MINIFOLDS 2: SURFACES AND MINIFOLDS OF GENERAL TYPE

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Abstract. We prove that the only homologically minimal manifold (minifold) of dimension two is the projective plane. The main step in the proof is to show that the possible alternatives, so called fake projective planes do not admit full exceptional collections. At the end of the paper we give some speculations about whether line bundles $\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2)$ form a (non-full) exceptional collection on a fake projective plane whose canonical class is divisible by 3 as a possible counterexample to a conjecture by Kuznetsov.

1. Introduction

Let X be a compact algebraic or Kähler manifold of dimension d. We call X *homologically minimal manifold* or simply a *minifold* if X admits a full exceptional collection in $\mathcal{D}^b_{coh}(\mathrm{X})$ of length $d+1$. It is conjectured that the only even dimensional minifolds are projective spaces. In this paper we give a proof of this statement in the case $d = 2$:

Theorem 1. *A compact Kähler surface X admitting a full exceptional collection in $\mathcal{D}^b_{coh}(X)$ of length 3 is isomorphic to the projective plane.*

Proof. By Corrollary 9 in [GM] we may assume X be to projective. From the existence of the exceptional collection of minimal length 3 on X we deduce that X has the same Betti numbers as a projective plane (see [GM], Proposition 10).

It follows then by a Theorem of Yau [Y] that X is either isomorphic to $\mathbb{P}^2$ or is a quotient of a complex two-ball $B^2 \subset \mathbb{CP}^2$. In the latter case X is called a fake projective plane and sits in one of the finitely many isomorphism classes of the that have been recently classified by [PY07] and [CS]. The case of a fake projective plane is ruled out by Corollary 3 which we prove in Section 2.

□

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2. Exceptional collections on a fake projective plane

Proposition 2. *Let X be an algebraic variety over a field such that $K_0(X)$ has no p-torsion. Then $Pic(X)$ has no p-torsion.*

Proof. We prove that if $Pic(X)$ has p-torsion, then so does $K_0(X)$.

Let L be a line bundle on X such that $L^{\otimes p} \cong \mathcal{O}_X$. Consider the element $[L] \in K_0(X)$, and let $N = [L]-1$. We first prove that $N \in K_0(X)$ is nilpotent. Indeed N being of rank zero, sits in the first term $F^1K_0(X)$ of the topological filtration on $K_0(X)$. The topological filtration is multiplicative, and hence $N^{dim(X)+1} \in F^{dim(X)+1}K_0(X) = 0$.

Let k be minimal such that $N^k = 0$. If $k = 1$, that is $N = 0$ and $[L] = 1 \in K_0(X)$, then $L \cong \mathcal{O}_X$ by the Riemann-Roch theorem without denominators.

We assume now that $k \geqslant 2$. We have

$$\begin{aligned} [L] &= 1 + N \\ 1 &= 1 + pN + N^2\alpha \\ 0 &= pN + N^2\alpha \end{aligned}$$

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and after multiplying by N^{k-2}:

$$pN^{k-1} = 0,$$

so that N^{k-1} is nontrivial p-torsion in $K_0(X)$.

$\square$

Corollary 3. *A fake projective plane does not admit a full exceptional collection in $\mathcal{D}^b_{coh}$.*

Proof. Assume that a fake projective plane X has a full exceptional collection of length R. It follows from [GM], Proposition 16 that $K_0(X)$ is free of rank R. Proposition 2 and [GM], Corollary 8 now imply that

$$H^2(X, \mathbb{Z}) \cong Pic(X)$$

is torsion-free. The Universal Coefficient Theorem states that

$$H^2(X, \mathbb{Z}) \cong \mathbb{Z} \oplus H_1(X, \mathbb{Z})^{tors}$$

which implies that $H_1(X, \mathbb{Z})$ must be torsion-free as well. On the other hand $h^{1,0}(X) = 0$ and hence $H_1(X, \mathbb{Z}) = 0$. The latter assertion however contradicts to [PY07], Theorem 10.1 which states that $H_1(X, \mathbb{Z})$ is nontrivial.

$\square$

Similarly to fake projective planes one can consider fake projective fourspaces, that is smooth projective fourfolds with the same Hodge numbers as those of $\mathbb{P}^4$ but which are not isomorphic to $\mathbb{P}^4$.

