Nuclear runaway of oxygen-neon-magnesium white dwarf as a Electron-Capture Supernova: Influence of Initial Conditions and Input Physics on the Collapse Conditions

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ABSTRACT

The Crab nebula and the population of low-mass branch of neutron stars have suggested the formation of neutron stars other than the standard collapse of an iron core. One proposal for explaining such difference is the electron capture supernova, originated from stars of masses 8 - 10 \(M_\odot\). In these stars, the electron captures \(^{20}\text{Ne}(e, \nu)\ ^{20}\text{F}(e, \nu)\ ^{20}\text{O}\) and \(^{24}\text{Mg}(e, \nu)\ ^{24}\text{Na}(e, \nu)\ ^{24}\text{Ne}\) can trigger the nuclear runaway in the form of oxygen-neon flame. However, whether such flame can already unbound the star as a Type Iax supernova is unclear. In this article, we model the evolution of a oxygen-neon-magnesium white dwarf in the deflagration phase with different configurations and input physics by two-dimensional hydrodynamics simulations. In particular, we present a series of models which explore the possible model parameters. They include the range of runaway density from \(10^{49} - 10^{52}\) g cm\(^{-3}\), flame structure of both centered and off-centered ignition kernels, special and general relativistic effects, turbulent flame speed formula and the treatments of laminar burning phase. We find that the ONeMg model given from stellar evolution model when the deflagration starts, has a high tendency to collapse into a neutron star, thus suggesting the electron capture supernova to be a candidate of the low-mass branch of the neutron star population. We further classify cases which undergo a direct collapse without explosion and then we derive the collapse conditions. At last we discuss the nucleosynthesis constraints and the possible observational signals of this class of supernovae.

1. INTRODUCTION

1.1. Formation and Evolution of O+Ne+Mg White Dwarfs

Main-sequence stars with masses from 8 to 10 \(M_\odot\) form an oxygen-neon-magnesium (ONeMg) core near the end of its stellar evolution. For stars more massive than \(12 \ M_\odot\) (Sugimoto & Nomoto 1980), hydrostatic burning up to silicon burning is carried out in the stellar core, creating an iron core exceeding the Chandrasekhar mass. In this case, the collapse of the iron core triggers the core collapse supernova. On the other hand, for a star with a mass below \(8 \ M_\odot\), the central density of the carbon-oxygen (CO) core is not high enough to trigger the electron capture, thus leaving a stable CO white dwarf without further nuclear reactions (Nomoto 1982). Stars in the mass range of \(M_{\text{up,Ne}} < M < 10 M_\odot\) become electron capture supernovae (Nomoto 1984), where \(M_{\text{up,Ne}} = 9 \pm 1 M_\odot\) is the upper mass limit of the progenitor of an ONeMg WD. \(M_{\text{up,Ne}}\) depends on also metallicity, which decreases with metallicity (Siess 2007; Pumo et al. 2009; Langer 2012).

The ONeMg core has a mass below the critical mass for neon burning at \(1.37 \ M_\odot\), no neon ignition can be triggered spontaneously. The electron degeneracy in the stellar core leads to a CO or hybrid CO+ONe WD for stars below \(8 \ M_\odot\).

After the core is depleted of helium, the neutrino emission in the carbon-oxygen core leads to a net energy loss which creates a temperature inversion (Nomoto 1987).

During the carbon burning, due to surface convection, the helium shell surrounding the carbon-burning core is dredged up. Depending on the mass loss rate from the SAGB star, a single O+Ne+Mg (ONeMg) white dwarf (WD) is formed, which could be the case for stars in the mass range of \(M_{\text{up,C}} < M < M_{\text{up,Ne}}\), where \(M_{\text{up,C}} = 6 \pm 3 M_\odot\) is the upper mass limit of the progenitor of a CO WD. In Nomoto (1987), the dredge up can take away the initial helium shell of mass about \(2 M_\odot\) and leaves behind a thin helium layer of mass \(\sim 10^{-3} M_\odot\). This provides the condition that the later helium burning in the envelope does not increase the carbon core mass beyond the Chandrasekhar mass. How-
ever, the exact helium envelope depends on the mass loss from SAGB star, which is uncertain.

At a central density \( \sim 10^9 \text{ g cm}^{-3} \), the electron matter becomes degenerate where its Fermi energy successively exceeds the thresholds for electron capture on nuclei produced by C-burning such as \(^{25}\text{Mg}, ^{23}\text{Na}, ^{24}\text{Mg}, ^{20}\text{Ne}, \) and \(^{16}\text{O} \). For nuclear pairs of odd mass numbers \( A = 23 \) and \( 25 \), i.e., \(^{25}\text{Mg} - ^{25}\text{Na}, ^{23}\text{Na} - ^{23}\text{Ne}, \) and \(^{25}\text{Na} - ^{25}\text{Ne}, \) the nuclear URCA processes electron captures and \( \beta \) decays occur because the transition from the ground state of the mother nucleus to the ground state of the daughter nucleus is no longer forbidden by quantum transition. The resultant URCA cooling of the O+Ne+Mg core is important and the related rates have recently been calculated carefully (Toki et al. 2013; Suzuki et al. 2016). Despite the possible importance of URCA process to the temperature evolution of the ONeMg core, due to the subtleties in its implementation, no direct modeling of URCA process simmering phase for the ECSNe has been modelled, unlike the Type Ia supernova counterpart (Schwab et al. 2017).

When the ONeMg core has a mass \( 1.376 M_\odot \) (at a central density of \( 10^{9.6} \times 10^9 \text{ g cm}^{-3} \)), the electron Fermi energy exceeds the threshold for electron captures \(^{24}\text{Mg}(e^-, \nu) ^{24}\text{Na} \). Thus, electron capture of \(^{24}\text{Mg}(e^-, \nu) ^{24}\text{Na} \) starts. But the rate is slow enough that a gradient in molecular weight is formed where semi-convection is possible. Also, the drop in \( Y_e \) makes the core further contract (Miyaji et al. 1980; Nomoto et al. 1982; Nomoto 1987). (Here \( Y_e \) is the electron mole number of the matter.)

In general, the electron capture in the core can trigger a \( Y_e \) gradient, which can trigger semi-convection. However, the accurate of semi-convection and also its efficiency is poorly constrained. The treatment of semi-convection with electron capture is complicated. Thus both Schwartzschild criterion (Miyaji et al. 1980; Nomoto 1987; Takahashi et al. 2013) and Ledoux criterion have been applied for convection (Miyaji & Nomoto 1987; Hashimoto et al. 1993; Schwab et al. 2015). In Miyaji et al. (1980), it can drastically raise the runaway density from \( 10^9 \) to \( 10^{10.2} \text{ g cm}^{-3} \) when using the Schwartzschild criterion (i.e. excluding the semi-convection) to \( 10^{10.2} \text{ g cm}^{-3} \) when using the Schwartzschild criterion. By including the semi-convection, the convective flow owing to the \( Y_e \) difference can take away the thermal energy from the thermal core, where thermal energy is deposited by the gamma-ray emitted during electron capture.

Electron captures of \(^{24}\text{Mg}(e^-, \nu_e) ^{24}\text{Na}(e^-, \nu_e) ^{24}\text{Ne} \) begins to participate when the ONeMg core reaches \( 1.38 M_\odot \) (or at a central density \( 10^9 \) g cm\(^{-3} \)). When central density reaches \( 10^9 \) g cm\(^{-3} \), another channel \( ^{20}\text{Ne}(e^-, \nu_e) ^{20}\text{F}(e^-, \nu_e) ^{20}\text{O} \) emerges. Both weak interaction channels generate a considerable amount of heat by gamma ray emission. This provides the first trigger of oxygen-neon flame in the stellar core.

We note that besides ONeMg core, another channel to form neon deflagration in an ONeMg WD is by Accretion Induced Collapse (AIC) (Canal & Schatzman 1976). It is the collapse of a white dwarf into a neutron star through mass accretion from its companion star. In this picture, the white dwarf accretes mass from its main-sequence companion star (Nomoto 1987), when the latter one evolved to the red giant stage so that its hydrogen and helium envelope are transferred to the WD through the Roche Lobe. Depending on the mass accretion rate, the accreted hydrogen carries out hydrostatic burning on the As a result, the WD can gain its mass to the Chandrasekhar mass, triggering the electron capture and the oxygen-neon deflagration.

We remark that AIC has also its theoretical difficulties, namely the mismatch of occurrence rate and expected rate. In contrast to the typical single degenerate scenario (Miyaji et al. 1980), models show that AIC can also occur in WD pairs which have a total mass beyond the Chandrasekhar mass (Mochkovitch & Livio 1989). Population synthesis has suggested that the super-Chandrasekhar mass white dwarf merger is one of the robust candidates for the AIC event, which has a rate comparable with Type-Ia supernovae (Yoon et al. 2007). However, there is not yet a direct observation of this class of supernova. One of the resolutions is that AIC is difficult to be observed due to its dim light and light-curve (Dessart et al. 2006).

1.2. Physics of neon deflagration in ONeMg WD and its collapse

In the ONeMg core of the 8 - 10 \( M_\odot \) main-sequence star, near its end of stellar evolution the central density is sufficiently high (\( 10^{9-10} \) g cm\(^{-3} \)) (Nomoto 1984), where weak interaction becomes important. The electron capture can lower the pressure efficiently in the matter after it is being swept by the deflagration wave (Nomoto & Kondo 1991). Energy is released during the burning of oxygen and neon to form iron-peaked elements. The pressure difference across the deflagration front, due to the degenerate electron at high density, is relatively lower than that of the carbon-oxygen matter, which is of lower density in general. The smaller density difference makes the buoyancy force of the hot burnt matter smaller. Thus the convective flow which spread the hot material to the surrounding becomes weaker. Furthermore, the typical electron capture rate in the ash, owing to its higher central density and its strong
density-dependence, is higher compared to typical CO WD. The burnt matter have a lower electron fraction in general. As the electron gas contributes to the majority of pressure in the matter, the drop of electron fraction not only suppresses the growth of flame, but also inhibit the expansion of the burnt core. These factors provide more time for electron captures to take place – the key to trigger the direct collapse.

After the core collapses into a neutron star, the stiff nuclear matter stops the infall matter in the core and creates a bounce shock that propagates outward. The shock provides strong compression heating to the outer matter, which loses its strength at the same time. When the shock reaches the surface, due to the large density gradient, the shock becomes stronger again, which ejects the low density matter in the form of wind. However, the conclusion of whether the bounce shock can lead to a healthy explosion is highly dependent on the input physics. In early work, it is shown that the shockwave is unable to unbind the star due to the highly endothermic process of photodisintegration of iron peak elements to helium (Bethe & Wilson 1985). It is believed that the bounce shock formed when the central density of the star reaches nuclear density, is stalled at about ∼ 100 km (Bethe & Wilson 1985). The neutrino emitted by the neutron star via neutron star cooling is indispensable for the explosion. The shock is shown to revive by neutrino energy deposition in the envelop. This can lead to either a direct explosion (Hillebrandt et al. 1984) or delayed revival (Mayle & Wilson 1988). Later multi-species AIC simulations show that neutrino indeed helps the explosion to sustain, but the explosion strength is weak (Dessart et al. 2006).

Despite the status that the collapse of ONeMg core is well studied for about two decades, so far there are only hints about the existence of electron capture supernova. This class of collapse is theoretically challenging. The first theoretical difficulty is the tension between the expected and observed occurrence rate of such collapse as an ECSN. On one hand, we expect the star with a mass 8 − 10M⊙ can undergo ECSN naturally. On the other hand, chemical composition during the nucleosynthesis of the ONe deflagration can bring the opposite perspective. The ejected matter of ECSN includes neutron-rich isotopes, such as 62Ni, 66Zn, 68Zn and so on (Woosley & Baron 1992). If the oxygen-neon deflagration can unbind the star without making the ONeMg core collapse into a neutron star, the occurrence rate of such explosion should be very low, compared to standard Type Ia supernovae (Fryer et al. 1999).

The observables of these collapse processes are well studied in the literature. Since the ONeMg core generates energy by nuclear burning towards iron-peak elements, it behaves comparable to Type-Ia supernovae due to its power source from 56Ni decay. But the light curve is comparatively dim (∼ 10^{41} erg/s) due to the small amount 56Ni (10^{-3}M⊙) (Baron et al. 1987; Metzger & Thompson 2008) and fast-evolving (2 − 4 days) due to the non-massive ejecta less than (10^{-1}−3M⊙), where the presence of magnetic field can amplify the ejecta mass (Dessart et al. 2006, 2007). The nickel-rich flow (Metzger et al. 2009a) also shows its signature in the spectra (Durliba et al. 2010). The neutrino signals of the AIC is studied in Dessart et al. (2006), where the rotation of the progenitor WD is closely related to the structure of neutrino cone and hence the deleptonized zone structure. The gravitational wave signal is also studied in Abdikamalov et al. (2010). They show that the gravitational signal can provide hints on the progenitor’s properties, such as the velocity structure inside the WD, which is correlated to the rotation profile of the progenitor.

The post-collapse scenario provides also information about the remnant neutron star. Matter which is not ejected by the explosion forms an accretion disk (Metzger et al. 2009b), which leads to jet formation (Livio 1999). This is believed to be one of the source of short gamma ray burst (Lee & Ramirez-Ruiz 2007). The collapse is also related to the formation of millisecond pulsar (Hurley et al. 2010; Freire & Tauris 2014) or magnetar (Duncan & Thompson 1992), signals in the radio and X-ray regime are expected (Norris & Bonnell 2006), which is also important to the population of NS. In a survey of NS mass through X-ray pulsars (Kniige et al. 2011), two populations of NSs are observed, with one group having a lower rotation period about 10 s, while the other group has a higher one about 300 s. This indicates the possibility of NS formation by two distinctive paths. The ONeMg core collapse, also known as the electron-capture supernovae (ECSNe), can naturally explain the qualitatively different class of NS from NS made from the standard core-collapse SN.

1.3. Motivation

The modeling of ONeMg core from stellar evolution is difficult owing to many numerical difficulties. There exists a clear uncertainty in the initial condition for the ONeMg when deflagration starts. The pre-runaway phase are critically related to the later explode/collapse condition because it influences on how the initial flame propagates and how the energy is balanced by neutrino loss. In Nomoto & Kondo (1991), one-dimensional model has been done to show that the effective flame speed owing to turbulence and convection are crucial in
determining the final fate of the star. The first three-dimensional realizations of this phase in Jones et al. (2016) show similar conclusion on the importance of the input physics. Their work shows that the choice of input physics, namely the inclusion of the Coulomb correction in the equation of state, the differences in the initial model, as a result of different convection model, and the resolution play a role in determining the final fate of the ONeMg core. However, due to the three-dimensional nature, only a small number of progenitor models and input physics are studied. Also, the expected chemical properties of the ejected matter is not comprehensively studied. In view of that, we make use of the two-dimensional hydrodynamics code with tracer particles to study this problem. The two-dimensional model allows us to span the parameter space much faster than three-dimensional models for the same computational time. Also, the tracer particles can predict the yield in the ejecta, which serve as an important constraint to the expected rate compared to other types of supernovae. Both features allow us to keep track of the post-collapse observable of the runaway.

