Lecture 4: The Levelt theorem (Lecture 6 in Bolibrukh)

(1) $\frac{dy}{dz} = B(z) \cdot y$ near $z = 0$, where we have a regular singular point

X - space of solutions and $\sigma^* : X \to X$ is the monodromy

weak Levelt ~~decomposition~~ basis :

$X = \bigoplus_{i=1}^{s} X_i$, X_i - (generalized) eigenspaces of σ^*

Recall $\varphi : X \to \mathbb{Z} \cup \{\infty\}$

$\varphi(y) = \sup \{ l \mid \lim_{z \to 0} \left| \frac{y(z)}{z^\lambda} \right| = 0 \text{ for all } \lambda < l \}$

e.g. $\varphi(z^{1.2}) = 1$, $\varphi(z^{1+i}) = 1$

$\varphi_i^1 > \varphi_i^2 > \dots > \varphi_i^{m_i}$

$0 \subset X_i^1 \subset X_i^2 \subset \dots \subset X_i^{m_i} = X_i$

$X_i^l = \{ y \in X_i \mid \varphi(y) \geq \varphi_i^l \}$

$Y(z) = [\, Y_1(z) \;\; Y_2(z) \dots \; Y_s(z) \,]$

where $Y_i(z) = [\, Y_{i,1} \,,\, Y_{i,2} \,, \dots Y_{i,m_i} \,]$ columns of $Y_i(z)$ form a basis of X_i

$Y_{i,1}$: form a basis of X_i^1 s.t. matrix of σ^* is uper triang.,

$Y_{i,2}$: project to a basis of X_i^2 / X_i^1 matrix of σ^* in X_i^2 / X_i^1 is upper triangular

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matrix of σ^* matrix of σ_i^* $X_i^1 \subset X_i^2 \subset \cdots \subset X_i^{m_i}$

$$G = \begin{bmatrix} G_1 & & 0 \\ & \ddots & \\ 0 & & G_s \end{bmatrix}, \quad G_i = \begin{bmatrix} G_i^{11} & G_i^{12} & \cdots & G_i^{1,m_i} \\ 0 & G_i^{22} & & \vdots \\ & & \ddots & \\ & & & G_i^{m_i m_i} \end{bmatrix}$$

$G_i^{\ell\ell}$ is the matrix of $\sigma_i^*: X_i^{\ell}/X_i^{\ell-1} \circlearrowleft$

it is upper triangular w/ diagonal entries λ_i, $1 \le i \le s$

$$\Rightarrow \quad E = \frac{1}{2\pi\sqrt{-1}} \ln G = \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_s \end{bmatrix}$$

$$E_i = \begin{bmatrix} E_i^{11} & E_i^{12} & \cdots & E_i^{1m_i} \\ & \ddots & & \vdots \\ & & E_i & \\ O & & & E_i^{m_i m_i} \end{bmatrix}, \quad E_i^{\ell\ell} = \rho_i \cdot I + N_i^{\ell\ell}$$

upper triangular

$$\rho_i = \frac{1}{2\pi\sqrt{-1}} \ln \lambda_i$$

$$0 \le \operatorname{Re} \rho_i < 1$$

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{bmatrix}, \quad A_i = \begin{bmatrix} \varphi_i^1 I & & 0 \\ & \ddots & \\ 0 & & \varphi_i^{\ell} I \end{bmatrix}$$

$$\varphi_i^{\ell} = \varphi(Y_{i,\ell})$$

$\Rightarrow$ the monodromy matrix of $Y(z)$ are upper triangular. More precisely, G and $E = \frac{1}{2\pi i}\log G$

Put

$$A_i := \begin{bmatrix} \psi_i^1 \cdot I & & 0 \\ & \ddots & \\ 0 & & \psi_i^{m_i} I \end{bmatrix}, \quad A := \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{bmatrix}$$

$$E = \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_s \end{bmatrix}, \quad E_i = \begin{bmatrix} E^i_{11} & \cdots & E^i_{1,m_i} \\ & \ddots & \vdots \\ 0 & & E^i_{m_i,m_i} \end{bmatrix}$$

$E^i_{\ell\ell} = \rho^i + N^i_{\ell\ell}$ ← upper triangular

$\rho^i = \frac{1}{2\pi\sqrt{-1}} \log \lambda^i_1, \ 0 \le \rho^i < 1$

then, as we proved last time

$$(2) \qquad Y(z) = U(z)\, z^A\, z^E,$$

where $U(z)$ is holomorphic at $z=0$.

