Lecture 12 | Birkhoff standard forms.

$$(12.1) \qquad z\frac{dy}{dz} = C(z)\,y$$

$$(12.2) \qquad C(z) = z^r \sum_{n=0}^{\infty} C_n z^{-n} \qquad C_0 \neq 0 \quad r \geq 0$$

r – Poincare rank. converges in $O_\infty = \{z \in \mathbb{CP}^1 : |z| > R\}$

$$(12.3) \qquad x = T(z)\,y$$

$$(12.1) \rightarrow z\frac{dx}{dz} = \tilde{C}(z)\,x \qquad (12.4)$$

$$(12.5) \qquad \tilde{C} = z\left(\frac{d}{dz}T\right)\cdot T^{-1} + T\cdot C\cdot T^{-1}$$

$T \in GL(p, O_\infty)$ – anal. transf. (Poincare preserves)

$T \in GL(p, \dot{O}_\infty)$ – merom. transf.
merom.

In 1913 Birkhoff proved any (12.1) after anal. tr. is (12.4) with $\tilde{C}(z) = \tilde{C}_r z^r + \dots + \tilde{C}_0$

Def (12.4), (12.6) is Birkhoff standard form for (12.1)

Example 12.1 (Gantmacher/1950's, counterexample to)

$$z\frac{dy}{dz} = \left(\begin{pmatrix}0&0\\0&1\end{pmatrix} + z^{-1}\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)y \qquad \leftarrow r=0$$

Proof Let $T(z) = T_0 + \frac{T_1}{z} + \dots$, $T_0 = Id$

$r=0 \Rightarrow \tilde{C}$ is constant, so (12.5) implies

$$\tilde{C}\left(1 + \tfrac{1}{z}T_1 + \dots\right) = -\tfrac{1}{z}T_1 + \left(1+\tfrac{1}{z}T_1\right)\left(\begin{pmatrix}0&0\\0&1\end{pmatrix} + z^{-1}\begin{pmatrix}0&1\\0&0\end{pmatrix}\right) + \dots$$

$$\Downarrow$$

$$\tilde{C} = \begin{pmatrix}0&0\\0&1\end{pmatrix} \qquad \begin{pmatrix}0&0\\0&1\end{pmatrix}T_1 = -T_1 + T_1\begin{pmatrix}0&0\\0&1\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$\begin{pmatrix}\cdot&\bullet\\\cdot&\cdot\end{pmatrix} \leftarrow$ gives $0=1$

Rem Birkhoff's proof works if monodromy is diagonalizable.

Def (12.1) is reducible if after anal. change $\tilde{C} = \begin{pmatrix}c' & *\\ 0 & c''\end{pmatrix}$ (12.7)

BSF is global problem since $\tilde{C}$ is merom.

Let $Y(z)$ – fund. sol. (12.1), $G = \begin{pmatrix} \rho & * \\ 0 & \end{pmatrix}$, $E = \frac{1}{2\pi i}\ln G$

$0 \le \rho_i < 1$

$$Y(z) = T(z)\, z^E \qquad (12.8)$$

$T(z)$ – 1-valued, $T(z) \in GL(\mathcal{O}_\infty)$

$$F = (\mathcal{O}_\infty, \mathcal{O}_0 = \mathbb{C}, g_{\infty 0} = T(z))$$

D given by $\omega_\infty = \frac{C(z)}{z}dz$ and $\omega_0 = \frac{E}{z}dz$

0 – log. sing

∞ – pole of order $r+1$

$$\frac{C(z)}{z}dz = dY \cdot Y^{-1} = dT \cdot T^{-1} + T\frac{E}{z}dz\, T^{-1}$$

$$\omega_\infty = d g_{\infty 0}\, g_{\infty 0}^{-1} + g_{\infty 0}\, \omega_0\, g_{\infty 0}^{-1}$$

Prop (10.2) $\to$ F is merom. triv. D given by $\left(\frac{\tilde{C}(z)}{z}\right)dz$

(e) – merom. base holom. outside 0

$$\tilde{C}(z) = \tilde{C}_r z^r + \dots + \tilde{C}_0 + \dots + \frac{\tilde{C}_{-k}}{z^k}$$

$[(12.9)$ $\quad \tilde{Y}(z) = \Gamma(z)Y(z) = \Gamma(z)T(z)z^E = U(z)z^E$

defined on $\mathbb{C}$,

(12.9) – no sing except $0, \infty$,

$\Downarrow$

$\tilde{C}$ is rational, pole of order $\cancel{0}$ at 0 and order r at ∞.