In Section 2 we briefly outline our hydrodynamics code and the updates employed to model the pre-collapse phase. In Section 3 we report our results of our parameter study. It includes the array of models we used to following the evolution of ONeMg core. This aims at studying the post-runaway evolution of the ONeMg core at different 1. central density, 2. initial flame structure, 3. initial flame position, 4. flame physics and 5. pre-runaway configuration. In Section 4 we discuss how our results can be understood collectively for future models given by stellar evolution. We also compare our results with some models presented in the literature. Then, we discuss the possible observational constraints on ECSN base on the nucleosynthesis yield. At last we give our conclusion. In the appendix we present briefly the possible observational results when the ECSN collapse based on the formalism of the advanced leakage scheme.

2. METHODS

We use the two-dimensional hydrodynamics code as developed for supernovae and nucleosynthesis. We refer the readers to Leung et al. (2015a,b); Leung & Nomoto (2017); Nomoto & Leung (2017b); Leung & Nomoto (2018) for a detailed description of the code and its applications in Type Ia supernovae. We also refer the refer the readers to Nomoto & Leung (2017a) for a detailed motivation of connecting AGB-star to electron capture supernovae. Here we briefly describe the physics included in the code.

2.1. Hydrodynamics

The code solves the Euler equations in cylindrical coordinate, namely:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1}
\]

\[
\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla P - \rho \nabla \Phi, \tag{2}
\]

\[
\frac{\partial \tau}{\partial t} + \nabla \cdot [\vec{n}(\tau + \rho)] = -\rho \vec{v} \cdot \nabla \Phi. \tag{3}
\]

Here, \(\rho, \vec{v}, P\) and \(\tau\) are the density, velocity, pressure and energy density of fluid defined as \(\tau = \rho \epsilon + \rho |\vec{v}|^2/2\), with \(\epsilon\) being the specific internal energy of the fluid. \(\Phi\) is the gravitational potential satisfying the Poisson equation \(\nabla^2 \Phi = 4\pi G \rho\). We use the fifth-order weighted-essentially non-oscillatory (WENO) scheme for spatial discretization (Barth & Deconinck 1999) and the five-step third-order non-strong-stability-preserving Runge-Kutta (NSSP RK) scheme (Wang & Spiteri 2007) for time-discretization. We use the Helmholtz equation of state (Timmes & Arnett 1999). This equation of state accounts for the contribution of ideal electron gas at arbitrarily degenerate and relativistic levels, ions in the form of a classical ideal gas, photon gas with Planck distribution and the electron-positron annihilation pairs. To track the deflagration wave we use the level-set scheme, where the deflagration front is represented by the zero-contour of a scalar field \(S\), which is advected by the fluid and propagate with the turbulent flame speed, i.e.

\[
\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = -v_{\text{turb}} \vec{n} \cdot \nabla S. \tag{4}
\]

The turbulent flame prescription is the same as the SNIa case, where the effective flame speed is proportional to the laminar flame speed \(v_{\text{lam}}\), with an amplifying factor due to the turbulence velocity. Mathematically, we have \(v_{\text{turb}} = v_{\text{lam}} f(\nu' / v_{\text{lam}})\), where \(\nu'\) is the velocity fluctuation owing to turbulence (See also Pocheau (1994); Niemeyer et al. (1995); Schmidt et al. (2006); Leung et al. (2015a) for the general notation of turbulent flame). In this work, we choose

\[
v_{\text{turb}} = v_{\text{lam}} \left[1 + C_t \left( \frac{\nu'}{v_{\text{lam}}} \right)^n \right]^{1/n}. \tag{5}
\]

We choose \(n = 2\) following Schmidt et al. (2006), which represents Gaussian distribution of velocity distribution and self-similarity. The value \(C_t\) is fixed by matching the asymptotic behaviour of turbulent flame from experiment. In (Schmidt et al. 2006), \(C_t\) is picked as 4/3. The laminar speed is a function of density and \(^{16}\)O mass fraction as given in Timmes & Woosley (1992). To estimate the velocity fluctuation in the sub-grid scale, the
use the one-equation model presented in Niemeyer et al. (1995). We define the specific kinetic energy density in the sub-grid scale \( q_{turb} = |\vec{v}|^2/2 \). This energy density is also regarded as a scalar and follows the fluid advection, and exchange energy with the specific energy of the fluid, namely

\[
\frac{\partial q_{turb}}{\partial t} + \vec{v} \cdot \nabla q_{turb} = \dot{q}_{turb},
\]

\[
\frac{\partial E}{\partial t} = -\dot{q}_{turb}.
\]

Depending on the context, \( \dot{q}_{turb} \) can contain different terms. In the ONeMg core, we have \( \dot{q}_{turb} = \dot{q}_{\text{prod}} + \dot{q}_{\text{diss}} + \dot{q}_{\text{comp}} + \dot{q}_{\text{RT}} + \dot{q}_{\text{diff}} \). The term on the right hand side stands for turbulence production by shear stress, turbulence dissipation, turbulence generation by compression, production by Rayleigh-Taylor instabilities and turbulent diffusion.

2.2. Microphysics

In this version, since we need to model the evolution of WD at high density, the previously implemented 7-isotope network is not sufficient because in the core region, the matter is most likely to appear in nuclear statistical equilibrium, where the density (electron fraction) is sufficient high (low) that neutron-rich isotopes become significant. To include the factor, we modified the burning scheme and isotopes following the prescription in Townsley et al. (2007). We introduce the quantities \( Y \), \( q_B \) and \( \phi_i \) (\( i = 1, 2, 3 \)). They represent the inverse of mean atomic mass (1/\( \bar{A} \)), binding energy and the burning progress variables. They are treated as scalars as the level-set function, which are transported by the fluid motion. We use the operator splitting in each step by separating the hydrodynamics and nuclear reaction. In the hydrodynamics phase, these quantities are also transported by the fluid, namely

\[
\frac{\partial \rho Y}{\partial t} + \vec{v} \cdot \nabla Y = \dot{Y},
\]

\[
\frac{\partial \rho q_B}{\partial t} + \vec{v} \cdot \nabla q_B = \dot{q}_B.
\]

After each step, the mean atomic mass and mean atomic number are reconstructed by \( 1/Y \) and \( Y_e/Y \). These two quantities are passed to the equation of state subroutine to find the derived thermodynamics quantities including the pressure and its derivative with respect to density and temperature.

In the nuclear reaction phase, for the oxygen-neon white dwarf, we assume \( \phi_1 \) to be the burning of \( ^{20}\text{Ne} \) its immediate product, \( \phi_2 \) to be the burning of fuel until NQSE and \( \phi_3 \) to be the burning from \( ^{28}\text{Si} \) to NSE. This scheme is coupled with the level-set that the region swept by the level-set method is considered as burning of \( ^{20}\text{Ne} \). In doing that we can directly apply the flame speed formula directly to the level-set scheme. To prevent burnt matter from repeatedly release energy due to numerical diffusion, all \( \phi_1, \phi_2 \) and \( \phi_3 \) are restricted to be monotonically increasing and \( \phi_2, \phi_3 \) is allowed to evolve only when \( \phi_2, \phi_3 \) burning has been completed. Their evolution also satisfies the following equations

\[
\frac{\partial \phi_i}{\partial t} + \vec{v} \cdot \nabla \phi_i = \dot{\phi}_i,
\]

where \( i = 1, 2, 3 \). We also apply operator splitting when solving these equations. We assume \( \phi_{1,2,3} \) follows the fluid advection. So, the advection terms are handled by the WENO and NSSP-RG scheme together with the Euler equations. The source terms are solved analytically.

2.3. NSE and weak interactions

In the simulation, grid with \( \phi_3 \) represents that all exothermic nuclear burning has finished and the matter enters NSE state. In this stage, we use the pre-computed table which consists of the mean atomic mass, binding energy and electron capture rate as a function of density, temperature and electron fraction. The NSE composition is computed based on the 495-isotope network as described in Timmes (1999), which includes isotopes from \( ^1\text{H} \) to \( ^{91}\text{Tc} \). The network also includes the Coulomb correction factor as described in Kitamura (2000). Similar to Townsley et al. (2007), the NQSE burning and the NSE conversion timescale are included to account for the reaction at low temperature, where the reaction timescale can be comparable or even much larger than the hydrodynamics timescale. This guarantees that at temperature higher than \( 5 \times 10^9 \text{K} \), the chemical composition follows the NSE evolution directly; while at temperature lower than that, only a partial conversion is allowed. We require the new composition \( X_{\text{new}} \), new temperature \( T_{\text{new}} \) and the new specific internal energy \( \epsilon_{\text{new}} \) satisfies

\[
\frac{\epsilon_{\text{new}} - \epsilon}{\Delta t} = N_A(m_n-m_p-m_e)\Delta Y_e + \dot{\epsilon}_\nu + q_B(X_{\text{NSE, new}}) - q_B(X_{\text{NSE}}).
\]

We remark that the NSE composition is a function of density, temperature and \( Y_e \) as \( X_{\text{NSE, new}} = X_{\text{NSE}}(\rho_{\text{new}}, T_{\text{new}}, Y_e_{\text{new}}) \). The source terms on the right hand side are the change of the binding energy due to composition change, the energy loss due to neutron-proton mass difference and the energy loss by neutrino emission during electron capture.

To obtain an accurate electron capture rate even for a low \( Y_e \), we follow the method in Jones et al.
(2016) to extend the electron capture rate table by including neutron-rich isotopes. The electron capture rate is computed following the prescription in Seitezahl et al. (2010) with the individual electron capture rates given in Langanke & Martinez-Pinedo (2001) and Nabi & Klapdor-Kleingrothaus (1999). We solve

\[
\frac{DY_e}{Dt} = \sum_i X_i \frac{m_p}{m_i} (\lambda_{i\text{ec}} + \lambda_{i\text{pc}} + \lambda_{i\text{bd}} + \lambda_{i\text{pd}}),
\]

where \(m_p\) and \(m_i\) are the baryon mass and the mass of the isotope \(i\). \(D/ Dt\) is the derivative in the rest frame of the fluid, \(\lambda_{i\text{ec}}, \lambda_{i\text{pc}}, \lambda_{i\text{bd}}\) and \(\lambda_{i\text{pd}}\) are the rates of electron capture, positron capture, beta-decay and positron-decay of the isotope \(i\) respectively in the units of s\(^{-1}\).

3. MODELS AND RESULTS

3.1. Initial Model

In this section we describe how we prepare the initial condition for the hydrodynamics run. In all the model, we prepare the ONeMg core following the simplified two-layer structure as derived from Schwab et al. (2015). The inner part imitates the zone where electron capture begins to take place, thus with a lower \(Y_e\) and a higher temperature, while the opposite case for the outer part. Here we choose \((Y_e, T) = (0.496, 4 \times 10^9 \text{ K})\) for the inner part and \((Y_e, T) = (0.5, 3 \times 10^9 \text{ K})\) for the outer part. We assume the chemical composition change is small enough that it remains \(X(^{16}\text{O}) = 0.55\) and \(X(^{20}\text{Ne}) = 0.45\) throughout the star.

We remark that Ideally, the initial profiles should be taken directly from stellar evolution models, which can minimize the uncertainty in the temperature, \(Y_e\) and isotope distributions, by controlling the pre-runaway evolution, e.g. the mass accretion rate, in each model. However, the change among models consist of changes of temperature, \(Y_e\) profile and chemical abundance. This makes the comparison of models, as well as the extraction of influences of each model parameter, difficult.

Notice that a few stages in the pre-runaway evolution can influence the final ONeMg structure. At \(\rho_e \sim 10^9 \text{ g cm}^{-3}\), the density becomes high enough so that the Fermi energy of the electron exceeds the chemical potential its pair-isotope (e.g. \(^{20}\text{Ne}, ^{20}\text{F}\) and \(^{24}\text{Mg}, ^{24}\text{Na}\)). Therefore, energetically the system prefers the electron capture to combine with proton in the nuclei and form neutron, even when \(m_n - m_p = 1.235 \text{ MeV}\). In the process, energetic gamma ray \(\sim \text{ MeV}\) is emitted while local \(Y_e\) drops.

Furthermore, during electron capture, it creates a discontinuity of \(Y_e\). Such discontinuity can trigger the local semi-convection (Miyaji et al. 1980; Lesaffre et al. 2004; Stein & Wheeler 2006; Schwab et al. 2017), which brings complexity in the \(Y_e\), temperature and abundance profile.

Then, the URCA process adds another complication. The electron capture of isotope \(^{23}\text{Na}\) transforms \(^{23}\text{Ne}\) becomes energetically favourable when the matter reaches a density of \(1.7 \times 10^9 \text{ g cm}^{-3}\) (corresponding to the threshold energy 4.38 MeV) (Lesaffre et al. 2004). At that region, the matter undergoes convection since C-burning takes place in the core. At that density, \(^{23}\text{Na}\) is able to capture electron and forms \(^{23}\text{Ne}\). At the same time, the daughter nucleus \(^{23}\text{Ne}\) move away from the core following the upward convective flow. When the \(^{23}\text{Ne}\) is sent to the outer place, with a lower density and thus a lower chemical potential, it becomes energetically favourable for the \(\beta\)-decay from \(^{23}\text{Na}\) to \(^{23}\text{N}\) to take place. This process is the most vigorous on the URCA shell, where one electron neutrino and one anti-electron neutrino are generated and they escape freely from the star. Since in the whole process no change in the chemical composition or \(Y_e\) is resulted, the neutrino emission and its corresponding energy loss can be continued for a long time. Such process can lower the core temperature, and hence delay the runaway.

To demonstrate the complexity of initial profile when including realistic physics, we show in Figure 1 the differences of initial model using more realistic stellar evolution model (Zha et al. 2018). In these models, the ONeMg core is evolved with the MESA code with mass accretion as the model parameters. To accommodate the pre-runaway electron capture related physics (including both electron capture of \(^{16}\text{O}\) and \(^{20}\text{Ne}\), and the URCA process of odd-number elements), a 37-isotope network from \(^{1}\text{H}\) to \(^{27}\text{Al}\) is used. The models are run until O-burning becomes runaway. We plot the upper left panel the density and temperature profiles of two contrasting Models A and B. Model A assumes a higher mass accretion rate (\(10^{-5} M_\odot \text{ year}^{-1}\)), while Model B assumes a lower one (\(10^{-9} M_\odot \text{ year}^{-1}\)). The change of mass accretion rate will bring multiple effects on the initial models. With a higher accretion rate, the central density increases faster. This can change the time duration for URCA process and hydrostatic O-burning to takes place. Thus, the temperature, chemical abundance and \(Y_e\) are expected to be different.