<u>Thm</u> [Levelt] The regular singular point $z=0$ is Fuchsian if and only if the matrix $U(0)$ is invertible.

<u>Pf.</u> Put $B(z) = \frac{B_0(z)}{z}$ and Substitute (2) in (1); then we get

$$(3) \qquad B_0(z)\,U(z) = z\frac{dU}{dz} + U(z)\,L(z), \quad \text{where } L(z) = A + z^A E z^{-A}$$

← holomorphic at 0!

$\Rightarrow$) <u>Assume $z=0$ is Fuchsian</u> $\Rightarrow B_0(z)$ is holomorphic at 0

$$B_0(0)\cdot U(0) = U(0)\,L(0)$$

$\Rightarrow L(0) : \operatorname{Ker} U(0) \longrightarrow \operatorname{Ker} U(0)$. Assume $\operatorname{Ker}(U(0)) \neq 0$ and

Let $\underset{\neq 0}{c} \in \operatorname{Ker} U(0)$ be an eigen-vector of $L(0)$. Put

$$y_c(z) = Y(z).c$$

We compute $\varphi(y_c(z))$ in two different ways.

1-st way:

$$L(z) = \begin{bmatrix} L_1(z) & & 0 \\ & \ddots & \\ 0 & & L_s(z) \end{bmatrix}, \quad L_i(z) = A_i + z^{A_i} E_i \, z^{-A_i}$$

$$= \begin{bmatrix} \psi_i^1 I \cancel{A_i} + E_i^{11} & E_i^{1,2} z^{\psi_i^1 - \psi_i^2} & \cdots & E_i^{1,m_i} z^{\psi_i^1 - \psi_i^{m_i}} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \psi_i^{m_i} I + E_i^{m_i m_i} \end{bmatrix}$$

$$L_i(0) = \begin{bmatrix} (\rho^i + \psi_i^1) I + N_i^{11} & 0 & 0 \\ \vdots & \ddots & \\ 0 & \cdots & (\rho^i + \psi_i^{m_i}) I + N_i^{m_i m_i} \end{bmatrix}$$

$$L(0) = \begin{bmatrix} L_1(0) & & 0 \\ & \ddots & \\ 0 & & L_s(0) \end{bmatrix} \Longrightarrow c = \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix}, \quad c_i = \begin{bmatrix} c_i^1 \\ \vdots \\ c_i^{m_i} \end{bmatrix}$$

c_i^l is a vector of size $\dim X_i^l / X_i^{l-1}$

c is an eigenvector of $L(0)$ only if $c_i^l \neq 0$ for precisely one pair (i, l), $1 \le i \le s$, $1 \le l \le m_i$ and the eigen-value of c is $\rho^i + \psi_i^l$

Note that $y_c = Y(z).c$ is a linear combin. of the solutions belonging to $Y_{i,l}$ (← they project to a basis X_i^l / X_i^{l-1})

$$\Longrightarrow \boxed{\varphi(y_c) = \psi_i^l}$$

2-nd way:

$$y_c(z) = U(z)\, z^A z^E \cdot c$$

Note that $E = R + N$, where $R = \begin{bmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_s \end{bmatrix}$

$R_i = \rho_i I$. We get $\left([R, N] = 0 \,! \right)$

$$y_c(z) = U(z)\, \underbrace{z^A z^N z^{-A}}_{} \underbrace{z^A z^R \cdot c}_{= z^{\rho_i + \psi_i^l} c} =$$

$$z^N = 1 + \sum_{k=1}^{\infty} \frac{(\ln z)^k}{k!} N^k$$

$$= z^{\rho_i + \psi_i^l}\, U(z) \left(1 + \sum_{k=1}^{p-1} \frac{(\ln z)^k}{k!} \left(z^A N z^{-A} \right)^k \right) \cdot c$$

