theory of last 3 lectures can be applied here (in coord $\xi_i = z - a_i$).
So we have matrices, evaluations, exponents, etc depending on i:

$$Y_{e_i} \to \underline{Y_i},\ \underline{U_i},\ \underline{A_i},\ \underline{E_i},\ \underline{\rho_i^j},\ \underline{\varphi_i^j},\ \underline{\beta_i^j}$$

$Y_i \to Y_i \cdot T(z)$ — does not depend on trivialization $F|_{O_i}$

$U_i \to U_i \cdot T(z)$

if $(s_1, \dots, s_p) \to (s_1, \dots, s_p) \cdot T^{-1}(z)$

<u>Def</u> $\{\beta_i^j\}$ are <u>exponents</u> of ∇ at a_i.

<u>Q1</u> What is ∇ if F is trivial v.b.?

<u>Q2</u> What is the relation between ~~How~~ Levelt's filtrations and exponents in different points?

<u>Q3</u> Conditions on $\{\beta_i^j\}$ for ∇ to be Fuchsian?

<u>Assume F is trivial.</u> In base of global holom. sections ∇ give.

$$dy = \omega\, y \qquad (7.2)$$

$\boxed{\Omega^1_{\mathbb{C}P^1} = \mathcal{O}_{\mathbb{P}^1}(-2}$

<u>Def</u> $B_i = \mathrm{Res}_{z=a_i}\, \omega$

If (7.2) is Fuchsian then $\omega - \sum_{i=1}^{n} B_i \frac{dz}{z-a_i} \in \Gamma(\mathbb{C}P^1, \Omega^1_{\mathbb{C}P^1}) = \mathbb{C}$

So (7.2) becomes in coord z: (7.3) $\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i}{z-a_i}\right) y$; $\sum_{i=1}^{n} B_i \left(= -\mathrm{res}_{\infty}\, \omega\right) = 0$

<u>Thm 7.1</u> ($\to$ Q2, Q3) • Sum of all exponents is non-positive integer.

$$\Sigma := \sum_{i=1}^{n} \sum_{j=1}^{p} \beta_i^j \le 0 \qquad (7.4)$$

• System with r.s.p. / $\mathbb{C}P^1$ is Fuchsian $\Longleftrightarrow \Sigma = 0$.

<u>Proof</u> Consider $\mathrm{tr}\, B(z)\, dz$. Let Y_i be fund. sol. w.r.to Levelt's base (e

$\propto$ $\boxed{\text{Exr. 3.2 and 2.5}}$

By Liouville's formula: $\det Y_i = c_0 \cdot \exp \int \mathrm{tr}\, B(z)\, dz$

in nbhd of a_i : $\mathrm{tr}\, B(z)\, dz = d \ln \det Y_i$

<u>T5.1</u> (5.3) implies $\det Y_i = h(z)\,(z-a_i)^{b_i + \sum_{j=1}^{p} \beta_i^j}$ $b_i := \mathcal{V}_{(z-a_i)} \det U_i(z)$

so $\mathrm{res}_{a_i} \mathrm{tr}\, B(z)\, dz = b_i + \sum_{j=1}^{p} \beta_i^j$ <u>$b_i \ge 0$</u> $h(a_i) \ne 0$

$\sum_i \mathrm{res}_{a_i} \omega = 0 = \sum_{i=1}^{n} b_i + \Sigma \Rightarrow (7.4)$

T5.2 $\Rightarrow$ Fuchsian $\Leftrightarrow b_1 = \dots = b_n = 0 \Leftrightarrow \sum b_i = 0$ $\boxplus$

Consider Fuchsian eq.

$$u^{(p)} + q_1(z) u^{(p-1)} + \dots + q_p(z) u = 0 \qquad (7.2)$$

$a_1, \dots, a_n$ – sing. points $\quad a_n = \infty$

$$\rho = \frac{1}{z} \qquad \frac{d^p u}{d\rho^p} + \tilde{q}_1 \frac{d^{p-1} u}{d\rho^{p-1}} + \dots = 0 \qquad (7.8)$$

$$\left(\frac{d}{dz}\right)^j = \left(-\rho^2 \frac{d}{d\rho}\right)^j = \sum_{i=1}^{j} c_i^j \rho^{j+i} \left(\frac{d}{d\rho}\right)^i \quad ; \; c_j^j = (-1)^j$$

$$\tilde{q}_1 = (-1)^p \rho^{-2p} \left(c_{p-1}^{p} \rho^{2p-1} + c_{p-1}^{p-1} \rho^{2p-2} q_1(\rho^{-1}) \right)$$

$$\tilde{q}_k = (-1)^p \rho^{-2p} \Big(c_{p-k}^{p} + c_{p-k}^{p-1} \rho^{2p-k-1} q_1(\rho^{-1}) + \dots$$

$$+ c_{p-k}^{p-m} \rho^{2p-k-m} q_m(\rho^{-1}) + \dots c_{p-k}^{p-k} \rho^{2p-2k} q_k(\rho^{-1}) \Big)$$

(7.2) Fuchsian at $\rho = 0$ $\Leftrightarrow$ $R_i(\rho) = \rho^{-i} q_i(\rho^{-1})$ is holomorphic $i = 1 \dots p$ at $\rho = 0$

$\Rightarrow \forall i \quad q_i(z) = \dfrac{r_i(z)}{(z-a_1)^i \cdots (z-a_{n-1})^i}$ (7.10) $\quad r_i(z)$ – holom on $\mathbb{A}^1(\mathbb{C})$

(Liouv. thm) $\quad r_i(z) \underset{\infty}{=} O(z^{k_i})$

$\Rightarrow r_i(z)$ is polynomial of degree k_i $\quad (n-1) i \geq i + k_i$.

