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MUTATIONS OF POTENTIALS

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Abstract. In two-dimensional case we develop the theory of mutations of potentials and prove the Laurent phenomenon. This is an extension of the theory of cluster algebras for case when number of directions of mutations could be (much) higher than number of variables, but at least one function remains Laurent polynomial after all mutations. The motivating examples are potentials that are Fukaya-Oh-Ohta-Ono mirror images of special Lagrangian tori on del Pezzo surfaces and Auroux wall-crossing formula relating invariants of different tori.

1. Introduction

Let x_i be coordinates on n-dimensional algebraic torus $T = \mathbf{C}^{*n}$. Fix logarithmic volume form

$$\omega = \frac{1}{(2\pi i)^n}\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n} = \frac{1}{(2\pi i)^n} d\log x_1\wedge\cdots\wedge d\log x_n \tag{1}$$

On the space $L = \mathbf{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ of Laurent polynomials $W \in L$ consider the functionals of constant term $Tr(W) = \int_{|x_i|=1} W\omega$, constant term of d-th power $g_d(W) = Tr(W^d)$, and their generating functions G, $\hat{G}$ sending Laurent polynomial $W \in \mathbf{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ to power series of one variable:

$$G_W(t) = \sum_{d\geqslant 0} g_d(W)\frac{t^d}{d!} = \int_{|x_i|=\epsilon} e^{t\cdot W}\omega \tag{2}$$

$$\hat{G}_W(t) = \sum_{d\geqslant 0} g_d(W)t^d = \int_{|x_i|=\epsilon} \frac{1}{1-tW}\omega \tag{3}$$

We are interested in the following problem

Problem 4. *Describe image and fibers of functional G i.e. given power series $G(t) \in L$ find the space $L_G = \{W \in L | G_W(t) = G(t)\}$.*

As a first step in solving it we address the following problem

Problem 5. *Find the group of fiberwise automorphisms of G, and more generally – describe transformations preserving G i.e. find when $G_W = G_{W'}$.*

Definition 6. *Special Cremona group* $SCr = SCr_n(\mathbf{C})$ is a subgroup of Cremona group $Cr_n(\mathbf{C}) = \operatorname{Aut}\mathbf{C}(x_1, \ldots, x_n)$ preserving volume form ω. Its elements are called special birational transformations. SCr_2 is also called *Symp*. SCr is subgroup of index 2 in $SCr^+ = \{f \in Cr | f^*\omega = \pm\omega\}$.

Definition 7. A pair $(W, f) \in (L, Cr)$ is called a *special (Laurent) pair* if f is a special birational transformation and $W' = f^*W$ is a Laurent polynomial.

Lemma 8. *If (W, f) is a special pair, then $G_W = G_{W'}$.*

Proof. Change of coordinate under the integral. □

Group Cr has maximal torus T and its normalizer $N(T)$ with "Weyl group" $N(T)/T = GL(n, \mathbf{Z})$ acting by monomial transformations. Group SCr^+ inherits the same maximal torus and normalizer, while for SCr we have $N_{SCr}(T)/T = SL(n, \mathbf{Z})$.

Symmetries of G contain the irrelevant continuous group T. To get rid of these symmetries we restrict our attention to subspace $L_N \subset L$ of normal Laurent polynomials: Laurent polynomial W is *non-degenerate* if its

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Newton polytope (convex hull of characters with non-zero coefficients) is n-dimensional and contains 0 (trivial character of T) inside interior.

Definition 9. Nondegenerate Laurent polynomial W is *normal* if all its coefficients at the *vertices* of Newton polytope are 1s.

Laurent polynomials we consider are normal.

Problem 10. *Given normal Laurent polynomial W, find transformations $f \in SCr$ such that f^*W is normal Laurent polynomial.*

There are obvious transformations of $SL(n, \mathbf{Z})$. In next section we describe non-trivial transformations for $n = 2$.

1.1. **Preliminaries.** Let us briefly recall the results of [12]. Let $S \subset \mathbf{Z}^2$ be the set of primitive vectors in $\mathbf{Z}^2$, i.e. vectors with coprime coordinates. For a vector $u \in S$ we define a <u>piecewise linear mutation</u> to be an automorphism of the set $\mathbf{Z^2}$ given by the formula:

$$\mu_u^{pl} : v \mapsto v + max(< u, v >, 0)u,$$

where $< u, v >$ is a antisymmetric bilinear form on $\mathbf{Z}^2$, normalized in a way that the value on generators is $< (1, 0), (0, 1) >= 1$.