Prasad and Yeung [PY09] have been considering *arithmetic* fake projective fourfolds that fake projective fourspaces which appear as quotients of a unit complex ball $B^4 \subset \mathbb{C}^4$ by a torsion-free arithmetic subgroup in $PU(4, 1)$. They construct explicitly four arithmetic fake projective fourfolds ([PY09],Theorem 3) and prove that the first integral homology group $H_1(X, \mathbb{Z})$ of any arithmetic fake projective fourspace is non-zero ([PY09],Theorem 4). Using the reasoning of Corrolary 3 we come to the same conclusion:

Corollary 4. *An arithmetic fake projective fourspace does not admit a full exceptional collection in $\mathcal{D}^b_{coh}$.*

Apart from the statement of Corollary 3 there is nothing we can say about the structure of the derived category of a fake projective plane X. Even the weaker question, that about $K_0(X)$ of a fake projective plane is equivalent to the Bloch conjecture on the zero-cycles on X and does not seem to be accessible at the moment.

We can, however, consider a subcategory, generated by the bundles that are constructed explicitly such as the canonical class of the fake projective plane. The situation is especially interesting when the canonical class K_X is divisble by 3, see [PY07], 10.4. In this setting we let

$$K_X = \mathcal{O}(3),$$

for an ample line bundle $\mathcal{O}(1)$ on X, and we ask the following question:

Question 5. *Do the line bundles $\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2)$ form an exceptional collection on X?*

We obviously can not hope for the collection to be full in view of the Corollary 3. Let us now explain why the Question 5 is not totally meaningless. For that we first compute all the sheaf cohomology groups of $\mathcal{O}(k)$ on X that are accessible using formal sheaf-theoretic considerations and show that the results agree with the answer "yes" to Question 5.

Lemma 6. *The Hilbert polynomial of $\mathcal{O}(1)$ on X is given by*

$$\chi(\mathcal{O}(k)) = \frac{(k-1)(k-2)}{2}.$$

Proof. By the Riemann-Roch formula we have

$$\begin{aligned} \chi(\mathcal{O}(k)) &= 1 + \frac{(O(k) \cdot O(k-3))}{2} \\ &= 1 + \frac{k(k-3)}{2} \cdot (O(1) \cdot O(1)) \\ &= \frac{(k-1)(k-2)}{2} \end{aligned}$$

since $(O(1) \cdot O(1)) = 1$, because on the fake projective plane we have

$$9 = K_X^2 = (O(3) \cdot O(3)) = 9(O(1) \cdot O(1)).$$

□

More information comes from Kodaira vanishing theorem which tells us that $h^i(\mathcal{O}(k)) = 0$ for $i > 0$, $k \geqslant 4$ and from the Serre duality $h^i(\mathcal{O}(k)) = h^{2-i}(\mathcal{O}(3-k))$. Let us put the known values of $h^i(\mathcal{O}(k))$ for $i = 0, 1, 2$ and small k in the table:

	$\mathcal{O}(-2)$	$\mathcal{O}(-1)$	$\mathcal{O}$	$\mathcal{O}(1)$	$\mathcal{O}(2)$	$\mathcal{O}(3)$	$\mathcal{O}(4)$	$\mathcal{O}(5)$
h^0	0	0	1	0	?	0	3	6
h^1	0	0	0	?	?	0	0	0
h^2	6	3	0	?	0	1	0	0

It is easy to see that all the values hidden behind the question marks are equal to each other and do not exceed 2. The Question 5 is in fact equivalent to the following:

Question 7. *Is $H^0(X, \mathcal{O}(2)) = 0$?*

One can hope that Question 7 can be answered using the construction of $(X, \mathcal{O}(1))$ as a quotient of $(B, \mathcal{O}(-1))$ where $B \subset \mathbb{P}^2$ is a two-ball by the arithmetic group action (see [PY07], 10.4).

A positive answer to the Questions 5, 7 would provide a counterexample to a conjecture of Kuznetsov which states that an admissible category with vanishing Hochschild holomology must itself be trivial, see [K], Conjecture 9.1 and Corollaries 9.2, 9.3.

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