In the figure, we plot in the upper left panel the comparison between Model A and Model B for the density and temperature profiles. Owing to the degenerate electron gas, the density profile are very similar for the two models. Also, in both models the core is the first place for O-runaway to take place. The runaway size is \(\sim 10^{-12} M_\odot\). However, the local temperature for Model A is always higher than Model B for the same
mass radius. The temperature structure is also more complicated than the two-layer structure.

In the upper right panel, we plot the $Y_e$ profile similar to the upper left panel. We can see that due to the different shell burning and convection, the $Y_e$ shows a four-layer structure. The exact position of the transition again depends on the accretion rate.

In the lower left and right panels, we plot the abundance profiles for Model A and Model B respectively. The composition also shows some differences, but at a level smaller than $Y_e$. The major isotopes $^{16}$O and $^{20}$Ne are mostly the same, except a small difference in the transition near the surface. Larger differences appear in minor isotopes. For example, the $^{12}$C is depleted in Model A but not in Model B. Also, in the surface, $^{12}$C has a significant amount in Model A but not in Model B.

From these qualitative comparison, it suffices to show that the comparison of more realistic models can be complex because each model differs from other in a few aspect. Furthermore, there exists uncertainty in the runaway treatment, for example the deflagration kernel. Therefore, in this work, we focus on the individual effects of each model parameters. Therefore, we begin all the model with same structure, and vary one of the parameter at one time. The global and systematic comparison of runaway models from different stellar evolution modeling will be carried out in future work.

3.2. Numerical Models

3.2.1. Connection to Stellar Evolutionary Models

The uncertainties in the evolution of ONeMg cores lead to the ambiguity of the final evolution of ONeMg cores. One is the semi-convection associated with electron captures. Depending on the efficiency of semi-convection, the ONe-deflagration density in the ONeMg can change from $\sim 10^{10.2}$ (Schwarzschild criterion) down to $\sim 10^{9.95}$ g cm$^{-3}$ (Ledoux criterion). Semi-convection is an over-stable convection, so that the oscillatory convective instability and the associated mixing grow with the timescale of heat-exchange. More mixing leads to a higher central density (Takahashi et al. 2013). Therefore, $10^{9.95}$ g cm$^{-3}$ set by the Ledoux-criterion is the lower limit to the deflagration density. In the present case, electron capture forms the extremely steep gradients in both temperature and $Y_e$, so that the analysis of semi-convective mixing requires careful treatment and essentially multi-dimensional simulations.

The second uncertainty is the initial flame structure. The development of the initial flame is sensitive to the internal motion of the star. However, in stellar evolution, which is modelled in one dimension, the non-radial motion of matter is neglected. In particular, local turbulence can provide velocity and temperature fluctuations, which can be important near the runaway phase. Without an exact knowledge about the internal motion of the star, one cannot derive the exact position of the first nuclear runaway, as well as its possible shape. Therefore, the initial flame of the ONeMg, similar to SNe Ia, is poorly constrained.

The third uncertainty is the relativistic effects. The impact of relativistic effects is unclear. In the ONeMg core, the density in the core is high enough that the electron becomes relativistic. In that case, the contribution of the pressure and internal energy as a gravity source can be non-negligible. One has to study how these components affect the dynamics, and whether the collapse criteria changes with them.

3.2.2. Model Description

The model parameters spanned in this work tries to cover the uncertainties left in the stellar evolution modeling. In Table 1, we tabulate the initial setting of our hydrodynamics models. In preparing the initial model, we make reference to the pre-deflagration model as computed in Schwab et al. (2015). They showed that the pre-deflagration WD consists of three parts, the outer envelop where no burning occurs, the outer core where hydrostatic burning of $^{24}$Mg carries out, and the inner core where electron capture and faster nuclear reactions occur. We follow their temperature profile that the inner core of higher temperature at $4 \times 10^9$ K and an outer core of lower temperature at $3 \times 10^8$ K. However, we do not resolve the most inner core around $10^{-4} M_\odot$ which is equivalent to less than a few grid points of our simulations as what is found in Schwab et al. (2015). Since we want to lower the acoustic noise which may also alter the subsequent dynamics, the inner burning core is neglected in this article and instead we patch an initial flame of similar size to mimic that deflagration in the central region has already started. However, we remarked that the precise structure of the initial deflagration require a full multi-dimensional simulations right after the first nuclear runaway has started. However, this becomes already computationally expensive due to the slow propagation of the laminar flame. To balance such uncertainty, we therefore implemented different flame structure so as to mimic the different possible outcomes. In particular, we include the $c3$, $blb$, $b1b$ and $b5$ flame. The $c3$ flame is the same “three-finger” structure as in Niemeyer et al. (1995). The finger shape is to enhance the development of Rayleigh-Taylor instabilities. Also, this phase can prevent the development of flow along the boundary, which might not be physical.
c3 flame located at center with an outer radius of about 20 km and an inner radius of 10 km. The flame structure is similar to what we use to start the deflagration phase in Leung et al. (2015a,b) with a smaller size. The b1b flame is assuming a ring of radius 15 km located at 50 km away from the center. The bubble can be regarded as a result of the initial hot spot that undergoes the fastest nuclear burning is being transported away by convective flow before runaway starts. Similarly, the b1a flame is similar to b1b but at a distance of 100 km. b5 is a collection of 5 rings distributed evenly at a distance of 35 and 60 km, which has a mean distance of 50 km. This structure is similar to the multi-spot ignition kernel as done in Jones et al. (2016). We also include variations of the c3 flame by magnifying or diminishing its size to achieve different initial burnt mass $M_{\text{burn,ini}}$. This attempts to balance the unresolved region where thermonuclear runaway where the hydrostatic oxygen burning takes place.

3.3. Benchmark Model

Figure 1. (upper left panel) The final density and temperature profiles for Models A (mass accretion rate = $10^{-5} M_\odot \text{ year}^{-1}$) and B ((mass accretion rate = $10^{-9} M_\odot \text{ year}^{-1}$). (upper right panel) Similar to the left panel but for the final $Y_e$ profiles. (lower left panel) The final chemical abundance profile for Model A. (lower right panel) The final chemical abundance profile for Model B.

Figure 2. The central density against time for two contrasting models.

3.3.1. Central Density and $Y_e$

In Figure 2 we plot the central density against time for the two models. In both models, the central density
Table 1. The initial configurations and the final results of the simulations. $\log_{10} \rho_c$ is the logarithmic of initial central density in units of g cm$^{-1}$. $Y_{e, \text{in}}$ and $Y_{e, \text{out}}$ are the initial electron fraction of the core and envelope. $Y_{\text{min}}$ is the minimum electron fraction reached in the simulation. $t_{\text{coll}}$ is the time lapse from the beginning of simulation to the moment where the central density exceeds $10^{11}$ g cm$^{-1}$. No $t_{\text{coll}}$ is given for models which do not collapse. $M$ and $M_{\text{burn}}$ are the initial mass and the amount of matter burnt by deflagration in units of $M_{\odot}$. $R$ is the initial radius of the star in $10^3$ km. $E_{\text{tot}}$ and $E_{\text{med}}$ are the final energy and the energy released by nuclear reactions in the units of $10^{50}$ erg. $E_{\text{tot}}$ are not recorded for models which do not explode. Results stands for the final fate of the white dwarf, where "C" and "E" stand for the WD in the state of collapse and expansion at the point when the simulation is stopped. Gravity means the use of Newtonian gravity source or the pseudo-relativistic gravity source.

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mildly increases for the first 0.2 s. However, after 0.2 s, the evolution of the two models deviate from each other. For the collapsing model, after 0.2 s, the increment in density resumes again. The process accelerates and the central density reaches as high as $10^{11} \text{g cm}^{-3}$ at 0.55 s. This shows that the core-collapse is proceeding. For the exploding model, the central density remains steady for about 0.5 s, and then it drops gradually. Its density drops to 1% of its initial value at about 1 s after the trigger of deflagration. This shows that the explosion occurs.

We first note the existence of a hierarchy in the timescale, the strong interaction (in particular NSE timescale) $t_{\text{NSE}}$, the weak interaction timescale $t_{\text{weak}}$ and the hydrodynamical timescale $t_{\text{hyd}}$. For ONeMg core, we have $t_{\text{NSE}} \ll t_{\text{hyd}} < t_{\text{weak}}$ in the core with a typical density $10^{10} \text{g cm}^{-3}$ and a temperature $10^{10} \text{K}$. As a result, the matter composition is almost simultaneously adjusted according to local $(\rho, T, Y_e)$. The highly relativistic and degenerate electron gas has an adiabatic index $\Gamma \approx 4/3$. Furthermore, the photo-disintegration of $^{56}\text{Ni}$ into $^4\text{He}$ or recombination of $^4\text{He}$ into $^{56}\text{Ni}$ further lowers the effective $\Gamma$.

In Figure 3 we plot the change in the central $Y_e$. As in the central density (Fig. 2), $Y_e$ of both models show similar evolution: they decrease quickly from 0.5 to 0.4 by electron capture.

These figures show that electron capture on NSE materials tends to induce contraction of the core by lowering $Y_e$. On the other hand, nuclear energy release due to ONe-deflagration tends to induce expansion of the core. In the present model, electron capture is slow so that these competing processes occur in almost hydrostatic equilibrium, and the outcome depends mostly on the central density.

It should be noted that the trigger of the collapse is very different from iron-core collapse. In the iron-core, the softening of core comes from photo-disintegration, which occurs in NSE timescale, thus the whole process is dynamical. While in ECSN, the softening process depends on the weak interaction, the drop of adiabatic index $\Gamma$ is determined by $t_{\text{weak}}$, and therefore the whole process is quasi-static.

### 3.3.2. Energetic

In Figure 4 we plot the nuclear energy production rate against time for two contrasting models, Model c3-09800-N and Model c3-09900-N. They correspond to the models of the same configuration but with a minor density different at center from $10^9.8$ to $10^9.9 \text{g cm}^{-3}$. They stand for the exploding and collapsing model respectively. Within such a small increment in central density, the energy production rate is very different. Here it is defined by $\Delta Q_i / \Delta t$, where $\Delta Q_i$ is the energy produced by channel $i$ during the given timestep $\Delta t$.

In the exploding model, namely the model with a lower central density, the total energy production rate is always positive, showing that the flame is sufficient rapidly propagating than the energy loss from endothermic nuclear reactions or escaped neutrino. The production rate grows until $t = 1$ s, then quickly drops to zero. This corresponds to the quenching of the flame by expansion.

On the other hand, in the collapsing model, the energy production rate drops below zero after $t = 0.9$ s, this suggests the system is rapidly losing energy, in par-
3.3.3. Flame Structure

To further understand how the collapse takes place, we plot in Figure 5 the flame structure with the temperature color plot. Most parts of the star remain cold while the inner parts, which are burnt in ash, have a high temperature about $9 \times 10^9$ K. The flame front reaches to about 400 km and has a slightly lower temperature ranging from $3 \times 10^9$ to $6 \times 10^9$ K. The flame still mildly preserves the $c_3$ character from the initial flame structure, while the middle "finger" has been polished. In contrast to typical Type Ia supernova models, the central part is not the place with the highest temperature. Instead, the outer part of the ash with a radius 100 - 300 km has the highest temperature. This is because the electron capture has become important that the neutrino can bring away significant energy from the core, which has the highest density.

In Figure 6 we plot the final $Y_e$ similar to the Figure 5. At the end of the simulation, the $Y_e$ has reached as low as $\sim 0.35$ within the innermost 100 km. From 100 to 200 km, the $Y_e$ gradually increases up to 0.42. From 200 to 500 km, the $Y_e$ increases up to 0.5. Outside the flame, where $r > 500$ km, no electron capture takes place and the $Y_e$ is everywhere 0.5.

3.3.4. Flame Propagation

At last, we plot successively how the flame propagates in time. We plot in Figure 7 the flame structure and temperature color plot of the collapsing model at 0, 0.125, 0.25 and 0.375 s respectively. The $c_3$ flame is used as the initial flame structure as shown in the upper-left plot. The whole ash has a rather uniform temperature due to the complete burning taking place at this high central density about $10^{10}$ g cm$^{-3}$. At the beginning the flame quickly propagates due to its high temperature, which extends to about 200 and 250 km at $t = 0.125$ and 0.250 s. At the same time, the "finger" structure becomes less obvious. This is related to the competition between electron capture and thermal energy released by deflagration. On one hand, the high temperature owing to ONe flame can boost the propagation of the finger, thus enhancing the instability feature of the flame. However, the large burning zone also allows rapid electron captures to take place, which lower the pressure of the degenerate electron gas. As a result, the elongated part experiences a stronger suppression in flame propagation along the radial direction. On the other hand, the suppression is weaker for the trough part of the flame, thus it can propagate slightly faster than the cusp part. As a result, the "finger" structure is polished. In the lower-right plot, the flame appears to be laminar with a wide segment of flame which shows no signs of perturbations.

3.4. Effects of Central Density

3.4.1. Model with a centered ignition kernel

In this part we discuss the global behaviour of the ONeMg core which is ignited by a centered flame for different central density.

First we compare the evolution of the flame to contrast the differences in flame propagation in models which explode as a thermonuclear runaway with models with collapse into a neutron star. In Figure 8 we plot the temperature color plot of the Model c3-09850-N from 0 to 1.25 s at an interval of 0.25 s. The high temperature in the core also corresponds to the matter being burnt by the O-Ne deflagration. This model shows a thermonuclear runaway which behaves like a weak Type Ia supernova. The $c_3$ flame is used in order to enhance the
development of Rayleigh-Taylor instability. Very soon after the flame is launched, the flame size grows to a radius 150 km in 0.5 s. Most of the ash is extremely hot, $> 9 \times 10^{9}$ K. At $t = 0.75$ s, the flame grows to a size of 450 km, but the expansion of the flame ash cools down the ash to $8 \times 10^{9}$ K. As time evolves, the nuclear ash leaves NSE at about 1.25 s, where most matter has a temperature below $5 \times 10^{9}$ K. The flame at that time is already 2000 km large, while substructure in the flame develops, which includes both the Rayleigh-Taylor instability by the injection of flame and the Kelvin-Helmholtz instability by the curly shape along the ash.