For a vector $u \in S$ we define a <u>mutation</u> in the direction u as a birational automorphism of $\mathbb{P}^2$ given by the formula:

$$\mu_{(m,n)} : x^a y^b \mapsto x^a y^b (1 + x^n y^{-m})^{an-bm}$$

In particular the mutation in the direction $(0, 1)$ is given by:

$$\mu_{(0,1)} : (x, y) \mapsto (x, \frac{y}{1 + x}).$$

The tropicalization of this morphism gives a map:

$$\mu_{(0,1)}^{pl} : (a, b) \mapsto (a, b - min(0, a)).$$

The geometric meaning of the tropicalization is the following. Suppose we have a toric surface X given by the fan T. Then $T' = \mu_v^{pl}(T)$ is another fan, defining toric surface X'. Let D_v be the toric divisor on X corresponding to the vector v, and s is the point on D_v with coordinate -1. Let D'_{-v} be the toric divisor on X' corresponding to the vector $-v$, and s' is the point on D'_{-v} with coordinate -1. Then by the results of [12], there is a surface $\widetilde{X}$ and maps

$$\pi : \widetilde{X} \to X,$$
$$\pi' : \widetilde{X} \to X',$$

where π is the blow-up of X at s, and π' is the blow-up of X' at s'. This gives a resolution of birational isomorphism

$$\mu_v = \pi' \circ \pi^{-1}.$$

Moreover strict transform of toric divisors from X to $\widetilde{X}$ equals strict transform of toric divisors from X'. The correspondence between toric divisors is given by the map μ^{pl}. Namely we have:

$$\pi_{st}^* D_t = \pi_{st}^{'*} D_{\mu_v^{pl}(t)},$$

where π_{st}^* denotes strict transform.

2. Mutations

2.1. **Properties of potential.** Consider a toric surface X with rational function W, called potential.

Let us introduce a curve C defined by the formula:

$$C - \sum_t n_t D_t = (W),$$

where $\sum_t n_t D_t$ is the part of (W) supported on toric divisors.

The open toric orbit has specific toric coordinates x, y, which we use as rational coordinates on X.

We denote D_t the divisor corresponding to the ray $t \in \mathbf{Z}^2$, as well as all its strict transforms. If $t = (a, b)$, then the function $\frac{x^b}{y^a}$ gives a rational function $D_t \to \mathbb{P}^1$, which we call the canonical coordinate. We consider it up to taking its inverse. Each toric divisor has the point, where canonical coordinate equals -1. We denote the set of all such points by Ω.

To such a pair (X, C) we associate a set of vectors $V \subset \mathbf{Z}^2$ with multiplicities, which will encode the way the curve C intersects toric divisors. If the curve C intersects divisor corresponding to a vector v transversally, then vector v enters V the number of times equal to the multiplicity of intersection. If the intersection of C with such divisor is not transversal, then we count the correct multiplicities using blow-ups. Let $s \in D_v \subset X$ be a point where the canonical coordinate equals -1, and C intersects D_v in s. Then we make a blow-up of X in s, and we denote E_1 the exceptional curve of the blow-up. Then we blow-up the point of intersection of E_1 and the strict transform of D_v, and we denote E_2 the exceptional curve of the blow-up. We continue by induction, so that E_k is the exceptional curve of the blow-up at intersection of E_{k-1} and the strict transform of D_v. We denote n_k the index of intersection of the strict transform of C with the curve $E_k \backslash (E_k \cap E_{k+1})$. In the last formula we just remove one point of intersection of E_k with E_{k+1}. Of course, there will be only finite number of E_k which intersect C, so we need to consider only finite number of blow-ups. Then vector kv enters set V with multiplicity n_k.

