~~Some interesting~~ nc-HS
~~Examples~~ of and
Exceptional Diophantine approximations.

My 2nd talk in Mainz (1st – 4 years ago right after PhD).

I was trying to understand ~~inerfions~~ ? 1) Γ-class conjecture. (why Γ-class?)

NB) In L_n $C \to 0$
$\left(C = \lim \sum_{k=1}^{n} 1/k - \log n\right)$

~~Another source~~ 2) Apery's Diophantine approximations.

3) Relate them to each other.

At same time (with Golyshev–Katzarkov) – think about „ℓ-adic" realizations

- „Mysterious dualities" | Why $\dim H^i(Gr(k,N)) = \dim ~~(Gr(k,N))~~$ $(H^0(Gr(k,N),\mathcal{O}(1))$?

Descl. Too many dualities involved. (etc.

van Straten (trying to use ~~for~~ commutations)

Apery $\zeta(2), \zeta(3)$ are very exceptional.

E. Frenkel and B. Gross (09)

$F = k(\mathbb{P}^1)$ $|k| < \infty$

G - simple group $/k$

Aut. rep of G over $\mathbb{A}_F$

local factors: $/\infty$ - Steinberg rep.

$/0$ - simple supercuspidal

$/\mathbb{G}_m$ - unramified

$\Downarrow$

Unique rep.

Langlands

Langlands

ℓ-adic rep.

$\pi : Gal(F) \to \check{G}$

π - unramified over $\mathbb{G}_m$

at ∞ - tame

at 0 - slightly wild

$V \in Rep(\check{G})$ - induced lisse ℓ-adic sheaf

What is π? Constructed by Deligne (~ 77) Katz (~ 90)

for

$\check{G} =$	SL_n,	Sp_{2n},	SO_{2n+1},	G_2
$\dim V =$	n,	$2n$,	$2n+1$,	7

Now $F = \mathbb{C}(\mathbb{P}^1)$ $X = \mathbb{P}^1$ curve

π replaced by $\check{G}$-bundle on X

i.e. ($\check{G}$-bundle $\mathcal{L}$, connection ∇)

- reg. sing at ∞ - reg. unipotent monodromy
- irreg. at 0 with slope $\frac{1}{h}$ (h - Coxeter # of $\check{G}$)

Expect that such $(\mathcal{L}, \nabla)$ is unique.

(Key point - oper structure)

Bundle - trivial. Reduction - trivial.

Let N - principal nilpotent element in Lie algebra of $\check{B}^{opp}$

E - basis vector for highest root space for $\check{B}$ on $\check{g} = Lie(G)$

The connection (FG):

$$\nabla = d + N\left(\frac{dt}{t}\right) + E\,dt$$

- connection on trivial adjoint representation

If $V \in Rep(\check{G})$ $\to$ induce ∇ on V.

In Katz examples - MUM, so scalar ODE of rk $\dim V$.

Another evidence - acyclicity ($\sim$ extremality).

Local system on $\mathbb{P}^1$ is rigid.

(of solutions)

Conj For simplicity $\check{G}$ is simply-laced (type E_n)

Identify V with $H^{\cdot}(G/P, \mathbb{C})$
$\check{X}$

Then $(V, \nabla) = QC(X)$

Corollary (thm, Golyshev-Manivel) - Quantum Satake.

$$QC(Gr(k,N)) = \Lambda^{k} QC(Gr(1,N))$$

in sense of Lie algebras

Reform ~~A~~ Q/n-a cars
Bert, Kim, C-F, Sabbah

Consider $(\mathbb{P}^{N-1})^k$

$QC((\mathbb{P}^{N-1})^k)$

Group S_k acts on $QC((\mathbb{P}^{N-1})^k)$ into irreducibles.

$\Rightarrow$ the connection decomposes into irreducibles.

Λ^k part is $QC(Gr(k,N))$

(In terms of J

$$J^{(q)}_{Gr(2,5)} = J^{(q_1,q_2)}_{(\mathbb{P}^4)^2, [\mathcal{O}(1,0)-\mathcal{O}(0,1)]} \Big|_{q_1=q_2}$$

$Q(Spin(2n))$
$\shortparallel$
Spinor/Quadric)

II Conj Frenkel-Gross connection $(\mathcal{H}, \nabla)$

is de Rham realization of nc-HS.