$\Rightarrow$ Thm 12.1 $\forall$ (12.1) after anal. is (12.9)

with reg. sing. point at 0.

$G \underset{S}{\mapsto} \begin{pmatrix} 0 & * \\ & \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} \lambda_1 \ge \lambda_2 \ge \dots & 0 \\ 0 & \ge \lambda_p \end{pmatrix}$

$$F^\Lambda = (\mathcal{O}_\infty, \mathcal{O}_0 = \mathbb{C}, g^\Lambda_{\infty 0} = T(z) z^{-\Lambda})$$

D^Λ given by $\omega^\Lambda_\infty = \frac{C(z)}{z}dz$, $\omega^\Lambda_0 = (\Lambda + z^\Lambda E z^{-\Lambda})\frac{dz}{z}$

$\mathcal{E}$ – set of these bundles (similar to Thm 8.1)

Thm 12.2 (12.1) has anal. tr. to BSF $\Leftrightarrow$ $\mathcal{E} \ni$ holom. triv. bundle.

similar to (11.1)

Thm 12.3 $(E, D) \in \mathcal{E}$, (12.1) – irreducible. Then

$$k_i - k_{i+1} \le r \quad \forall i$$

P3

Proof) Thm 9.1 $\Rightarrow$ $\exists\ T \in H^0(\mathcal{O}_\infty),\ U \in H^0(\mathbb{C})$ s.t.

$$T \cdot g_\infty^\wedge = T \cdot T \cdot z^{-\Lambda} = z^{-K} \cdot U \qquad K = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_p \end{pmatrix}$$

(12.4) is $\dfrac{\tilde{C}(z)}{z}\,dz = \dfrac{-K}{z}\,dz + z^{-K}\omega z^{K}$, $\omega = dU U^{-1} + U(\Lambda + z^{\Lambda} E z^{-\Lambda}) U^{-1}\frac{dz}{z}$

$$y'(z) = z^{-K} U(z)\, z^{\Lambda} z^{E} \qquad (12.11)$$

ω has log. sing. at 0.

Assume $k_\ell - k_{\ell+1} > r$ $\qquad \omega'_{ij}(z) = \omega_{ij}(z) \cdot z^{k_j - k_i}$

for $i > \ell$, $j \le \ell$ $\quad \omega'_{ij} = 0$ $\quad$ $\mathrm{ord}_0 \le r-1$, $\mathrm{ord}_\infty \le r-1$

$$\tilde{C}(z) = \begin{pmatrix} C' & * \\ 0 & C'' \end{pmatrix} \Rightarrow (12.1) \text{ is reducible.}$$

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similar to Thm 10.4

Thm 12.4 Irreducible (12.1) can be anal. transf. to BSF.

Proof) Λ-adm, $F \in \mathcal{E}$

$\lambda_i - \lambda_{i+1} \ge r(p-1)$

L.10.2 $\Rightarrow$ $\exists\ T'(z)$ holom. inv. at 0

$$T'(z) z^{-K} U(z) = U''(z) z^{D} \qquad D = \text{transpose}(-K)$$

U'' - hol. inv. in $\mathbb{C}$

(12.10) $\Rightarrow$ $|d_i - d_{i+1}| \le r(p-1)$

$\Rightarrow H = D + \Lambda$ is adm. $\quad h_i > h_{i+1}$

$y'' = T' \cdot y'$; then $y'' = U'' \cdot z^{H} z^{E}$

So it is Fuchsian at 0.

Sing. points are 0 and ∞, so $C''(z)$ holom at $\mathbb{C}$.

Order of pole of C'' at ∞ $= r$.

$\Downarrow$

$C''(z)$ is polyn. of degree r.