Example 11. We consider $\mathbb{P}^2$ with potential $W_9 = x + y + \frac{1}{xy}$. The curve defined by the equation $W = 0$ is an elliptic curve, intersecting toric divisors at toric points. Let us consider a toric surface X_0 given by fan:

$$(2,-1), (1,-1), (0,-1), (-1,-1), (-1,0), (-1,1), (-1,2), (0,1), (1,0).$$

This surface is a blow-up of $\mathbb{P}^2$ at 6 points, and the strict transform of $W = 0$ is the smooth elliptic curve C_0 that intersects transversally 3 toric divisors $D_{(2,-1)}, D_{(-1,-1)}, D_{(-1,2)}$. In particular, the set V for the pair (X_0, C_0) is $V_0 = \{(2,-1), (-1,-1), (-1,2)\}$.

Definition 12. By analogy with cluster mutations, we define the *seed* to be a triple (X, W, V), where X is a toric surface, W is a rational function on X, called potential, and $V = \{v_1, \ldots, v_n\}$ is a tuple of vectors in $\mathbf{Z}^2$.

The seed can be mutated in either of n directions v_i.

Definition 13. The cluster mutation μ_i of seed (X, W, V) in i'th direction is a new seed (X', W', V') defined as follows.
$V' = \mu_i(V) = \{v'_1, \ldots, v'_n\}$ such that $v'_i = \mu^{seed}_{v_i}(v_i) = -v_i$ and $v'_j = \mu^{seed}_{v_i}(v_j) = \mu^{pl}_{v_i}(v_j)$ for $j \neq i$.
X' is the toric surface, whose fan is obtained from the fan of X by applying $\mu^{pl}_{v_i}$.
The function W' is the pull-back of W under birational isomorphism μ_{v_i}.

Note, that if compose mutation in direction v with mutation in direction $-v$, then we obtain the seed, which is related to the original seed by the action of a unipotent element of $SL(2, \mathbf{Z})$.

We choose initial seed (X_0, W, V_0), and then we start to apply mutations in different directions. In this way we obtain the set of seeds.

Definition 14 (Property U)**.** We say that seed (X, W, V) satisfy property U if the following conditions hold

1. C is an effective divisor i.e. W is a Laurent polynomial;
2. $C = A + B$, where A is the irreducible non-rational curve and B is supported on rational curves;
3. The intersection of C with toric divisors belongs to the set Ω;
4. If $t \in V$, then the intersection index $k_t = (C \cdot D_t) \geqslant n_t$
5. For toric divisor D_t intersection index $(A \cdot D_t)$ equals to number of i such that $v_i = t$

Given a seed (X, W, V) we can define curve C by the equation

$$C - \Sigma_t n_t D_t = (W),$$

where $\Sigma_v n_v D_v$ is the part corresponding to toric divisors. Recall, that there is surface $\widetilde{X}$ and maps

$$\pi : \widetilde{X} \to X,$$

$$\pi' : \widetilde{X} \to X',$$

where π is a blow-up of the point on $D_v \subset X$, and π' is a blow-up of the point on $D_{v'} \subset X'$.
For the seed (X', F', V') we have the curve C' given by

$$C' - \Sigma_t n'_t D_t = (W').$$

Now we prove the following

Lemma 15. *If seed (X, F, V) satisfy property 14 then its mutation (X', F', V') also satisfy property 14.*

Proof. The birational transformation $\mu_u : X \to X'$ is decomposed as a blow-up π and blow-down π'. Let E be the exceptional curve of π, and E' the exceptional curve of π'. After blowing up π at the intersection of C and D_u we have:

$$(\pi^* F) = \pi^*_{st} C + (k_u - n_u)E + \Sigma_v n_v D_v.$$

The divisor $C'' = \pi^* C + (k_u - n_u)E$ is effective, because $k_u \geqslant n_u$. From the other side

$$(\pi^* F') = (\pi^* F) = (\pi')^*_{st} C' + (k'_{-u} - n_{-u})E' + \Sigma_{v'} n_{v'} D_{v'}.$$

In [12] we proved, that canonical coordinates on toric divisors are preserved by μ_u. It implies that the set $\Omega \subset X$ of points with coordinate -1 maps by μ_u to the corresponding set on X', except for points on divisors D_u, D_{-u}, where we are making blow-ups. But C' can intersect D_u, D_{-u} only at the set Ω, which proves the first statement of the lemma.