We have:

$$z^A N z^{-A} = z^A (E - R) z^{-A} = z^A E z^{-A} - R = L(z) - A - R =$$

$$= L(0) - A - R + O(z)$$

and $(L(0) - A - R) \cdot c = 0$ (since c is an eigenvector of $L(0)$ w/ eigenvalue $\rho_i + \psi_i^l$)

$$\Rightarrow y_c(z) = z^{\rho_i + \psi_i^l} \Big(\underbrace{U(z) \cdot c}_{= \underbrace{U(0)\cdot c}_{0} + O(z)} + O\big(z (\ln z)^{p-1}\big) \Big) = z^{\rho_i + \psi_i^l + 1}\, O\big((\ln z)^{p-1}\big)$$

if $\lambda < \psi_i^l + 1$ then $\lim_{z \to 0} \frac{y_c(z)}{|y|^\lambda} = 0 \Rightarrow \varphi(y_c(z)) \geq \psi_i^l + 1$

contradiction. ■

$\Leftarrow$) From formula (3) we have

$$B_0(z) = z \frac{dU}{dz} U^{-1}(z) + U(z) L(z) U^{-1}(z)$$

if $U(0)$ is invertible then the RHS is holomorphic. □

Corollary. The Levelt's thm holds for Levelt's basis as well.

Pf. e' – Levelt's basis $\quad 0 \subset X^1 \subset X^2 \subset \dots \subset X^m = X$

$$Y_{e'}(z) = U'(z)\, z^{A'} z^{E'} = U(z)\, z^{A} z^{E} \cdot S$$

S – const. matrix (invertible)

Note that $\det(z^{E'}) = z^{tr(E')} = z^{tr(\sigma^*)\frac{1}{2\pi\sqrt{-1}}} = \det(z^E)$

$$\Rightarrow \det(U'(z)) \cdot \det z^{tr A'} = \det(U(z))\, z^{tr A} \cdot \det(S)$$

$$0 \subset X^1 \subset X^2 \subset \dots \subset X^m = X$$

$$\underbrace{X^1 \cap X_i = \dots = X^{k_i^0} \cap X_i}_{0} \subset \underbrace{X^{k_i^0+1} \cap X_i = \dots = X^{k_i^1} \cap X_i}_{X_i^1} \subset \dots \subset \underbrace{X^{k_i^{m_i-1}+1} \cap X_i = \dots = X^{k_i^{m_i}} \cap X_i}_{X_i^{m_i}}$$

$$\Rightarrow tr A_i = \sum_{l=1}^{m_i} \psi_i^l \dim\left(X_i^l / X_i^{l-1}\right) = \sum_{l=1}^{m} \psi^l \cdot \dim\left(X^l \cap X_i / X^{l-1} \cap X_i\right)$$

$$\Rightarrow tr A = \sum_{i=1}^{s} tr A_i = \sum_{l=1}^{m} \psi^l \cdot \dim\left(X^l / X^{l-1}\right) = tr A'$$

$$\Rightarrow \det(U'(z)) = \det(U(z)) \det(S) \quad \square$$

Lecture 5: The global theory (Lecture 7 in Bolibrukh)

1. Exercises.

Def: If $\frac{dy}{dz} = \frac{B_0(z)}{z^k} \cdot y$, $B_0(0) \neq 0$ has a regular singular point; then k is called Poincaré rank of the singularity.

$b := \varphi(\det U(z))$.

Claim 1: $b \geq r$.

Claim 2 [Saavage]: If $U(z)$ is holomorphic around $z=0$, invertible outside 0 (i.e. for $z \neq 0$); then $\exists$ $\Gamma(z)$ is holomorphic at $z=0$ and $c_1 \geq c_2 \geq \dots \geq c_p = 0$ s.t.

$$\Gamma(z) U(z) = z^C V(z)$$

where $C = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_p \end{bmatrix}$, $V(z)$ is holom. and $V(0)$ is invertible.

Claim 3. Prove that $b \leq \frac{p(p-1)}{2} r$.

Hint: Use Claim 2. and prove $c_i - c_{i+1} \leq r$ $\forall i$.