...

$\mathbb{P}^1\times\mathbb{P}^2$	– 0, 1	(4+2)	How to polarize by $\mathcal{O}(1,1)$?
$Gr(2,5)$	– 0, 2	(7+3)	r·2 sing points
$OGr(5,10)$	– 0, 3	(11+5)	r·~~3~~2 sing points
E_6/P	– 0, 4, 8	(17+9+1)	r·2 sing points ~~…~~ (and 0)
E_7/P	– 0, 5, 9	(28+18+10)	r·3 sing points (and 0)

Minuscule

FEC on $Gr(k,N)$ – known (Kapranov)
$OGr(5,10)$ – " – (Kuznetsov, Polishchuk)
E_6/P – Manivel (num), Faenzi–Manivel
E_7/P – not known (

$$\frac{r}{\dim} = 1 - \frac{1}{k}$$

~~k = …~~ $k = d+1 = 10 - r$

$$X_n = G/P \qquad H^0(X_n, \mathcal{O}(1)) = V$$

				$r(G/P)$	$\dim(G/P)$	$\dim H^0(G/P)$
The GUT			E_8	29+1 29	(56+1)	(240+8)
Freudenthal variety			E_7	18	27	56
Cayley Plane		E_6 	E_6	12	16	27
Spin(10) $OGr(5,10)$		D_5 	$SO(10)$	8	10	16
$Gr(2,5)$		A_4 	$SU(5)$	5	6	10
$\mathbb{P}^1\times\mathbb{P}^2$	Standard model	 $A_1\times A_2$	$(U(1)\times)$ $SU(2)\times SU(3)$	(2,3)	3	6
$\bullet \sqcup \mathbb{P}^1$	Broken model				(0,1)	3

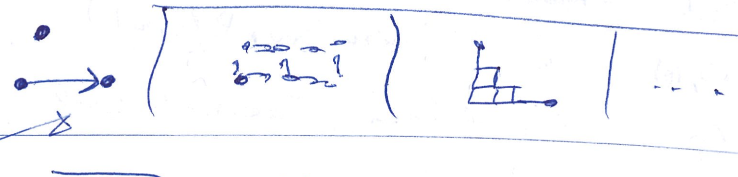
~~The~~ A Mirror Image → Inductive construction from ~~scratch~~ dirt $(\bullet \sqcup \mathbb{P}^1)$!!!

Origin	– EHX 96	slim diagrams 4
Development	– B, CF, K, vS	
Revised	– Bondal – G.	

Prop. $\text{Poset}^{\text{Bruhat}}(X_n) = \text{Ideals}(\text{Bruhat}(X_{n-1}))$

Hasse diagram, ...

Start from



Γ_Z , $\overline{\Gamma_Z}$ Hasse

$$W_X = \sum_{\substack{\text{edges } (\overline{\Gamma_Z}) \\ e}} \frac{x_{head(e)}}{x_{tail(e)}}$$

$X \xleftrightarrow{\text{descent}} Z$

$X_n \to X_{n-1}$

W corresponds to Gonciulea–Lakshmibai toric deg. $X \mapsto X_0$

Miracle $\overline{\Gamma_Z}$ can be embedded in S^2.

Dual Quiver Q. $X_0 =$ ~~Moduli~~ $($Q

$\mathcal{M}_{Rep}(Q, \text{dim}=(1,\dots,1), \text{stability})$

Q What is S^2? (Moduli space of what?)

Q. Why $GW(X_{n+1})$ and $GW(X_n)$ are related in this way,

$Z = \mathbb{P}(X \cap T_x X)$ = Fano var. of lines on X passing through x.

The linearizations

$\boxed{\text{Mirror}}$

We want (universal) map : „Shapes" → „~~Linear~~ "motive" vector spaces"

→ ↓ „realization"

nc-HS (de Rham, Betti), ℓ-adic sheaves, crystals, ...

In A-model

(symp. geom OR ~~alg. geom~~ projective varieties up to deformations)

Good shapes — • $Fuk(X)$ is smooth, proper.
• $QH(X)$ converges
• ...

Contain all ~~smooth~~ Fano manifolds.