For models with heading of c3 (Models c3-09800-N, c3-09850-N, c3-09900-N, c3-09925-N, c3-09950-N, c3-09975-N, c3-10000-N), we compute the deflagration phase of WD at different central densities but with the same flame structure of c3 using the Newtonian gravity. In this series of model, when central density increases, the total mass increases from 1.38 to 1.39 $M_{\odot}$. Only a mild rise in mass is observed due to the highly degenerate electron gas. On the other hand, the radius decreases from $1.54 \times 10^{3}$ to $1.36 \times 10^{3}$ km. The opposite variations in the mass and radius are consistent with the WD model. The minimum $Y_e$ also drops when $\rho_c$ increases, because the electron capture at high $Y_e$ increases as density increases. The collapse time, which is related to the drop of $Y_e$ also drops. Similar, we observe a drop in the burnt mass. For models which explode, Models c3-09800-N and c3-09850-N, about 1 $M_{\odot}$ is burnt. For the collapsing models, the faster the
collapse it has, the smaller amount of fuel is burnt. In the exploding models, due to an efficient energy loss by neutrino emission, the final energy is much lower than typical Type Ia supernovae, which is around $10^{49}$ erg, in contrast to the nuclear energy release, which is in the order of $10^{50}$ erg. In Fig. 4, we plot the luminosity against time for Models c3-09800-N and c3-09900-N. The two models represent a typical explosion and a direct collapse. For both Models c3-09800-N and c3-09900-N, the luminosity gradually increases as the flame propagates outward. But that of Model c3-09900-N starts to drop at 0.8 s due to the neutrino emission by the electron capture process. Quickly at 0.9 s, the energy by neutrino emission becomes sufficiently efficient that the whole WD loses energy. This cools down the core, slows down the expansion and then makes the core collapse.

In Fig. 9 we plot the central densities for Models c3-09800-N, c3-09850-N, c3-09900-N, c3-09925-N, c3-09950-N, c3-09975-N and c3-10000-N. Refer to Table 1 for the details of the configurations.

In Fig. 10 we plot the central electron fraction as a function of time for the six models as in Fig. 9. Unlike the central densities, the central electron fraction drops drastically for about 0.5 second until it reaches some equilibrium value. The equilibrium $Y_e$ decreases while the initial central density increases. For the models which show a direct collapse, the drop of central $Y_e$ slows down at $Y_e \approx 0.38$ around 0.3 to 0.5 s then, it further decreases to 0.36, as the central densities of these models further increase to $10^{11}$ g cm$^{-3}$. For models which show an explosion, the central electron fraction drops as similar to the models which directly collapse, but they reach a higher equilibrium $Y_e$ compared to those models. In particular, Models c3-09800-N and c3-09850-N show an equilibrium $Y_e$ of 0.39 and 0.40 at the time of 0.7 to 0.8 s after the deflagration has started. Following the expansion of the star, the central $Y_e$ gradually increases and reaches another equilibrium value of about 0.40 at $t \approx 1.1$ s.

In Fig. 11 we plot the total burnt mass against time for the same set of models as Figure 9. For models which can explode, including Model c3-09800-N, c3-09850-N and c3-09900-N, the whole star is swept by the deflagration wave within 1.5 s. About 1 - 1.4 $M_\odot$ is burnt by O-Ne deflagration. For Models c3-09925-N, c3-09950-N
and c3-10000-N, which directly collapse into a neutron star, the deflagration wave stops burning extra material at 0.4 - 0.6 s. This shows that the flame cannot propagate faster than the contraction speed of the star. About 0.3 - 0.5 $M_{\odot}$ of the matter, mostly the matter of the inner core, is burnt.

3.4.2. The b1a Series

For models with heading b1a (Models b1a-09800-N, b1a-09875-N, b1a-09900-N, b1a-09925-N, b1a-09950-N and b1a-10000-N), they are the WD models similar to above, but with an initial flame b1a, which means a flame bubble (a ring in the three-dimension visualization) of size 15 km at 50 km away from the WD center. Since the initial configurations are prepared from a WD at hydrostatic equilibrium. The initial masses and radii are the same as those of the c3 series. Models b1a-09800-N, b1a-09875-N, b1a-09900-N, b1a-09925-N are exploding while the others are collapsing. In general, the trend of the $Y_e$ at the end of simulations are similar that a higher $\rho_c$ implies a lower $Y_e$. However, for models with the same central density, $Y_e$ is higher for the b1a flame than the c3 flame. Also, less mass is burnt and the direct collapse occurs faster, for the same central density, with an exception of Model b1a-09875-N. Due to a shorter time for the deflagration wave to sweep the fuel before the core collapse, less energy is released by nuclear reactions as central density increases. It can be seen that the general pattern for the b1a series is comparable with the c3 series.

In Figure 12 we plot the central density against time similar to Fig. 9. Due to the off center burning, there is not any change of central density before 0.1 second. Once the flame reaches the center, the central density drops abruptly due to the expansion of matter. Then, the central densities of all six models begins to increase. Models with initial central densities greater than $10^9 \, 925 \, g \, cm^{-3}$ begins the collapse at 0.4 - 0.7 s. Again, the collapse time decreases when central density increases. On the contrary, Models b1a-09800-N, b1a-09875-N and b1a-09900-N expand at about 0.5 - 0.7 s. In particular, the central density of Model b1a-09900-N can reach as high as $10^{10} \, g \, cm^{-3}$, before the expansion takes place.

In Figure 13 we plot similar to 12 but for the central $Y_e$. Similar to the central density, there is no change in central $Y_e$ before 0.1 second, when the flame has not reached the core. After that, it quickly drops with a rate proportional to the central density, and slows down after it reaches about 0.38 - 0.41. For models which directly collapses, the central $Y_e$ quickly resumes its fall again and reaches 0.35 - 0.36 at the end of the simulations. In Models b1a-09925-N and b1a-09950-N, there are mild bumps in central $Y_e$ at $t \approx 0.6$ s. This is because the off center burning has led to uneven distribution of $Y_e$. Unlike the Models with c3 flame, the central ignition allows that the matter with the higher densities to be burnt for the longer time, thus having a longer time for electron capture and a lower $Y_e$. This creates a distribution of increasing $Y_e$ as moving away from the core. For the b1a cases, the region which undergoes the longest duration of electron capture is away from center, this creates an non-monotonic distribution of $Y_e$, where during the collapse, matter with different $Y_e$ contributes. However, for Model b1a-10000-N, the direct collapse occurs without reaching the equilibrium $Y_e$. Therefore, the electron capture around all the region is similar. There is no jump in $Y_e$.

In Figure 14 we plot the burnt mass against time for the same set of models as Figure 12. Models which can explode, including Models b1a-09800-N, b1a-09875-N and b1a-09900-N, the O-Ne deflagration can consume most material in the star ($1 - 1.4 \, M_{\odot}$) within 1.5 s, the central density of Model b1a-09900-N can reach as high as $10^{10} \, g \, cm^{-3}$, before the expansion takes place.

In Figure 13 we plot similar to 12 but for the central $Y_e$. Similar to the central density, there is no change in central $Y_e$ before 0.1 second, when the flame has not reached the core. After that, it quickly drops with a rate proportional to the central density, and slows down after it reaches about 0.38 - 0.41. For models which directly collapses, the central $Y_e$ quickly resumes its fall again and reaches 0.35 - 0.36 at the end of the simulations. In Models b1a-09925-N and b1a-09950-N, there are mild bumps in central $Y_e$ at $t \approx 0.6$ s. This is because the off center burning has led to uneven distribution of $Y_e$. Unlike the Models with c3 flame, the central ignition allows that the matter with the higher densities to be burnt for the longer time, thus having a longer time for electron capture and a lower $Y_e$. This creates a distribution of increasing $Y_e$ as moving away from the core. For the b1a cases, the region which undergoes the longest duration of electron capture is away from center, this creates an non-monotonic distribution of $Y_e$, where during the collapse, matter with different $Y_e$ contributes. However, for Model b1a-10000-N, the direct collapse occurs without reaching the equilibrium $Y_e$. Therefore, the electron capture around all the region is similar. There is no jump in $Y_e$.

In Figure 14 we plot the burnt mass against time for the same set of models as Figure 12. Models which can explode, including Models b1a-09800-N, b1a-09875-N and b1a-09900-N, the O-Ne deflagration can consume most material in the star ($1 - 1.4 \, M_{\odot}$) within 1.5 s,
with the Model b1a-09900-N has the highest burnt mass at the end of simulation. There is a clear trend that the final burnt mass increases with the central density. On the other hand, for models which collapse, including Models b1a-09925-N, b1a-09950-N and b1a-10000-N, the deflagration wave can sweep from 0.3 - 0.7 $M_\odot$ of the ONeMg core before contraction suppresses the flame propagation. This occurs at 0.4 - 0.7 s, where star models with a lower initial central density has a later flame-stopping time. Again, a clear trend of $M_{\text{burn}}$ can be observed that when the central density increases, the final burnt mass decreases. This is consistent with the picture that at a higher central density, the electron capture rate is faster so that the star can trigger its collapse at early time, resulting in a lower burnt mass.

### 3.4.3. The b1b Series

In this series we further study the density dependence of ONeMg WD with an off-centered flame at 100 km from the origin. The models include Models b1b-09900-N, b1b-09950-N and b1b-10000-N. The flame structure in this series of models is similar to b1a, but the "flame ring" is located at 100 km apart from the core. Similar to the b1a series, the initial profiles are exactly the same as the c3 series that they share the same mass and radius for the same central density. In this series, Models b1b-09900-N and b1b-09950-N are exploding while the others are directly collapsing. Similar to the two series above, the higher the central density the WD has, the lower the final $Y_e$ at the end of simulation and a faster collapse it has. Also, less nuclear energy is released owing to a smaller mass of fuel is burnt by deflagration wave.

In Figure 15 we plot the central density against time for the four models similar to Figures 9 and 12. With a flame bubble located farther from center, the flame needs about 0.3 s to reach the center, which creates a small drop in central density. At around 0.5 s, Models b1b-09975-N and b1b-10000-N starts its collapse. The central density of Model b1b-09950 also increases above $10^{10}$ g cm$^{-3}$ after 0.5 s, but drops again when the star expands. Model b1b-09900 shows almost no contraction when the electron captures take place at the core.

In Figure 16 we plot the time evolution of central $Y_e$ for the same series of models similar to Figures 10 and 13. There is no change in $Y_e$ at the first 0.3 s. After the deflagration wave has swept across the center, $Y_e$ immediately drops with a rate proportional to the initial central density. In Models b1b-09750-N and b1b-10000-N, the electron captures mildly slow down when $Y_e \approx 0.37$, and then the drop resume again until the end of simulations, down to a value of $\approx 0.36$. In contrast, $Y_e$ show an obvious equilibrium value at 0.39 and 0.42 for Models b1b-09900-N and b1b-09950-N. The latter one
Figure 17. Same as 15, but for the total burnt mass. The asterisk stands for the moment the central density of the star reaches $10^{10.5} \text{ g cm}^{-3}$.

remains the same value after the expansion starts, while the former one slight increases to 0.41, as the matter in the core begins to mix with the surrounding material, which has a higher $Y_e$.

In Figure 17 we plot the total burnt mass against time for the same set of models as Figure 15. The b1b flame model has a similar trend as b1a model. Models which explode, including Models b1b-09900-N and b1b-09950-N, has a higher final $M_{\text{burn}}$, which ranges from 1.0 - 1.4 $M_{\odot}$. Again the whole nuclear runaway lasts for about 1.5 s. For models which collapse, including b1b-09975-N and b1b-10000-N, they have a significantly lower $M_{\text{burn}}$ of 0.7 and 0.5 $M_{\odot}$ respectively, where the flame stops consuming new fuel at $\approx 0.5$ s.

3.5. Effects of input physics

As remarked in the introduction, the influence of relativistic effects on the collapse condition is not yet well studied in the literature. In order to understand whether this component is important in the simulations, we study the counterpart models for Models c3-10000-N and c3-10200-R, namely c3-10000-N, c3-10200-R. We choose these two models because they have the most compact ONeMg among all the models we have in this study, we therefore expect the relativistic effects will be the most pronounced. In general to embed the physics of relativistic gravity requires a complete restructure of the code due to the new set of conserved quantities and the emerge of the metric tensor. However, a trivial extension for this type of simulations is not feasible owing to the extensive use of electron capture table and the nuclear statistical equilibrium table, which can introduce error during interpolation and make the conversion from conservative variables (i.e. local mass-energy density, momentum density and energy density) to primitive variables (i.e. rest-mass density, velocity and specific internal energy) difficult. In view of that, we use a model that can approximate the enhancement of gravity within the Newtonian gravity framework. We follow the prescription in Kim et al. (2012). Based on the Poisson equation for the gravitational potential $\nabla^2 \Phi = 4\pi G \rho$, where $\Phi$ and $\rho$ are the gravitational potential and matter mass density. We replace $\rho$ by $\rho_{\text{active}}$, where

$$\rho_{\text{active}} = \rho h^2 \frac{1 + v^2}{1 - v^2} + 2P,$$

(13)

where $P$, $v^2$ are the fluid pressure and the magnitude square of the velocity. $h = 1 + \epsilon + P/\rho$ is the specific enthalpy of the matter. In this sense, the extra mass-energy owing to the internal energy and the kinematic of the matter are included.

To demonstrate the effects of including the relativistic correction in the gravitational potential, in Figure 18 we plot the central density for the models with central flame of burnt mass $8.56 \times 10^{-4} M_{\odot}$ and with initial central densities $10^{10.0}$ and $10^{10.2}$ g cm$^{-3}$ respectively.

In Figure 19 we plot similar to 18 but for the central electron fraction. In both cases a direct collapse is observed. The evolution of the central density is not sensitive to the general relativistic corrections to the gravitational potential. The two curves are overlapping each other for both pairs of models at all time. Similar results can be found for the electron fraction. They agree with each other that owing to the almost identical evolution of the density, which plays an important role in the equilibrium $Y_e$, the evolution of $Y_e$ is also independent of the presence of relativistic effect. This provides the confidence in preparing the initial condition of the collapsing phase of the ONeMg WD, a Newtonian treatment.
We plot the mass being burnt by the deflagration wave for the three models. For Model bc3-09950-N the $M_{\text{burn}}$ smoothly rise to $\approx 1.3M_\odot$ at $t = 1$ s, showing that almost the ONeMg core is quickly burnt.

![Figure 19](image1.png)  
**Figure 19.** Similar to Figure 18, but for the central electron fraction.

is sufficient in determining the density and $Y_e$ profiles accurately.