From (2.1) we deduce that

$$C'' = (\pi')^*_{st} C' + (k'_{-u} - n_{-u})E'.$$

As divisor C'' is effective, we have that $k'_{-u} \geqslant n_{-u}$. The intersection index of C' with D_t for $t \notin \{u, -u\}$ is the same as the corresponding intersection of C. This implies the second statement of the lemma. Moreover as strict transform of C' is effective, then C' is effective as well.

We also have:

$$C' = \pi'_* \circ (\pi^*_{st} C + (k_v - n_v)E),$$

which implies that C' contains elliptic curve $A' = \pi'_* \circ \pi^*_{st}(A)$, and possibly additional rational curve $\pi'(E)$, which proves the third statement. C' intersects toric divisors

The strict transform $\pi^*_{st}(A)$ only intersects divisors D_v, D_w. So divisor $A' = \pi'_* \circ \pi^*_{st}(A)$ can only intersect $D_{u'}, D_{v'}, D_{w'}$. □

Theorem 16. *If seed (X, W, V) satisfies property 14 then function W in all the seeds is a Laurent polynomial.*

Proof. Lemma 15 implies that the divisor of W defines effective curve on the open toric orbit, in other words it has poles only on the locus of toric divisors. Therefore, W is a Laurent polynomial. □

3. Examples

Consider next 10 normal Laurent polynomials:

(17)
$$W_9 = x + y + \frac{1}{xy}$$

(18)
$$W_Q = x + y + \frac{1}{x} + \frac{1}{y}$$

(19)
$$W_8 = x + y + \frac{1}{xy} + xy$$

(20)
$$W_7 = (1 + x + y)(1 + \frac{1}{xy}) - 1$$

(21)
$$W_6 = (1 + x)(1 + y)(1 + \frac{1}{xy}) - 2$$

(22)
$$W_5 = (1 + x + y)(1 + \frac{1}{x})(1 + \frac{1}{y}) - 3$$

(23)
$$W_4 = \frac{(1 + x)^2(1 + y)^2}{xy} - 4$$

(24)
$$W_3 = \frac{(1 + x + y)^3}{xy} - 6$$

(25)
$$W_2 = \frac{(1 + x + y)^4}{xy} - 12$$

(26)
$$W_1 = \frac{(1 + x + y)^6}{xy^2} - 60$$

Proof of the next lemma is straightforward

Lemma 27. *Laurent polynomials satisfy property U 14 with the last condition properly modified.*

Theorem 16 implies that

Corollary 28. *All consecutive mutations of potentials W_n (from example 17) are Laurent polynomials.*

Projective plane and Markov's triplets. Triplets of positive integer numbers (x, y, z) satisfying Markov's equation

$$x^2 + y^2 + z^2 = 3xyz \tag{29}$$

are in charge of two numerologies for the projective plane $\mathbb{P}^2$.

These numbers are the ranks of exceptional bundles for full exceptional collections in the derived category of coherent sheaves ([6]) and their squares are the weights of Prokhorov–Hacking's degenerations of the plane to weighted projective plane $\mathbb{P}(x^2, y^2, z^2)$ ([7]).

We don't know yet an explicit construction relating degenerations of the surface with exceptional collections (however there is one proposed in [8]).

In this article we introduce two more interrelated hierarchies ruled by Markov's triplets — a hierarchy of *Laurent polynomials* and a derived hierarchy of *cluster collections*. The origin of our hierarchies is a Laurent polynomial $x + y + \frac{1}{xy}$ mirror dual to $\mathbb{P}^2$.

Although our main theorem and material in sections 1, 2 is of purely algebro-geometric origin and do not involve mirror symmetry or degenerations, our motivation lies in these two fields so we briefly describe the history and impact of the problem in the next paragraph.

Toric degeneration ansatz. Batyrev proposed a method to construct Laurent polynomials mirror dual to Fano varieties using *small toric degenerations* ([2]).

By generalized *toric degeneration ansatz* we mean the following:

Conjecture 30. *Given a* good *toric degeneration X of smooth Fano variety Y one may construct its* mirror dual reflection *as a Laurent polynomial W associated with X.*

Definition 31. We say that Laurent polynomial W is associated with toric variety X if Newton polytope of W is the fan polytope of X.

To make 30 explicit one has to specify the definitions of good toric degeneration, mirror duality and (hopefully) provide a constructive way to find the coefficients of W.