$X \sqcup Y$, $X \times Y$

Ring

$\mathbb{Z}[\text{Fanos}] = \bigoplus_{d \geq 0} R^{(d)}$

$\cup$

$Lazar = \mathbb{Z}[\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3, \dots]$

$\mathbb{Z}[~~\text{Fanos}~~]$ ~~is~~ $R^{(d)}$ are f.g., ~~finite~~

but ring is too big yet.

The construction (Hori 02, Auroux 07, ...)

Pick D-brane of type A

$W = \sum_{\substack{D \subset X \\ \supset D \subset L}} x^{[\partial D]}$

not well-defined

wall-crossing

~~$w: (\mathbb{C}^\times)^n \to \mathbb{C}$~~

$W: (\mathbb{G}_m)^n \to \mathbb{A}^1$

$\cap$

→ $W: M \to \cancel{\mathbb{A}^1}$

(wish it to be proper)

• W should have $\mathbb{Z}$, non-negative coeff.

• $Newton(W) \ni$ origin

Corollary $\exists!$ (distinguished) critical point of W with $(\mathbb{R}_+)$-coordinates. C

• It is Morse.

• $W(C) = T \in \mathbb{R}_+$

$T = \frac{1}{\text{Radius of conv.}}$ (i.e. $T = \max |u_i|$ $\max Re\, u_i$)

R

• $(\mathbb{R}_+)^n$ is a sLag in ~~$\mathbb{A}$~~ $Fuk(M)$

• vanishing sLag in $FS(M, W)$

$(\mathbb{R}_+)^n \cap W^{-1}(a)$ is sLag sphere.
$a > T$

• $|u_i| = T \Rightarrow u_i^{r(X)} = T^{r(X)}$

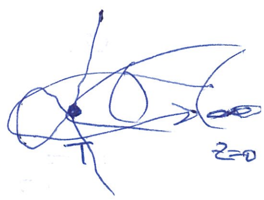
Conj U 1) no other singular points with value T (manifold-essential)

2) R = Mirror Image $(\mathcal{O}_X)$!
(at least numerically)

Conj U allows us to use $QC(X)$ for constructing explicit Diophantine approximations of ζ-values!

$\boxed{\text{Remark}}$

Fourier ↦ Fuchsian conn.
$(W_* \mathbb{C})$



Mon_T $s(E)$

$s(E) + \chi(\mathcal{O},E)\, s(\mathcal{O})$

$\shortparallel$

$s(E + \chi(\mathcal{O},E)\,\mathcal{O})$

Apply to $E = \mathcal{O}_{point}$

$s(\mathcal{O}_p) \to s(\mathcal{O}_p) + s(\mathcal{O})$

$s(E) \to s(E) + \chi(\mathcal{O},E)\, s(\mathcal{O})$

in Betti base

↓

ζ-values!

Another trick

$$\lim \frac{s(\cancel{\$}\Psi^{-1}(\gamma))}{s(\Psi^{-1}(pt))}$$

$\gamma \cup c_1(X) = 0$

$\Downarrow$

$s(\Psi^{-1}(\gamma))$ is holomorphic

$$\sum^{4} t^k \frac{a_k}{k!}$$

$$\lim \frac{\$\,\Phi_\gamma(t)}{\Phi_{pt}(t)} = \lim \frac{\Phi_\gamma^{(n)}}{\Phi_{pt}^{(n)}} \leftarrow \text{n-th coeff.}$$

$\Psi : K^0_{top}(X) \to H^*(X, \mathbb{C})$

$E \mapsto ch(E) \cup \hat{\Gamma}_X$

$\Psi(\mathcal{O}_p) = [pt]$

$\Psi^{-1}([pt]) = [\mathcal{O}_p]$

$\Psi(\mathcal{O}) = \hat{\Gamma}_X$

$\Psi^{-1}(1) = \dots$

~~$\langle A, B \rangle$~~

$\chi(E,F) = (s(E), s(F)|_{(-z)})$

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denominator of $\Phi_\gamma^{(n)} \sim D_n^{\deg \gamma}$

Fast convergence - Explicit inequality

on $\deg \gamma$, ~~and~~, $r(X)$ and T, T_2

$(T_2 = \max |u_i|, \text{s.t.} |u_i| < T)$

Ex case 3-dim space of hol at t solutions can extend through 2 points (Zudilin-like).