By combining these models, it can be seen that even when the GR effects are included, namely the contribution of pressure and internal energy as a mass source and with the Lorentz factor included, for the highest density model $\rho_c = 10^{10.2} \text{ g cm}^{-3}$, GR effects are almost negligible which poses no changes to the evolution of central density and $Y_e$, and thus the collapse condition.

3.6. Effects of Initial Flame Size

Before the runaway of the ONe deflagration, besides the hydrostatic burning of O and Ne forming Mg, the electron capture of $^{20}\text{Ne} \rightarrow ^{20}\text{F} \rightarrow ^{20}\text{O}$ is another major energy source for providing heat in the core as the trigger.

However, how the actual flame starts and its size depend on many parameterized processes, such as semi-convection, convective URCA process and the growth rate of the ONeMg core mass. The semi-convection becomes important during electron capture in the core. Electron capture creates a mean molecular weight difference between the core with a lower $Y_e$ and the envelope which remain unchanged. Depending on the efficiency of the semi-convection, the runaway size may vary. In general, efficient semi-convection leads to a faster transport of heat produced during electron capture. Also mixing leads to a wider region with low $Y_e$, which promotes contraction. As a result, the core reaches a higher central density before the temperature becomes high.

We emphasize that the uncertainties of semi-convection imply that $10^{10.95}$ given by Ledoux criterion (i.e., completely no mixing) is the lower limit. URCA shell cooling is important to cool down the core and leads to the higher deflagration density.

These uncertainties make the exact central density and temperature at which the deflagration takes place inconclusive, although a higher deflagration density is indicated. Different flame structure should be tested, to exhaust different possibilities where the flame may appear. This includes varying the position of the flame and the size of the flame.

To achieve the actual size such processes require multi-dimensionally modeling of the small scale fluid motion prior to the ignition. To characterize the effect of initial runaway size, we compare three different sizes of flame, c3, mc3 and bc3. The latter two flame structures are the same as the standard c3 flame, but with its size 2 times and 4 times larger.

In Figure 20 we plot the evolution of central density against time for models of different initial flame mass. For comparison, all the models are fixed to have the same central density $10^{9.95} \text{ g cm}^{-3}$. The flame is assumed to be burnt at center with a c3-flame shape but with its size scaled up or down, so that the initial mass being burnt $M_{\text{burn,ini}}$ in NSE ranged from $10^{-4}$ to $10^{-2}M_\odot$. For Model c3-09950-N with $M_{\text{burn,ini}} \sim 10^{-4}M_\odot$, the central density increases for the first 0.1 s, and then it remains constant for the next 0.4 s. Beyond $t = 0.7$ s, the ONeMg core collapses. On the other hand, when $M_{\text{burn,ini}} \sim 10^{-3}M_\odot$, a similar evolution occurs but the collapse starts earlier, at 0.5 s after the simulation. When $M_{\text{burn,ini}} \sim 10^{-2}M_\odot$, a similar contraction occurs at the beginning, but after $t = 0.5$ s, the star central density decreases, showing that the ONeMg WD fails to collapse. Instead, it explodes weakly similar to a Type Iax supernova.

![Figure 20](image2.png)  
**Figure 20.** The time evolution of the central density of Models c3-09950-N, mc3-09950-N and bc3-09950-N. 

In Figure 21 we plot the mass being burnt by the deflagration wave for the three models. For Model bc3-09950-N the $M_{\text{burn}}$ smoothly rise to $\approx 1.3M_\odot$ at $t = 1$ s, showing that almost the ONeMg core is quickly burnt.
Figure 21. Similar to Figure 20 but for the burnt mass. The asterisk stands for the moment the central density of the star reaches $10^{10.5}$ g cm$^{-3}$.

Figure 22. Similar to Figure 20 but for the burnt mass.

by the deflagration and almost the whole star is burnt. On the other hand, for Models c3-09950-N and mc3-09950-N, $M_{\text{burn}}$ rises in a slower rate, and they reach their asymptotic value of 0.4 $M_\odot$ at 0.6 and 0.5 s respectively. This shows that prior to the collapse, the flame itself is already halted due to the contraction of the star. Also, both models share a similar asymptotic $M_{\text{burn}}$ when collapse starts, showing that the collapse depends on less sensitively on the initial flame properties.

In Figure 22 we plot similar to Figure 20 but for the central $Y_e$. Similar to previous models, all three models show a rapid drop of $Y_e$ once the core is burnt to NSE. It drops to about 0.39 within 0.3 s, and the capture rate slows down. The equilibrium $Y_e$ of c3-09950-N is slightly higher than Models mc3-09950-N and bc3-09950-N. Then at $t = 0.5$ and 0.6 s, $Y_e$ drops rapidly again for the two collapsing models, Models c3-09950-N and mc3-09950-N. However, in Model c3-09950-N, due to the failure of collapse, its expansion allows mixing of the core matter with the matter of the outer zones, which has on average a higher $Y_e$. Its $Y_e$ slowly increased to 0.39 and remains unchanged after $t = 0.8$ s.

Combining these figures, it can be seen that the initial flame size also play a role in determining the collapse condition. In particular, a small flame $\sim 10^{-4} - 10^{-3} M_\odot$ will favour more to the collapse, while a large flame favour more to the explosion. This is because the large flame has a larger surface area, which can balance the contraction effects due to the electron capture.

3.7. Effects of Flame Physics

In modeling turbulent flame, the formula describing the relation between the turbulent velocity $v'$ and the effective flame propagation speed $v_{\text{turb}}$ is necessary. However, only a statistical description is available due to the stochastic nature of turbulent motion. Also, the terrestrial experiment is not yet capable in reproducing such extreme environment experimentally. As a result, how the turbulent motion can enhance the propagation of flame and also the effective flame speed remain unclear. In the literature of Type Ia supernova using the turbulent flame model, the typical formula assumes self-similar flame. With the renormalization scheme (Pocheau 1994), the general formula writes

$$v_{\text{turb}} = v_{\text{lam}} \left[1 + C_n \left( \frac{v'}{v_{\text{lam}}} \right)^n \right]^{1/n}, \quad (14)$$

with $v_{\text{lam}}$ being the laminar flame propagation speed while $C_n$ and $n$ corresponds to the constant derived from experiments and $n$ describes the velocity spectra of the turbulence structure. This formula has two asymptotic properties which are expected experimentally. 1. The effective propagation speed reduces to the laminar flame speed, when $v' \rightarrow 0$. This corresponds to the case that when there is no perturbation to the surface structure of the flame, the flame propagates as a laminar wave. 2. The effective propagation speed has a limit $\approx \sqrt{C_n v'}$. This means when the fluid motion is highly turbulent, the flame no longer depends on laminar flame speed, but follows only the velocity fluctuation inside the fluid.

However, one shortcoming in this model is that in deriving this formula, an isotropic turbulence is assumed due to its assumption of renormalization procedure. However, the existence of gravity confines the direction for the turbulent motion on the iso-potential surface. Also, as the flame is insusceptible towards Rayleigh-Taylor instabilities, the radial direction (pointing from the ONeMg core center) can be different from the other two orthogonal directions.
Figure 23. The evolution of central density against time for Model bc3-09950-N, bc3-09950-N-vf050 and bc3-09950-N-vf025.

Numerically, one has different $C_n$ and $n$ based on the context. In Peter (1999); Schmidt et al. (2006), $C_n = 4/3$ and $n = 2$ which corresponds to the Gaussian distribution of velocity fluctuations. In Hicks (2015), it is shown numerically that for a premixed flame with a one-way reaction such as $H_2$-air mixture, the relation has a best fit of $C_n = 0.614$ when $n = 2$, while $v_{turb} = v_{lam}(1 + 0.4321\tilde{v}_{1.997})$ is the best fit with $\tilde{v}$ being the dimensionless $v'$. The variations of this formula demonstrate that the scaling factor $C_n$ and the scaling power $n$ remain poorly constrained.

To understand the effects of this quantity to the ONeMg core, we vary the original value of $C_n$ (denoted as $C_n^0$) by considering $C_n = 0.25C_n^0$ and $0.50 C_n^0$. They correspond to the turbulent flame where turbulent production is less effective in disturbing the flame structure.

In Figure 23 we plot the central density against time for Models bc3-09950-N, bc3-09950-vf050 and c3-09950-vf025. In this series of models, Models bc3-09950-N explodes while bc3-09950-vf050 and bc3-09950-vf025 collapse. In terms of central density, the central density of all three models mildly increase for the first 0.3 s. For the collapsing models, the increase of $\rho_c$ resumes at $t \approx 0.4$ s. On the other hand, $\rho_c$ in Model bc3-09950-N slowly drops till $t = 0.6$ s. Then accompanying the expansion, its central density rapidly drops after $t = 0.6$ s. At $t = 1$ s, the central density drops to about 1% of its original value.

In Figure 24 we plot similar to Figure 23 but for the burnt mass $M_{burn}$. The exploding model bc3-09950-N has its matter up to $1.3M_\odot$ being burnt within 1 s. On the other hand, the two collapsing models, have their $M_{burn}$ about 0.7 and $< 0.1M_\odot$ when $C_n$ reduced to 50% and 25% of the original value. They do not show any asymptotic values, showing that the deflagration is still actively expanding its size when the core begins its collapse.

In Figure 25 we plot the central $Y_e$ similar to Figure 23. In the exploding Model bc3-09950-N, the central $Y_e$ again quickly drops from 0.5 to 0.38 within 0.3 s. Unlike previous test on the effects of flame size, the large initial flame we used is less changed by the surrounding. Beyond $t = 0.3$ s, the collapsing models restart their electron capture and the central $Y_e$ drops below 0.37 at $t = 0.5$ s. On the other hand, the failed-collapse model, during its expansion, has its central $Y_e$ slightly relaxed towards 0.39 as its final value beyond $t = 0.8$ s.

Combining these three plots, it can be seen that the effective formula of the turbulent flame prescription also plays a role in the ONeMg collapse condition similar to the initial flame size and the properties of the flame kernel. In particular, models tend to collapse (explode) when the flame is slow (fast). This is because the slower
3.8. Effects of Laminar Flame Propagation

In all simulations described above the turbulent flame is used. In this sense, we assumed that the nuclear flame in ONeMg core behaves like the turbulent flame in Type Ia supernovae (Reinecke et al. 1999; Schmidt et al. 2006), that the eddy motion in the sub-grid scale can smear the flame front, which distorts the laminar structure and enlarges the surface area. We notice that in some other literature of Type Ia supernova (See e.g. Plewa (2007)), the turbulence smearing effect is not considered in flame propagation. In those works, buoyancy due to the density contrast across the flame surface is the only source which perturb the flame surface. This results in a much slower flame because the Rayleigh-Taylor instabilities depend on the strength of gravity. Therefore, the flame basically propagates like laminar flame in the stellar core.

To extract the effects of the flame nature, namely whether it depends on the smearing effects by eddy motion, we consider another limit for the flame propagation, i.e. the laminar flame limit. This also corresponds to the work in Nomoto & Kondo (1991) to contrast the effects of flame nature on the explode-collapse bifurcations. Notice that if the flame can only be smeared buoyancy, the flame propagates like laminar flame until the flame reaches to a outer region ~ 100 km where local gravity becomes important. This is because Rayleigh-Taylor characteristic velocity scales as \( \sim \sqrt{gr} \). For a constant density sphere, \( g(r) \sim 4\pi Gr/3 \), which tends to zero at \( r \rightarrow 0 \). This means that there is no flame distortion and hence no flame acceleration when the flame starts in the core.

In this section, we explore this effect by considering models of the same flame structure but with laminar flame propagation. We repeat the numerical experiments at different central density to examine whether the transition density changes accordingly. However, we do not carry out simulations on the \( c_3 \) flame with laminar flame propagation. It is because we observe that in all models we have explored, the flame creates Rayleigh-Taylor instabilities along the symmetry axis in all density, suppressing the collapse to occur even for the highest density model \( \rho_c = 10^{10.2} \) g cm\(^{-3}\). When this instability takes place, the flame can be much enhanced along the two symmetry axis and propagate much faster compared to other direction. Therefore, we choose only off-center flame structure to isolate the effects of slow laminar flame.

In Models b1b-09950-N-Lam, b1b-09975-N-Lam and b1b-10000-N-Lam we run three contrasting models which can be compared with Models b1b-09950-N, b1b-09975-N and b1b-10000-N respectively. As described above, the models with an ending "-Lam" corresponds to the flame which only propagate in laminar way. We can see that Models b1b-09950-N-Lam, b1b-09975-N-Lam and b1b-10000-N-Lam collapse into neutron star. In contrast, we can compare with Model b1b-09950-N, which has the same configuration but with turbulent flame prescription, this model directly explodes like a Type Iax supernova. On the other hand, when laminar flame prescription is used, the star direct collapses. This shows that the nature of the flame is important for determining the final result of the ECSN. In particular, whether the flame interacts with sub-grid eddy motion, or only interacts with buoyancy smearing, will change the collapse-explode bifurcation of the benchmark model \( \rho_c = 10^{9.95} \) g cm\(^{-3}\).

To characterize the difference of flame propagation by the turbulent flame and laminar flame, we plot in the Figure 26 the temperature colour plot of Model b1b-09950-N from 0 to 1.25 s at an interval of 0.25 s. The hot elements also trace the flame structure. Here the b1 structure is used. As described above, we do not want the Rayleigh-Taylor instability along the symmetry axis affects the results, which is unphysical in nature, so we choose an off-centered flame such that the flame has shorter time to interact with symmetry boundary. The turbulent flame allows the structure to grow rapidly. Within the first 0.5 s, there is a two bump structure developed and the size has grown to about 450 km. At \( t = 0.75 \) s onwards, the large-scale structure freezes and one can see the two "fingers" shape has evolved. Again, the gap between the "fingers" appears when the cold fuel sinks faster into the hot ash than the flame propagation. At \( t = 1.0 \) s, the flame expands rapidly to 2000 km, where the surface becomes more perturbed that the hydrodynamics instabilities become well pronounced.

In Figure 27 we plot similar to Figure 26 but for Model b1b-09950-N-Lam from 0.2 - 1.2 at selected time. A qualitative comparison of the flame structure already demonstrates drastic differences between the propagation of laminar flame and turbulent flame. At early time before 0.4 s, the fluid motion has largely reshaped the original spherical flame. Many small scale "mushroom shape" swarmed out as a manifestation of the Rayleigh-Taylor instabilities. At \( t = 0.6 \) s, the flame has finally anchored at the core, where one can see a hot core of size 150 km. After that, core does not grow in a visible size.
Figure 26. The temperature colour plot of the Model b1b-09950-N at 0, 0.50, 1.00 and 1.25 s of the simulations. The hot region also represents the region being burnt by the O-Ne deflagration.