2. Fuchsian systems on $\mathbb{P}^1$.

F – holomorphic v.b. on $\mathbb{P}^1$ w/ merom. connection ∇

$a_1, \dots, a_n \in \mathbb{P}^1$ the set of sing. points ($\infty \notin \{a_1, \dots, a_n\}$)

O_i : small neighbhood of a_i $\Rightarrow$ horiz. sections of ∇ are given by

$$\frac{dy}{d\xi_i} = B_i(\xi_i)\cdot y \quad , \quad \xi_i = z - a_i$$

we have local invariants $\quad Y_i(z) = U_i(z)\, z^{A_i} z^{E_i}$

ρ_i^j, $1 \le j \le m$, φ_i^ℓ, $1 \le \ell \le m$ $\quad , \quad \beta_i^j = \rho_i^j + \varphi_i^j$ Levelt's exponents

<u>Questions 1:</u>

1) What is ∇ for trivial v.b. F

2) What are the relations between Levelt's filtrations and exponents in different points.

3) Conditions on (β_i^j) for ∇ to be Fuchsian.

Assume F is trivial and ∇ is Fuchsian. Then the system looks:

$$dy = \omega \cdot y$$

where ω is a 1-form on $\mathbb{P}^1$

<u>Define</u> $\quad B_i = \operatorname{res}_{z=a_i} \omega \quad \Rightarrow \quad \omega - \sum_{i=1}^{n} \frac{B_i}{z-a_i} dz \in \Gamma(\mathbb{P}^1, \overset{\mathcal{O}(-2)}{\overset{\shortparallel}{\Omega_{\mathbb{P}^1}}}) = 0$

$$\Rightarrow \quad \omega = \left(\sum_{i=1}^{n} \frac{B_i}{z-a_i} \right) dz \quad , \quad \sum_{i=1}^{n} B_i = 0$$

<u>Thm 7.1.</u> If ∇ is a ~~[illegible]~~ connection w/ regular singular points on a trivial bundle (on $\mathbb{P}^1$); then

(a) $\quad \Sigma := \sum_{i=1}^{n} \sum_{j=1}^{p} \beta_i^j \le 0$ and $\Sigma \in \mathbb{Z}$

(b) ∇ is Fuchsian $\Leftrightarrow$ $\Sigma = 0$.

Pf:

$$\det Y_i(z) = c_0 \exp\left(\int \operatorname{tr} B_i(z)\, dz\right)$$

$$\| $$

$$\det U_i(z) \cdot (z-a_i)^{\operatorname{tr} A_i + \operatorname{tr} E_i} = h(z)\, (z-a_i)^{b_i + \sum_{j=1}^{p} \beta_i^j}$$

$$\Rightarrow \operatorname{tr} B_i(z)\, dz = d\log\left(\det Y_i(z)\right) \qquad b_i = \operatorname*{ord}_{z=a_i}\left(\det\left(U_i(z)\right)\right)$$

$$\Rightarrow \operatorname*{res}_{z=a_i} \operatorname{tr} B_i(z)\, dz = b_i + \sum_{j=1}^{p} \beta_i^j$$

$$\Rightarrow 0 = \sum_{i=1}^{n} \operatorname*{res}_{z=a_i} \operatorname{tr} B(z)\, dz = \sum b_i + \Sigma$$

$$\Rightarrow \Sigma = -\sum_{i=1}^{n} b_i \leq 0 . \quad \square$$

3. Fuchsian equations.

$$u^{(p)} + q_1(z)\, u^{(p-1)} + \cdots + q_p(z) \cdot u = 0$$

$a_1, a_2, \ldots, a_n = \infty$ singular points

In coordinate $\zeta = z^{-1}$

$$(\partial_z)^j = (-\zeta^2 \partial_\zeta)^j = \sum_{i=1}^{j} c_i^j \zeta^{i+j} \partial_\zeta^i$$

$\Rightarrow$ we get

(7.8) $$\left(\partial_\zeta^p + \tilde{q}_1(\zeta)\, \partial_\zeta^{p-1} + \cdots + \tilde{q}_p(\zeta)\right) \cdot u = 0$$

(7.8) is Fuchsian at $\zeta = 0$ iff $R_i(\zeta) = \zeta^{-i} q_i(\zeta^{-1})$ is holom. at $\zeta = 0$ $\quad 1 \leq i \leq p$.