In original Batyrev's setup good stays for small toric degeneration: that is a degeneration to a toric Fano variety with Gorenstein terminal singularities and an extra condition of preserving Picard group.

This notion for good is too restrictive — in particular all terminal surfaces are smooth, so small toric degeneration of surfaces cover only trivial case of smooth toric del Pezzo surfaces (i.e. 5 del Pezzo surfaces of degree $d \geqslant 6$).

One may weaken the restriction by considering toric degenerations with Gorenstein non-terminal singularities, or weaken even further by considering toric degenerations with canonical singularities. In case of surfaces these two conditions are equivalent [1]

In [5] we have shown that indeed toric degeneration ansatz for surfaces can be generalized to this setting and proven ad hoc: there are 16 toric del Pezzo surfaces with du Val singularities (including 5 smooth), every smooth del Pezzo surface of degree $d \geqslant 3$ admits degeneration to one of these, for every of these degenerations one may construct a Laurent polynomial that is mirror dual to smooth del Pezzo surface in question (in the sense of variations of Hodge structures).

Gorenstein degenerations for quadric and surfaces of degrees 4,5,6,7 are not unique and this ambiguity provides us with the key observation

Conjecture 32 ([5])**.** *Laurent polynomials constructed from different toric degenerations of the same surface* [2] *are related to each other by birational transformations preserving the logarithmic volume form $\omega = \wedge \frac{dx_i}{x_i}$.*

Unfortunately del Pezzo surfaces of degree 1 and 2 do not admit any Gorenstein toric degenerations. However if one completely omits any restriction on the singularity (except normality [3]) then each family of del Pezzo surfaces

[1] Moreover, toric du Val singularities are just singularities of type A_n.

[2] Similar equivalence between mirror pairs obtained from some degenerations of partial flag manifolds constructed by Alexeev and Brion was observed by Rusinko ([11]).

[3] Despite degenerations to reducible varieties proved to be extremely useful and powerful tool , all degenerations we consider in this article have normal singularities.

admits *infinitely many* toric degenerations, and it turns out *all* of them are suitable for constructing a mirror dual Laurent polynomial. In other words, we will show that for del Pezzo surfaces *any* degeneration to normal toric Fano surface is good: in what follows we justify the toric degeneration ansatz for all possible toric degenerations of the projective plane i.e. we prove the following

Theorem 33. *For every $\mathbb{Q}$-Gorenstein toric degeneration of the projective plane there is reflection of the projective plane associated with the degenerate toric surface.*

Definition 34. We say that Laurent polynomial W is a reflection of projective plane if for all integer $d \geqslant 0$ constant term of W^d equals $\frac{3d!}{d!^3}$.

We construct polynomials W by systematically exploiting the observation 32. Special birational transformations were studied in the work [12]. Once one correctly defines the transformations in question (we call them *mutations*) the only remaining issue is to prove the *Laurent phenomenon* that states

Theorem (Theorem 16 in section 2)**.** *All mutations of initial potential are also Laurent polynomials.*

It turns out potential should satisfy rather strong conditions 14 that are interesting on their own.

Whole hierarchy of toric degenerations is known only in the case $\mathbb{P}^2$. In this we can prove the toric degeneration ansatz 30

Proof. Let W' be a consecutive mutation of W_9 (from example 17). Theorem 16 implies W' is Laurent polynomial. Lemma 8 implies $Tr(W'^d) = Tr(W_9^d) = \frac{3d!}{d!^3}$. It remains to compute the Newton polytope of W'.

Suppose that (u, v, w) are vectors from the seed V in the clock-wise order. Consider the triple of positive integers

$$(a, b, c) = (< u, v >, < v, w >, < w, u >).$$

Mutation μ_u sends (u, v, w) to $(v, -u, w+ < u, w > u)$. The triple (a, b, c) goes to

$$(a, c, ac - b).$$

Lemma 35. *(a, b, c) are positive numbers for all the seeds.*

Proof. For the starting seed (X_0, W, V_0) we have $(a, b, c) = (3, 3, 3)$. Note, that transformation $(a, b, c) \mapsto (a, c, ac-b)$ is the same, as the law for producing Markov numbers. This triple verify the formula:

$$a^2 + b^2 + c^2 = abc.$$

For fixed a, c it is a quadratic equation on b. So we can find another root by formula: $b' = ac - b$ or $b' = \frac{a^2+c^2}{b}$. The second formula implies that this numbers are always positive. □