Figure 27. The temperature colour plot of the Model b1b-09950-N-Lam at 0.2, 0.6, 1.0 and 1.2 s of the simulations. The hot region also represents the region being burnt by the O-Ne deflagration.
However, there is some hot flow along the rotation axis. This is the mentioned enhancement due to Rayleigh-Taylor instability along the symmetry boundary. However, this enhancement does not affect the results, that within a further 0.2 s, the core directly collapses.

We further study their evolution by the stellar quantities. In Figure 28 we plot the central density against time for Models b1b-09950-N-Lam, b1b-09975-N-Lam, b1b-10000-N-Lam. Model b1b-09950-N and Model b1b-09950-N-Lam are the same before $t = 0.4$ s, when the flame has not arrived the core. Once it reaches the core, namely at $t = 0.4$ s for Model b1b-09950-N and at $t = 0.8$ s for Model b1b-09950-N-Lam, they deviate from each other. Both models show an increment in central density due to the softening effect by electron capture. However, for Model b1b-09950-N, the central density starts to drop beyond $t = 0.6$ s, showing that the turbulent flame has released sufficient energy to support against the inward flows. On the other hand, in Model b1b-09950-N-Lam, the increment in central density leads to the collapse where there is no sign for the core to reach a temporary equilibrium. A similar evolution can be seen for Model b1b-10000-N-Lam. After $t = 0.8$ s where the flame reaches the core, the increment in central density further triggers the collapse.

In Figure 29 we plot similar to Figure 28 but for the central $Y_e$. After the flame has reached the core, which can be noted by the sudden drop of $Y_e$, the exploding b1b-09950-N stops electron capture at $t \approx 0.5$ s at the value 0.3. It later returns to a high value as it starts to mix with higher $Y_e$ material in outer radius. On the other hand, the $Y_e$ does not reach any equilibrium value once the core is burnt. The effective electron capture rate is slowed down at $t = 0.9$ s. Model b1b-10000-N-Lam also has a similar pattern. But it shows a short equilibrium $Y_e$ at 0.8 s, showing that the center does not prefer to collapse straight away; while the outer matter, which continues to flow inwards, as implied by the growth of the central density, trigger further electron capture which makes the collapse.

In Figure 30 we plot similar to Figure 28 but for the total burnt mass. The asterisk stands for the moment the central density of the star reaches $10^{10.5}$ g cm$^{-3}$. Different from all previous models, the Models b1b-09950-N-Lam and b1b-10000-N-Lam have a drastically lower $M_{\text{burn}}$. For Model b1b-09950-N which can explode as a thermonuclear runaway, it burns about 1 $M_\odot$ at the end of simulation. On the other hand, without all smearing effects from hydrodynamics instabilities and eddy motions, the flame behaves like as being confined in the core, with only about 0.02$M_\odot$ matter is burnt when $M_{\text{burn}}$ reaches its asymptotic value. The whole nuclear runaway lasts for about 1 s, which is almost double to the collapse cases in the same b1b-series described in previous section. One can note that
at $t = 0.8$ s in Figure 27 the Rayleigh-Taylor instability begins to appear along the $z$-axis. It does not affect the $M_{\text{barn}}$ against time significantly. This clarifies our previous motivation that using an off-center flame can avoid the artificial Rayleigh-Taylor instability to over-estimate the fuel-consumption rate by the nuclear flame.

3.9. Effects of Pre-Runaway Time Lapse

In our simulations, we put in the initial flame by hand at the beginning of our simulations. As described in previous sections, the position and size of the initial runaway depends strongly on the local heat diffusion properties and multi-dimensional fluid motion, which remains unclear in stellar evolution. Owing to the limit of computational size, the initial flame we put must be at least a few times of the grid size ($\approx 10$ km). Following the analysis of Timmes & Woosley (1992), one requires in fact a very small amount of matter ($10^{-2} - 10^{17}$ g), depending on the local temperature, such that the runaway can carry out spontaneously. In all of these ranges, it has a typical size much smaller than the typical resolution ($\sim$ km) when the first nuclear runaway starts. To develop into the flame that we put in by hand, there could be a time-lapse between the initial runaway (flame size $\ll$ km) and the current flame that we put in ($\sim$ 10 km). The exact time difference depends on multiple factors, including the inner fluid motion of the core, local turbulence strength and the initial runaway size (a random process owing to the probabilistic nature of the decay process). As a result, the $Y_e$ inside the initial runaway, due to the high temperature ($\sim 10^{10}$ K), can be different because electron capture and isobaric expansion can already occur inside these runaway zone. In the current procedure, by putting a flame directly in the simulation, it is possible that the electron fraction in the initial ash have already lowered. With a lower initial $Y_e$, the suppression of flame propagation will be weaker, which may lower the critical density for the collapse-explode bifurcation.

As an approximation of these effects, we prepared some more models with a much smaller c3 flame (for a few grids in order to make the flame shape well resolved by the level-set method). Such flame does not follow the fluid advection, to mimic in early time that at very small scale the turbulent effect is not important. We allow the flame to only grow self similarly. Meanwhile, all nuclear reactions, such as photo-disintegration of $^{56}\text{Ni}$ into $^4\text{He}$, and electron capture, can proceed. The process continues until the flame becomes the size of the c3-flame we used in previous simulations. Meanwhile, all nuclear reactions, such as photo-disintegration of $^{56}\text{Ni}$ into $^4\text{He}$, and electron capture, can proceed. At that point, we allow the flame to follow fluid advection.

In this series of model, we change the initial size of the flame from 25 % to 75 % of the original flame used in the c3 Model series. We choose the largest flame model because we want to contrast the effects of time lapse in the initial laminar phase.

In Figure 31, we plot the colour plot of Model c3-09950-N-B050 with ONe deflagration contour.

Figure 32. The evolution of central density against time for Model bc3-09950-N, bc3-09950-N-B075, bc3-09950-N-B050 and bc3-09950-N-B025.
In Figure 32 we plot the evolution of central density against time for Models bc3-09950-N, bc3-09950-N-B075, bc3-09950-N-B050 and bc3-09950-N-B025. Model bc3-09950-N explodes as described in previous sections while the other three models collapse. At the first 0.2 s, all four models show a similar $\rho_c$ evolution. However, beyond that time point, the density in the three models are slightly higher, which leads to its later collapse at 0.5 - 0.6 s.

In Figure 33 we plot similar to Figure 32 but for the $M_{\text{burn}}$. For the collapsing models, they can only burn about half of what the exploding model bc3-09950-N has totally burnt, about 0.4 - 0.6 $M_{\odot}$. Notice that the growth of $M_{\text{burn}}$ is later when the initial flame is scaled smaller because the notation $t = 0$ stands for the moment where flame is put, instead of the moment where the flame is allowed to be advected. All four models show a similar growth rate, showing that even the inner core has a very different $Y_e$ distribution, it does not change the flame propagation. However, when the initial flame becomes smaller, the final $M_{\text{burn}}$ decreases. It is because the core settles to a higher density prior to the free propagation of the flame.

In Figure 34 we plot similar to Figure 32 but for the central $Y_e$. The three collapsing models show a qualitatively similar pattern as those in previous section. But they all share a lower $Y_e$ compared to the exploding model bc3-09950-N. This is related to the difference in relaxation of the initial flame by isobaric expansion. From the results shown in this section, it shows that the ONeMg core evolved from stellar evolutionary model is likely to collapse into a neutron star and forms ECSN, but the exact details will still strongly depend on the pre-runaway scenario, where the electron captures in the sub-grid scale are important for the initial $Y_e$ profile and also its subsequent dynamics.

4. DISCUSSION

4.1. Global Properties of ONeMg WD

A typical ONeMg evolution is in general more complicated than Type Ia supernova explosion, despite the similarity of deflagration process. This is because the initial flame can be much localized compared to Type Ia supernova due to the weaker convection, which requires a finer grid size to resolve. Also, the higher den-
Table 2. The explode-collapse transition density for our models. Flame stands for the initial flame shape, which includes the centrally ignited kernel (the c3 flame structure), and off-center kernels (the b1 and b5 configuration). See Reinecke et al. (1999) for the graphical illustration of these flame structure. $M_{\text{burn}}$ is the initial mass assumed to be already burnt by deflagration prior to the beginning of simulations. Relativity stands for the inclusion of relativistic effects in the gravitation potential term. $\rho_{\text{crit}}$ is the critical density in g cm$^{-3}$ above which the models exhibit a direct collapse.

<table>
<thead>
<tr>
<th>Series</th>
<th>Flame</th>
<th>$M_{\text{burn}}$</th>
<th>Relativity</th>
<th>$\log_{10}(\rho_{\text{crit}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>c3</td>
<td>$8.56 \times 10^{-4}$</td>
<td>No</td>
<td>9.900</td>
</tr>
<tr>
<td>2</td>
<td>c3</td>
<td>$8.56 \times 10^{-4}$</td>
<td>Yes</td>
<td>9.900</td>
</tr>
<tr>
<td>3</td>
<td>c3</td>
<td>$6.67 \times 10^{-3}$</td>
<td>No</td>
<td>9.925</td>
</tr>
<tr>
<td>4</td>
<td>c3</td>
<td>$4.80 \times 10^{-2}$</td>
<td>No</td>
<td>9.975</td>
</tr>
<tr>
<td>5</td>
<td>b1</td>
<td>$1.39 \times 10^{-3}$</td>
<td>No</td>
<td>9.925</td>
</tr>
<tr>
<td>6</td>
<td>b1</td>
<td>$2.59 \times 10^{-3}$</td>
<td>No</td>
<td>9.975</td>
</tr>
<tr>
<td>7</td>
<td>b1</td>
<td>$2.59 \times 10^{-3}$</td>
<td>Yes</td>
<td>9.975</td>
</tr>
<tr>
<td>8</td>
<td>b5</td>
<td>$1.04 \times 10^{-2}$</td>
<td>No</td>
<td>9.975</td>
</tr>
<tr>
<td>9</td>
<td>b5</td>
<td>$1.04 \times 10^{-2}$</td>
<td>Yes</td>
<td>9.975</td>
</tr>
</tbody>
</table>

The modeling can take much longer time for one single model. In view of that, we study the global properties of the ONeMg in order to derive conditions such that the collapse/explosion bifurcation can be estimated without making explicit simulations.

To further quantify our conclusion, we plot in Figure 35 the average electron fraction against time for our models. Notice that this time we only take the matter which is already burnt into account. The fuel electron fraction is ignored since they are not related to the softening of the core. These models contain both those which explode or collapse. Different flame pattern, flame size, central density are covered in this ensemble. Notice that whether a WD collapses or explodes can be roughly classified into two groups, separated by the $Y_e$ boundary. For models with its mean average above (below) that critical value, all models explode (collapse). We find the critical mean $Y_e \sim 0.444$. Notice that this value is obtained from a range of possible progenitors.

In Table 2 we tabulate the transition density from collapse to explode for the flame configuration and input physics we have presented in this work. For models with a centrally ignited kernel, the transition density increases when the initial burnt mass increases. This is consistent with our previous argument that the strength of the ONe-deflagration is directly related to the explosion criteria of ECSN.

4.2. Comparison with literature

Here we briefly compare our methods and results with those reported in the literature. Compared to the variety of Type Ia supernova explosion models and parameter survey conducted in the literature, much fewer works are found in ECSN of ONeMg core.

Jones et al. (2016) is the only one literature similar to this work in the multi-D scenario. We treat their work as a benchmark to justify our numerical results. In Table 3 we tabulate the difference of input configuration and physics models adopted in the simulations. We have attempted to include as much overlap of the microphysics as possible for a better comparison. This includes the nuclear reaction network, electron capture rates and the flame capturing scheme (level-set method). However, the general infrastructure, including the hydrodynamics solvers and equation of states are different. Also, as described in our methodology, we use the 3-step nuclear reaction to represent the energy input from nuclear reaction and the NSE composition change.

Then we compare the global properties of our results. In our simulations, we find that in general an ONeMg WD while central ignition can collapse to form a neutron star when $\rho_c \geq 10^{9.9}$ g cm$^{-3}$. But the transition density increases when the initial ignition kernel enlarges. However, for off-center burning, the corresponding transition density becomes higher that it needs about $\rho_c \geq 10^{9.975}$ g cm$^{-3}$ in order for the ONeMg core to collapse. In Jones et al. (2016), they have presented a series of 6 models, with central densities of $10^{9.90}$, $10^{9.95}$ and $10^{10.2}$ g cm$^{-3}$ respectively. All models start with an off-center ignition kernel made of a few hundred flame bubbles distributed within 50 km, with a total mass about $2 \times 10^{-3} M_\odot$. Variations of models, for comparing the effects of input physics, consist of varying the resolution or including Coulomb correction. They find that no collapse occurs for all models with central densities of $10^{9.9}$ and $10^{9.95}$ g cm$^{-3}$ except the model with a central density $10^{10.2}$ g cm$^{-3}$. Our results are consistent with theirs for the choice of flame with similar mass and central densities. Also, their (our) model show the $10^{10.2}$ g cm$^{-3}$ model has a collapse time around 0.3 (0.26) s, which also agrees with each other (See for example Figure 18 for the evolution of central density). At last we compare the morphology of the flame. In their work, they show the flame structure in Figure 6 and the cross-section cut in Figure 7. We compare this with our results in Fig. 37. The outburst of flame in the spherical shape with Rayleigh-Taylor instabilities induced small-scale sub-structure can be seen. It can resemble with Figure 6 in their work that the structure is spherical with many small scale wrinkles on the surface. Because
our model has a coarser resolution compared to their work, the flame structure in our model show fewer sub-structure as theirs.

At last we compare the performance of the sub-grid turbulence model. In Figure 36 we plot the speed of sound, laminar flame speed and turbulent flame speed of the exploding model. At the beginning, before turbulence is developed, the effective flame speed is dominated by laminar flame speed. When the fluid motion becomes turbulent, the turbulent flame speed quickly overrides the slow laminar flame and reaches an equilibrium value of about a few percents of the speed of sound. This figure can be compared with Figure 4 in Jones et al. (2016) G13 model but with three differences. First, their model is done in three-dimensional Cartesian coordinates, while ours is in two-dimensional cylindrical coordinates. Second, the SGS model is based on the formalism in Schmidt et al. (2006) while ours is based on the scheme in Niemeyer et al. (1995). Both models belong to the class of one-equation model but with different prescription in the numerical turbulent viscosity, eddy production rate and decay rate. Third, their models start from a number of bubbles off-center, while, due to symmetry, we choose a centered flame as the initial flame structure. It can be seen that, even with a very different approach in modeling the sub-grid turbulence, we still have quantitatively similar results to the general behaviour of the sub-grid turbulence, such as the asymptotic value and the range of turbulent flame speed found in the simulation. One major difference is the time when turbulence becomes saturated owing to our choice of initial flame. We choose the c3 flame as done in Reinecke et al. (1999). The finger structure can enhance the generation of turbulence at the very beginning because of the asymmetry. On the other hand, the bubble structure, where bubbles are geometrically isolated at the beginning, makes the generation of turbulence slower because of the initially isotropic expansion of the bubble. Also, in our simulations, the reflective inner boundaries of both planes can create boundary flows, which also perturb the SGS turbulence production.