$$q_i(z) = \frac{r_i(z)}{\left[(z-a_1)\cdots(z-a_n)\right]^i} \qquad \text{where } r_i(z) \text{ is holom. } ~~\text{on}~~ \text{ in } \mathbb{C} \subset \mathbb{P}^1$$

at $z=\infty$, $r_i(z)$ has a polynomial growth z^{k_i}, $k_i \le (n-2)i$

$r_i(z) = q_i\left[(z-a_1)\cdots(z-a_n)\right]^i$ has degree $(n-2)i$

$\Rightarrow$ r_i is a polynomial of degree k_i+1

$\Rightarrow$ # of parameters is

$$\boxed{N = \sum_{i=1}^{p} (k_i+1) = (n-2)\frac{p(p+1)}{2} + p}$$

<u>Thm 2.</u> For Fuchsian equations we have:

$$\sum_{i=1}^{n} \sum_{j=1}^{p} \beta_i^j = (n-2)\frac{p(p-1)}{2}$$

<u>Pf.</u> Assume $z=\infty$ is not a singular point. Switch to a system:

$$y^\ell = \prod_{i=1}^{n} (z-a_i)^{\ell-1}\, \partial_z^{\ell-1} u, \qquad 1 \le \ell \le p$$

$\Rightarrow$ new system is Fuchsian w/ same exponents at $a_1, \ldots, a_n$

Choose a basis $\{e_1, \ldots, e_p\}$ for the equation $\Rightarrow$

$$Y(z) = \Gamma(z)\cdot W(z), \qquad W = \begin{bmatrix} e_1 & \cdots & e_p \\ e_1' & \cdots & e_p' \\ \vdots & & \vdots \\ e_1^{(p-1)} & \cdots & e_p^{(p-1)} \end{bmatrix}$$

$$\Gamma(z) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \left(\prod_{i=1}^{n}(z-a_i)\right)^{p-1} \end{bmatrix}, \qquad W(z) = \Gamma_1(z)\,\Gamma_2(z)\,V(\zeta)$$

Wronskian w.r.t. ζ

invertible near $\zeta=\infty$ for all ζ.

$$\Gamma_1(z) = \begin{bmatrix} 1 & & 0 \\ & z^{-2} & \\ & & \ddots \\ 0 & & z^{-2(p-1)} \end{bmatrix}, \qquad \Gamma_2(z) = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ * & & & \ddots \end{bmatrix}$$

$$\Rightarrow \varphi_{\zeta=\infty}\left(\det(W(z))\right) = p(p-1), \qquad \varphi_{\zeta=\infty}(\det\Gamma) = -n\frac{p(p-1)}{2}$$

<u>Def.</u> Degree of a vector bundle F on $\mathbb{P}^1$ is

$$c_1(F) = \sum_{\substack{i: \text{sing.} \\ \text{of the connection}}} \operatorname{res}_i \det \nabla$$

(independent of the choice of a meromorphic conn. ∇ on $\mathbb{P}^1$)

We have similar results for non-trivial bundle F:

<u>Thm.</u> If ∇ is a connection on F w/ regular singularities then

(a) $\Sigma := \sum_{i=1}^{n} \sum_{j=1}^{p} \beta_i^j \le c_1(F)$;

(b) ∇ is logarithmic iff $\Sigma = c_1(F)$. $\boxed{\text{M}}$

Exercises.

1) Exponents does not change under $\mathrm{Aut}(\mathbb{P}^1)$

2) Use Claim 1 and 3 ~~that~~ to prove that

$$-\frac{p(p-1)}{2} \sum_{i=1}^{n} r_i \le \sum_{i=1}^{n} \sum_{j=1}^{p} \beta_i^j \le -\sum_{i=1}^{n} r_i$$

r_i – Poincaré rank at a_i

3) Hypergeometric equation is a Fuchsian equation w/ $n=3$ $\{0,1,\infty\}$, $p=2$ and exponents $\beta_0^1 = \beta_1^1 = 0$

$\beta_0^2 = 1-\gamma$, $\beta_1^2 = \gamma-\alpha-\beta$, $\beta_\infty^1 = \alpha$, $\beta_\infty^2 = \beta$