This lemma implies, that vectors (u, v, w) from the seed are not colinear. From the other side Lemma (14) implies that elliptic curve A intersects toric divisors only at D_u, D_v, D_w. Let e_u, e_v, e_w be the corresponding indexes of intersection. Then the intersection theory on toric surfaces implies, that

$$e_u u + e_v v + e_w w = 0.$$

As we know that (u, v, w) are not colinear, we deduce that e_u, e_v, e_w are non-zero, thus A has non-zero intersection with D_u, D_v, D_w. In particular, vectors (u, v, w) can be reconstructed from (X, F).

□

For other del Pezzo surfaces full classification of degenerations is known only for degenerations to surfaces with Picard number 1 ([8]). We can show that all smoothable toric del Pezzo surfaces with $\rho = 1$ are associated with some of Laurent polynomials described in 28. Moreover we conjecture that answers to degeneration problem and Laurent phenomena problem coincide

Conjecture 36. *Let W' be a Laurent polynomial derived from polynomial W_d (listed in 17) by a sequence of mutations. Then toric surface associated with W' is smoothable to S_d. Conversely, if del Pezzo surface of degree d has a degeneration to toric Fano surface X then X is associated with one of mutations W' of W_d.*

We conclude with some open problems, conjectures and (counter)examples our investigation has lead us to.

Next problem is related to Conjectures 1 and 2 of [12] (now theorems of Jeremy Blanc [3]).

Problem 37 ("Sarkisov program")**.** *Show or disprove that any special Laurent pair $f^*W = W'$ can be decomposed into a chain of mutations.*

Problem 38 (Integrality of coefficients)**.** *Why coefficients of Laurent polynomials remain integral?*

One of the ways we tried to settle positivity led us to formulate the

Problem 39 ($SCr_d(\mathbf{Z})=?$). *Define the notion of special Cremona group over ring of rational integers, then show or disprove that H (group generated by mutations in all directions) is a group of finite index there.*

Problem 40 (Positivity of coefficients). *Why coefficients of Laurent polynomials remain positive?*

Positive solution to the next problem will also settle 38 and 40

Problem 41. *Find interpretation of Laurent polynomials as $\mathbf{Z}^n$-graded dimensions of some vector space.*

Provisional vector space should be related to some geometry of varieties in question.

Problem 42 (special birational is mirror to deformation). *Given a special Laurent pair $f^*W = W'$, show (or disprove) that respective toric Fano varieties are deformation-equivalent. That is construct a flat family over connected base with one fiber isomorphic to $T(W)$ and another fiber isomorphic to $T(W')$.*

Example 43 (42 for plane). Consider two degenerations $X_0 = \mathbf{P}(x^2, y^2, z^2)$ and $X_0' = \mathbf{P}(x^2, y^2, z\prime^2)$ with associated Laurent polynomials related by mutation (so $zz' = x^2 + y^2$). Then X_0 and X_0' are elements in 1-parameter family of degree $z \cdot z'$ surfaces in $\mathbf{P}(x^2, y^2, z, z')$.

Next problem was suggested by Hori. from physics viewpoint Landau–Ginzburg model should have compact compact Calabi–Yau fibers. Katzarkov argued that all relatively *minimal* compact smooth models of mirror should be equivalent (related by fiberwise flops). Nevertheless one wants to have some canonical construction of mirror symmetry: given a Fano variety to construct a smooth variety with potential and compact fibers. By definition algebraic variety is glued from affine charts along some glueing morphisms that are defined on some open parts of the charts (i.e. birational transformations). So it is very natural to consider tori of different toric degenerations as open charts and special birational transformations as glueing morphisms.

Problem 44 (Glueing). *Construct a fiberwise-compact canonical mirror of a Fano variety as a glueing of open charts given by (all) different toric degenerations.*

Problem 45 (Make toric degeneration ansatz explicit). *Given smooth Fano variety Y and its toric degeneration X provide explicitly a normal Laurent polynomial associated with X and mirror dual to Y.*

We believe it is possible to provide ansatz for surfaces using only Y, but for higher dimensions next example shows one has at least to use at least some direction in the deformation space of X.

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