4.3. Case for Failed-Collapse

As remarked in Woosley & Baron (1992), Type Ia supernova explosion for high central density model is unfavoured. The core can produce a significant amount of neutron rich isotopes. They can pollute its surrounding and lead to severe overproduction of these isotopes, which is inconsistent with solar abundance. Based on their models, they estimate the collapse of WD by accretion to occur at a rate of one per one thousand years. Here we first discuss the properties of the exploding models, and then we analyse the chemical signature of the exploding models.

For models which can explode, we expect they behave like a Type Ia supernova, albeit with a much lower explosion energy. In Figure 36 we show a temperature profile of one of the models which successfully exploded. We note that other flame profiles in the successfully exploded model has a similar structure as this flame profile. Despite we start with a three-finger structure, during the expansion of the flame, the inflow of fuel along the diagonal direction has suppressed the growth of the middle "finger". The other "fingers" on the two ends have successfully developed to a structure with a clear
As mentioned, along the diagonal line, we can see a clear signature of fuel injection by the blue zone about 1000 (500) in the z- (r-) direction. In the core, the initially hot ash which has a maximum temperature around $10^{10} \, K$ has cooled down to around $5 \times 10^{10} \, K$, which is barely to maintain NSE. In the outer part of the flame about 3000 km from center, the outer region has cooled down to around $2 \times 10^{9} \, K$, where all exothermic reactions has ended, except the slow recombinations of $\alpha$-particle to the $\alpha$–chain elements. Near the flame surface in the outer part, due to the developed turbulent flow at low density, clear signatures of the fluid perturbation at the surface can be seen. The "mushroom" shaped Rayleigh-Taylor instabilities around the top of the flame and the spiral shaped Kelvin-Helmholtz instabilities can be clearly seen. These processes are important to collect fuel from the unburnt zone inside the flame. But due to the low density, incomplete burning proceeds, which leaves trace of oxygen and neon in the outer part of flame. At last, there is a narrow band of green area outside the flame. This is an artifact of the simulation which comes from the flame. Notice that we put in the initial flame by hand at the first place. This disturbs the initial equilibrium profiles which we set at the beginning. As a result, the disturbed star reaches a new equilibrium state after having excited the atmosphere.

In all the models, owing to its final energy very close to zero, showing that the WD is marginally unbounded. In this case, as discussed in Jones et al. (2016), only part of the WD is ejected after the WD expands. In particular, when the hot ash bursts out from the core, it transfers part of its energy and momentum to the matter in the envelope by shock compression. As a result, most of the outer matter, which is rich in $^{16}\text{O}$ and $^{20}\text{Ne}$ and part of the inner matter, which is $^{28}\text{Si}$- and $^{56}\text{Ni}$-rich are mixed in the ejecta. The neutron-rich ones, which is produced in the innermost layer, tends to be trapped after the star has expanded. We will calculate the exact nucleosynthesis pattern in future works.

### 4.4. Conclusion and Future Work

In this article we study the evolution of oxygen-neon-magnesium (ONeMg) core using the two dimensional hydrodynamics simulations. We follow the oxygen-neon deflagration phase and study the conditions where the core start to collapse to form a neutron star.

We survey ONeMg core models of various configurations. This includes a central density of $10^{9.80} - 10^{10.20} \, g \, cm^{-3}$, and different flame structure of mass from $10^{-4} - 10^{-2} \, M_\odot$ in a centered or off-centered ignition kernel. We also explored the effects of input physics, which include the general relativistic effects, turbulent flame speed formula and treatment of laminar deflagration phase. We find that except the general relativistic effects, the later two input physics are highly influencing to the collapse condition. In general we find that the ONeMg core with a central density above $10^{9.90} \, g \, cm^{-3}$ is capable to start its collapse after the ONe deflagration has consumed the core and triggered the electron capture. This is consistent with the current picture of stellar evolution that the electron capture supernova of the 8 - 10 $M_\odot$ is likely to be the origin of the lower-mass branch of neutron star population.

To further characterize the effects of the initial configuration and input physics, we list their effects individually.

- The central density $10^9 - 10^{10.2} \, g \, cm^{-3}$ determines primarily the collapse condition. A higher density favour collapse of an ONeMg core;
- The flame structure (centered or off-centered ignition kernel) also determines primarily the collapse condition. When the initial runaway position becomes further from the origin, it becomes less likely to collapse;
- The GR effects is a less important factor. In the density range we considered, the GR effect is negligible to the final evolution and collapse condition;
- The Initial flame size determines primarily the collapse condition. A smaller flame favours collapse of the ONeMg core.
- The properties of turbulent flame determines primarily the collapse condition. A slower flame favours collapse of the ONeMg core;
- The initial treatment of the laminar flame phase determines primarily the collapse condition. A conditioned flame favours makes the ONeMg core more likely to collapse.

We have also studied globally how to determine the collapses condition. We study the mean electron fraction of the burnt matter inside the star. We find that, after averaging different progenitor models, a star is more likely to collapse once its mean electron fraction drops below $\approx 0.444$. At last, we also compute the typical nucleosynthesis yield of ONeMg core which fails to collapse at the first place. The yield is compared with solar abundance and thus the relative rate of this scenario is derived.

5. ACKNOWLEDGMENT

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APPENDIX

A. A BRIEF SUMMARY OF THE ADVANCED LEAKAGE SCHEME

In the post-bounce phase, we use the advanced leakage scheme (ALS) from A. Perego (2016). This scheme is an improved version of predecessor leakage scheme (Rosswog & Liebendoerfer 2003), combined with the essence from the isotropic diffusion source approximation (IDSA) (Liebendoerfer et al. 2009). We enlist the important equations which we have used in modeling the collapse phase. We refer the readers to the original instrument paper for the detailed derivation and the spirit of these approximations. In this scheme, the neutrinos are not only emitted by matter as in the leakage scheme, but can be transported and absorbed in IDSA. The major approximation for the neutrino-transparent zones is that, it evolves to asymptotically to the solution of the diffusive regime. This greatly simplifies the root-finding section by iteration as common in neutrino solver including IDSA, where in scenarios such as with strong dynamical motions or strong discontinuities, it is possible for the root-finder to fail in finding the new thermodynamics states for a given density and temperature, especially when there are man-made adjust of the internal energy and density when coupled with the hydrodynamics. For example, in multi-dimensional Eulerian hydrodynamics simulations, the advection of matter and sound wave always bring noises to the system. The residue error can lead instability of the root-finding subroutines.

First, we introduce two variables $Y_{\nu,i}$ and $Z_{\nu,i}$ which represent the mean neutrino number fraction and mean neutrino particle energy for the $i^{th}$-type neutrino ($i = e, \bar{e}$), defined by

$$Y_{\nu,i} = \frac{4\pi}{(hc)^3} \frac{m_B}{\rho} \int f_{\nu,i}^{\nu} E^2 dE,$$

$$Z_{\nu,i} = \frac{4\pi}{(hc)^3} \frac{m_B}{\rho} \int f_{\nu,i}^{\nu} E^3 dE.$$  \hspace{1cm} (A1)

$f_{\nu,i}^{\nu}$ is the occupation number of the $i^{th}$-type neutrino, $m_B$ is the nucleon mass. These variables satisfy the transport equations

$$\frac{\partial Y_{\nu,i}}{\partial t} + \vec{v} \cdot Y_{\nu,i} = Y_{\nu,i} \frac{\partial Z_{\nu,i}^{2/3}}{\partial t} + \vec{v} \cdot Z_{\nu,i}^{2/3} = \frac{3}{4} \rho^{3/4} \dot{Z}_{\nu,i}. $$  \hspace{1cm} (A3)
By assuming ideal degenerate Fermi gas for the neutrino equation of state in the neutrino-opaque zones, \( Y_{\nu, i} \) and \( Z_{\nu, i} \) are related to the temperature \( T_\nu \) and chemical potential of neutrinos \( \mu_\nu \) by

\[
Y_{\nu, i}(T_\nu, \mu_\nu) = \frac{4 \pi}{(hc)^3} \frac{m_B}{\rho} F_2 \left( \frac{\mu_\nu}{k_B T_\nu} \right),
\]

(A4)

\[
Z_{\nu, i}(T_\nu, \mu_\nu) = \frac{4 \pi}{(hc)^3} \frac{m_B}{\rho} F_3 \left( \frac{\mu_\nu}{k_B T_\nu} \right).
\]

(A5)

In calculate the source, we again divide the system into two parts, the neutrino-transparent and neutrino-opaque regime, characterized by the total mean free path at a position \( r \) from the core and for neutrino with an energy \( E \), \( \lambda_{\nu, tot}(E, r) \) and the energy mean free path \( \lambda_{\nu, en}(E, r) \). The mean free paths are the inverse of the scattering cross sections from neutrino-nucleon interactions. The total mean free path includes both elastic and inelastic processes and the energy mean free path only includes the inelastic processes. The related optical depth is defined as

\[
\tau_{\nu, i}(E, r) = \int_r^{\infty} \frac{1}{\lambda_{\nu, i}(E, r)} dr,
\]

(A6)

where \( i = \text{tot or ene} \). We include the neutrino processes same as Liebendoerfer et al. (2009). They include the absorption and emission processes

\[
\nu_e + n \rightarrow p + e^-,
\]

(A7)

\[
\bar{\nu}_e + p \rightarrow n + e^+,
\]

(A8)

and the elastic scattering processes

\[
N + \nu_i \rightarrow N + \nu_i,
\]

(A9)

with \( N \) being the nucleon (proton or neutron) and \( \nu_i \) being all types of neutrinos. In this work, we only model electron neutrino \( \nu_e \) and anti-electron neutrino \( \bar{\nu}_e \). The physics of neutrino-nuclei is neglected because the important part of the neutrino transport goes to the core, while they are free from nuclei in the density and temperature range we are considering.

To calculate the source terms, we write

\[
\dot{Y}_1 = \dot{Y}_e + \dot{Y}_{\nu_e} - \dot{\bar{Y}}_{\nu_e},
\]

(A11)

\[
\dot{u}_\nu = \dot{q}_\nu + \frac{1}{m_B} (\dot{Y}_e + \dot{Y}_{\nu_e} - \dot{\bar{Y}}_{\nu_e}).
\]

(A12)

The terms on the left hand side are the net gain/loss of neutrino and specific internal energy for a given mesh, while \( \dot{Y}_e \) and \( \dot{q}_\nu \) are those for the matter. The remaining terms are those for the neutrinos. The two equations governs the conservations of lepton and energy during its interaction with electrons and nucleon.

The terms \( Y_{\nu_e} \) and \( \dot{Y}_{\nu_e} \) are the change of neutrino number fraction and average energy calculated by

\[
\dot{Y}_\nu = \frac{\dot{Y}_{\nu, t+\Delta t} - Y_\nu}{\Delta t},
\]

(A13)

\[
\dot{Z}_\nu = \frac{\dot{Z}_{\nu, t+\Delta t} - Z_\nu}{\Delta t}.
\]

(A14)

Here \( \dot{Y}_{\nu, t+\Delta t} \) and \( \dot{Z}_{\nu, t+\Delta t} \) are the estimated new trapped components at \( t + \Delta t \) for each neutrino species. To do the estimation, we reconstruct the neutrino energy spectra by

\[
f_\nu^{\text{eq}}(E) = \gamma(r)(f_\nu(E, r))_{\text{eq}} \left( 1 - e^{-\tau_{\nu, en}(E, r)} \right)
\]

(A15)

where \( \gamma(r) \) is the normalization factor such that

\[
Y_\nu(r) = \frac{4 \pi}{(hc)^3} \frac{m_B}{\rho \rho(r)} \int f_\nu^{\text{eq}}(E, r) E^2 dE
\]

(A16)
coincide with the value $Y_e \Delta t$. As remarked (A. Perego 2016) a similar normalization procedure can be applied to $Z_\nu$ instead of $Y_e$ which guarantees the conservation of energy instead of lepton number. $(f_\nu(E, r))_{eq}$ is the Fermi-Dirac distribution in equilibrium for a given neutrino energy $E$ at a radius $r$. Here, the anti-electron neutrino is assumed to have the same degeneracy parameter but of opposite sign.

Then we calculate the change of the neutrino energy distribution by considering the processes of neutrino production and diffusion.

\[
\frac{df_\nu^{tr}}{dt} = j_{\nu, \text{prod}}^{tr} + j_{\nu, \text{diff}}^{tr}.
\]

(A17)

Assuming both terms always leads to some equilibrium states, we have

\[
j_{\nu, \text{prod}}^{tr} = \frac{(f_\nu)_{eq} - f_\nu^{tr}}{\max(t_{\nu, \text{prod}}, \Delta t)} \exp\left(-\frac{t_{\nu, \text{prod}}}{t_{\nu, \text{diff}}}ight)
\]

(A18)

and

\[
j_{\nu, \text{prod}}^{tr} = \frac{(f_\nu)_{eq} - f_\nu^{tr}}{\max(t_{\nu, \text{prod}}, \Delta t)} \exp\left(-\frac{t_{\nu, \text{prod}}}{t_{\nu, \text{diff}}}ight)
\]

(A19)

These two expressions makes sure when production dominates the neutrino process, namely when $t_{\nu, \text{prod}} \ll t_{\nu, \text{diff}}$, the neutrino spectra evolves to the new equilibrium, while when diffusion dominates the process, neutrino decays exponentially. The corresponding timescale is given in the following

\[
t_{\nu, \text{prod}}(E, r) = \frac{1}{j_\nu(E, r)},
\]

(A20)

and

\[
t_{\nu, \text{diff}}(E, r) = \frac{\Delta x_\nu}{c\tau_{\nu, \text{tot}}(E, r)},
\]

(A21)

where

\[
\Delta x_\nu = \alpha_{\text{diff}}\tau_{\nu, \text{tot}}(E, r)\lambda_{\nu, \text{tot}}(E, r)
\]

(A22)

is the effective length scale of diffusion. Having known the rate of change of $f_\nu^{tr}$, the new tapped component neutrino energy spectra can be calculated and the $f_{\nu, \text{tot}}^{tr, \Delta t}$ can be obtained. This provides the new $Y_\nu$ and $Z_\nu$ accordingly.