$k_i \leq (n-2) i$ so $(k_i + 1)$ parameters.

Since $\displaystyle\sum_{i=1}^{p} (k_i + 1) \leq (n-2) \frac{p(p+1)}{2} + p \Rightarrow$

Prop 7.1 Fuchsian scalar DE$/_{\mathbb{P}^1}$ of order p with n sing points has

$$N_{eq} = \frac{(n-2)p(p+1)}{2} + p \quad \text{parameters.}$$

Thm 7.2 (Fuchs condition) For (7.2)

$$\sum_{i=1}^{n} \sum_{j=1}^{p} \beta_i^j = \frac{(n-2)p(p-1)}{2} \qquad (7.11)$$

Proof Let ∞ be non-sing. point. eq (7.2) $\to$ system (7.1): $y^l = \prod_{i=1}^{n} (z-a_i)^{l-1} \left(\frac{d}{dz}\right)^{l-1} u$

Lecture 6 (6.1 and (6.15)) $\Rightarrow$ system is Fuchsian with same exponents

Choose base $e_1 \dots e_p$ for X. Fund matrix $Y(z) = \Gamma(z) W(z)$

$W(z)$ – Wronskian $= \begin{pmatrix} e_1 & \dots & e_p \\ e_1' & \dots & e_p' \\ e_1^{(p-1)} & \dots & e_p^{(p-1)} \end{pmatrix}$ $\quad \Gamma(z) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & (\prod (z-a_i))^{p-1} \end{pmatrix}$

Since $d_z^i u = z^{-2j} \sum c_i^j z^{j-i} d_\rho^i u$

$W(z) = \Gamma_1(z) \Gamma_2(z) V(\rho) \quad \Gamma_1(z) = \begin{pmatrix} 1 & & & 0 \\ & z^{-2} & & \\ & & \ddots & \\ 0 & & & z^{-2(p-1)} \end{pmatrix}, \; \Gamma_2(z) = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$

$V(\rho)$ – Wronskian w.r.to $\rho = z^{-1}$. $\quad V(\rho) \in GL(p, \mathbb{C})$ at $\rho = 0$ $\qquad c_j^j = (-1)^j$

$\det Y(z) = \det(\Gamma(z) \Gamma_1(z) \Gamma_2(z) V(\rho)) = z^b h(z)$, $b = (n-2) + 2(n-2) + \dots + (p-1)(n-2) = \dfrac{(n-2)p(p-1)}{2}$

$h(z) \in \mathcal{O}^\times$ at ∞

As in thm 7.1 consider $\sum$ res tr $B(z)\,dz$

$$res_\infty = -b = -\frac{(n-2)p(p-1)}{2}$$

$$\sum_i res_{a_i} = \sum_{i,j}\beta_i^j \quad \Rightarrow \quad b = \sum \beta_i^j \quad \blacksquare$$

Def 7.1 Degree $c_1(F) = \sum$ res $\det\nabla$

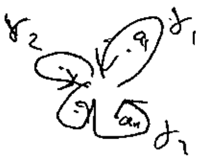
Thm 7.1 $\Rightarrow$ Cor 7.1 • $\Sigma \le c_1(F)$ (7.13)

• ∇ is logarithmic Fuchsian $\Longleftrightarrow$ $\Sigma = c_1(F)$ (7.14)

Monodromy $\chi : \pi_1(B\setminus\{a_1,\dots,a_n\}, z_0) \to GL(p,\mathbb{C})$ (7.15)

$S = S'\cdot G_\gamma$ $\quad \gamma \to G_\gamma$

$B = \mathbb{CP}^1 \Rightarrow \pi_1(B\setminus\{a_1,\dots,a_n\}, z_0)$ — free group with $(n-1)$ gen.



$\gamma_n \cdot \ldots \cdot \gamma_1 = Id$

In case F has hol. flat conn. monodromy can be expressed ~~as follows~~ in terms of g_{ij}

Exr 7.1 exponents doesn't depend on $PSL(2,\mathbb{C})$

7.2 Use E6.2 and Exr 6.4 and (7.6) to prove E. Corel's theorem:

$$-\frac{p(p-1)}{2}\sum r_i \le \sum\sum \beta_i^j \le -\sum r_i \qquad (7.17) \Downarrow (7.4)$$

r_i — Poincaré's rank at a_i

$r_i = k_i - 1$ $\quad \frac{B_i(z-a_i)}{z^{k_i}}$ $\quad B_i(0) \in GL$

Exr 7.3 Def HG eq. is Fuchsian eq $p=2$, $n=3$, $\alpha,\beta,\gamma \in \mathbb{C}$

$\beta_0^1 = \beta_1^1 = 0 \quad \beta_0^2 = 1-\gamma \quad \beta_1^2 = \gamma-\alpha-\beta;\ \beta_\infty^1 = \alpha;\ \beta_\infty^2 = \beta$

Use Prop 6.3 and 7.1 to show it has the form:

$$u'' + \frac{\gamma-(\alpha+\beta+1)z}{z(1-z)}u' - \frac{\alpha\beta}{z(1-z)}u = 0$$

7.4 Prove $\forall$ Fuch. eq. $p=2$, $n=3$ is HG up to $Aut(\mathbb{CP}^1)$ and $v = z^a(1-z)^b u$

7.5 Use Exr 6.2, 6.4 and (7.6) to prove

$$c_1(F) - \frac{p(p-1)}{2}\sum_{i=1}^n r_i \le \sum\sum \beta_i^j \le c_1(F) - \sum_{i=1}^n r_i$$