Then we also need to calculate the net change of lepton number $\dot{Y}_l$ and total specific energy $\dot{u}$ as results of neutrino emission and absorption in optically thin zones. They are given by

\[
\dot{Y}_l = m_B(R_{\nu_e}^0 - R_{\nu_e}^0 - H_{\nu_e}^0 + H_{\nu_e}^0),
\]

(A23)

and

\[
\dot{u} = -(R_{\nu_e}^1 + R_{\nu_e}^1 + H_{\nu_e}^1 + H_{\nu_e}^1).
\]

(A24)

$R_\nu^k(r)$ is the specific emission/absorption rate and its corresponding energy emission/absorption rate, defined by

\[
R_\nu^k(r) = \int_0^\infty r_\nu(E, r)E^{2+k} dE,
\]

(A25)

\[
h_\nu^k(r) = \int_0^\infty h_\nu(E, r)E^{2+k} dE.
\]

(A26)

Only $k = 0, 1$ are needed for our calculations.

The energy-dependent emission rate and diffusion rate are

\[
r_{\nu, \text{prod}}(E, r) = \frac{4\pi}{(hc)^3} \frac{j_\nu(E, r)}{\rho(r)}.
\]

(A27)

\[
r_{\nu, \text{diff}}(E, r) = \frac{4\pi}{(hc)^3} \frac{1}{\rho(r)} \frac{(f_\nu)_{eq}(E, r)}{t_{\nu, \text{diff}}(E, r)}.
\]

(A28)

The combination of $r_{\nu, \text{prod}}$ and $r_{\nu, \text{diff}}$ are chosen such that the production term is dominant in the optically thin region while diffusion term is dominant in the opaque region. The interpolation formula is set as

\[
r_\nu(E, r) = (1 - \alpha_{\nu, \text{blk}})\tilde{r}_\nu(e, r)\Psi(r)\exp(-\tau_{\nu, \text{cut}}(E, r)/\tau_{\text{cut}}),
\]

(A29)
where
\[
\tilde{r}_\nu = \frac{r_{\nu,\mathrm{prod}}(E,r) \times r_{\nu,\mathrm{diff}}(E,r)}{r_{\nu,\mathrm{prod}}(E,r) + r_{\nu,\mathrm{diff}}(E,r)}
\] (A30)
is the interpolation commonly used in grey leakage schemes. \(\Psi_\nu\) is the local normalization factor
\[
\Psi_\nu(r) = \frac{\int_0^\infty \tilde{r}_\nu(E,r) E^2 dE}{\int_0^\infty \tilde{r}_\nu(E,r) \exp(-\tau_{\nu,\mathrm{en}}(E,r)/\tau_{\mathrm{cut}}) E^2 dE}.
\] (A31)
This concludes the calculation of the emission term in the optically thin zone.

To calculate the absorption rate in the optically thin zone, we have
\[
h_\nu(E, r) = \frac{n_{\nu,\tau_\nu \lesssim 1}(E, x)}{\rho(r)} \chi_{\nu, ab}(E, r) F_{\nu}(E, r) H(E, r).
\] (A32)
\(F_{\nu}\) is the Pauli-blocking factor for electron or positrons in the final state including the neutron-proton mass difference \(\approx 1.293\,\text{MeV}\). \(n_{\nu,\tau_\nu \lesssim 1}(E, x)\) is the propagation-angle corrected local neutrino number density
\[
n_{\nu,\tau_\nu \lesssim 1}(E, x) = \frac{l_\nu(E, r)}{4\pi r^2 c \mu_\nu(E, r)}
\] (A33)
with
\[
\nu_\nu = \frac{1}{2} \left(1 + \sqrt{1 - \left(\frac{R_{\nu}(E)}{\max(R, R_{\nu}(E))}\right)^2}\right)
\] (A34)
being the analytic approximation formula for the spectral flux factor. The neutrino luminosity \(l_\nu(E, r)\), defined outside the neutrinosphere for a given energy, evolves spatially as
\[
\frac{dl_\nu(E, r)}{dr} = 4\pi r^2 \rho(r) r_\nu(E, r) - \frac{\chi_{ab}}{c} H(E, r) l_\nu(E, r).
\] (A35)
The concludes the calculation of the absorption term in the optically thin zones.

At last, to calculate the neutrino stress, we also classify into two regions. In neutrino opaque zones, neutrino behaves as part of the fluid which gives a stress
\[
\dot{v}_{\tau_\nu > 1} = \nabla \frac{P_{\nu,\mathrm{tot}}}{\rho},
\] (A36)
where the total neutrino pressure \(P_{\nu,\mathrm{tot}}\) is
\[
P_{\nu,\mathrm{tot}} = \frac{\rho}{3 m_b} (Z_{\nu_e} + Z_{\bar{\nu}_e}).
\] (A37)
In neutrino optically-thin zones, we have
\[
\dot{v}_{\tau_\nu < 1}(r) = \frac{1}{c} \int_0^\infty h(E, r) \mu_\nu(E, r) E^3 dE.
\] (A38)

B. POSSIBLE OBSERVATIONAL SIGNALS FOR THE COLLAPSING MODEL

For models which goes to a direct collapse, they collapse into a neutron star, releasing a significant amount of energy by neutrino, leaving a proto-neutron star with an envelope. The envelope late falls onto the neutron star and cools down. Here we carry out estimation of the possible observational signals, including neutrino signals from these models. We remap our models, which shows a direct collapse, from the 2-dimensional cylindrical grid to 1-dimensional spherical grid by doing angular average. Then we carry out one-dimensional hydrodynamics simulations from the collapse until bounce occurs. In Table 4, we list the input physics for doing the 1D estimation about the collapse phase. We refer interested reader of the detailed calculations in (Leung et al. 2018). In the 1D simulation, we use the same WENO 5th order shock-capturing scheme and the 3-step 3rd order NSSP RK scheme for spatial and temporal discretization. For the EOS, since we need to follow the evolution of the core until the formation of proto-neutron star, the original Helmholtz EOS is insufficient because it does not contain nuclear physics. Instead, we use the HShen
Table 4. The input physics and the choices of physics models in simulations.

<table>
<thead>
<tr>
<th>Input physics</th>
<th>Physics model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spatial discretization</td>
<td>5th order Weighted Essentially Non-Oscillatory Scheme (Barth &amp; Deconinck 1999)</td>
</tr>
<tr>
<td>Time discretization</td>
<td>5-step 3rd order Non-Strong Stability Preserved Runge-Kutta Scheme (Wang &amp; Spiteri 2007)</td>
</tr>
<tr>
<td>Baryonic matter EOS</td>
<td>HShen EOS (Shen et al. 1998)</td>
</tr>
<tr>
<td>Pre-bounce electron capture</td>
<td>Fitting table from direct Boltzmann transport (Dessart et al. 2006; Abdikamalov et al. 2010)</td>
</tr>
<tr>
<td>Pre-bounce neutrino transport</td>
<td>Parametrized neutrino transport (Liebendoerfer 2005)</td>
</tr>
<tr>
<td>Post-bounce neutrino transport</td>
<td>Advanced leakage scheme (ALS) (A. Perego 2016)</td>
</tr>
</tbody>
</table>

EOS (Shen et al. 1998), which is based on the relativistic mean-field model to describe the homogeneous phase of matter. The table includes extension with the Thomas-Fermi approximation to describe the inhomogeneous matter composition. The parameter for the incompressibility of nuclear matter is 281 MeV and the symmetry energy has a value of 36.9 MeV. Before bounce occurs, we use the parametrized neutrino transport scheme (Liebendoerfer 2005). This scheme treats the electron capture as the only neutrino source and simplifies the neutrino transport by only including an instantaneous absorption/emission. The neutrino also affects the hydrodynamics through its pressure in the neutrino-opaque region as an ideal degenerate Fermi gas. To estimate the electron capture at high density, we use the fitting table in Abdikamalov et al. (2010), which contains the $Y_e$ as a function of density. The electron fraction of the matter is instantaneously converted to the value given by the table, where the net change of electron capture is treated as neutrino source. After bounce, the simplified neutrino transport scheme is not sufficient, so we switch to the Advanced Leakage Scheme (A. Perego 2016). This scheme can be regarded as the extension of the leakage scheme (Rosswog & Liebendoerfer 2003), but is a simplified scheme of the Isotropic Diffusion Source Approximation (IDSA) (Liebendoerfer et al. 2009). It is because this scheme treats the neutrino number fraction and mean energy as independent variables as in IDSA. But in evolving to the new state, in the neutrino sector, it always assumes the new state inclines towards to the diffusion limit in the optically thin zones or the trapped limit in the optically thick zones. This guarantees that the scheme can approach asymptotically to a solution for an arbitrary timestep. This can bypass the difficulty of finding a new state in the original version of IDSA where occasionally no solution is found in zones where rigorous motion or discontinuities exist. In our simulations, we use 10 energy bands of neutrino from 3 MeV to 300 MeV in a logarithmic increasing band size. Since we want to understand the fundamental properties of how the collapse takes place, we include only $\nu_e$ and $\bar{\nu}_e$ in our calculation with only 2 absorption/emission channels and 4 scattering channels, namely:

$$n + \nu_e \leftrightarrow p + e^-,$$

$$p + \nu_e \leftrightarrow n + e^+,$$

for the absorption/emission, and

$$n + \nu_i \leftrightarrow n + \nu_i,$$

$$p + \nu_i \leftrightarrow p + \nu_i,$$

$$\alpha + \nu_i \leftrightarrow \alpha + \nu_i,$$

$$\text{ion} + \nu_i \leftrightarrow \text{ion} + \nu_i,$$

respectively. We use the rate formulae given in Bruenn (1985). Pair neutrino and neutrino bremsstrahlung are not included in this calculation. But these processes are less important compared to the channels included, although we note that for a long term simulation such as neutron star cooling, these two channels gradually replace the first two absorption-emission channels.

Since the advanced leakage scheme does not include neutrino cooling, which is an important channel for the proto-neutron star to lose energy effectively after the neutrinosphere has been settled, we only run the simulations until 50 ms after bounce, to extract the neutrino signals as well as the ejecta mass.

In the left panel of Figure 38 we plot the density profiles of one of the collapse model c3-10000-N at the beginning of the one-dimensional simulation, at bounce, 25 ms and 50 ms after bounce. At the beginning (end of the two-
we plot the velocity profiles for the same model similar to the left pane

of Figure 38. (left panel) The density profiles of the Model c3-10000-N at the start of simulation (the same profile as it ends in

the 2-dimensional simulation), at the bounce, 20, 50, 100, 200 and 300 ms after bounce. (right panel) Similar to left panel, but for the velocity profiles.

dimensional simulations in the deflagration phase), the core starts with a flat density profile. But the inner core first contracts to reach nuclear density due to the loss of pressure by electron capture. At bounce, a stiff core made of nuclear matter at density around $3 \times 10^{14}$ g cm$^{-3}$ is formed. The inner envelope shows a steep density gradient showing that it is still falling onto the neutron star. The outer envelope does not change much. At 20 ms after bounce, the neutron star core reaches an equilibrium state in density, while the accretion of matter of the inner envelope creates a layer outside the neutron star. At around $10^{12}$ g cm$^{-3}$, strong fluctuations of density appear due to the tension between the infalling matter from the outer envelope and the stabilized inner envelope. At 50 ms after bounce, the neutron star has a static state envelope about 200 km. The remained envelope has also contracted significantly to about 500 km, about half of its initial radius $\approx 1200$ km. At 100 ms after bounce onward, no significant change in the density profile of the neutron star up to 200 km. But there is still observable motion of the surface showing expansion. The cusps in the profiles also disappear.

In the right panel of Figure 38 we plot the velocity profiles for the same model similar to the left panel. At the beginning, the star is having a homologous contraction with a maximum velocity about $1.3 \times 10^{-2}c$ at about 500 km. At bounce, we can see the a neutron star core close to static is formed with a size of about 15 km. Outside the neutron star there is an infalling envelope with a maximum velocity about 0.2 c. The infalling envelope preserves also the homologous velocity profile Through shock heating, the material fallen on the neutron star quickly finds an hydrostatic equilibrium state. By examining the velocity profile at 20 ms after bounce, the bounce shock reaches about 100 km from the core, with a slightly lower infalling velocity about 0.16 c. There are outgoing matter in the profile at 50 ms after bounce. This shows the shock has reached to region where density is low enough for the density gradient becomes large enough, so that the shock strength increases again when it propagates. The infalling velocity has decreased to $\approx 0.12c$. Once the accretion shock reaches the surface, since there is no further matter suppressing the expansion of matter, it creates a high velocity flow on the surface. Some of the matter in the outermost layer even has a velocity exceeding the escape velocity, which makes the collapse of the ONeMg a dim and rapidly transient due to its high velocity and low ejecta mass. After the ejection of high velocity matter is ejected, the material becomes bounded and the star only exhibits oscillations with a homologous expanding profile in the envelope.

In Figure 39 we plot the entropy profiles similar to Figure 38. At the beginning, the whole star has almost a constant entropy $\approx 0.5$ k$_{\text{B}}$ per baryon, except near the surface. This is related to the initial flame put in by hand. The initial flame perturbs the initial hydrostatic equilibrium of the star, whose effects are reflected by the surface matter. At bounce, the whole star reaches a constant entropy about 3 k$_{\text{B}}$ per baryon. There is a cusp near the neutron star owing to the shock interaction. Again, similar to the velocity profiles, the quasi-static neutron star core has a constant entropy. At 20 ms, there is a significant rise of entropy to about 10 k$_{\text{B}}$ per baryon in the newly accreted layer from 10 - 80 km. The high entropy region can be compared with the velocity profile, which is the region which comes to a rest after falling on the neutron star surface but is slightly lagged behind. At 50 ms, the shock has reached 200 km and we can see a high entropy domain is formed up to about 110 km. This picture is consistent with the literature that the neutrino heating is essential in producing high entropy matter, which is supposed to be found in the ejecta. At 100
ms onwards, there is no significant change to the entropy profiles where an almost constant entropy zone is created in the envelope. At 100 ms after bounce, one can see that the ejecta can have an entropy as high as \( \approx 20 \, \text{kB per baryon} \).

In the right panel of Figure 39 we plot the \( Y_e \) profiles of the ECSN model similar to previous plot at the same time slice. The beginning \( Y_e \) profile is directly imported from the collapsing model in the main text. So, the core has reached a minimum of \( \sim 0.35 \) and gradually increases at 100 km up to 0.45. At no electron capture takes place beyond 200 km, where the deflagration has not yet reached the matter (See e.g. Figure 5). At bounce, the core \( Y_e \) reaches 0.2 and gradually increases to 0.35 at \( \sim 60 \) km, and up to 0.5 at 80 km. The locally higher \( Y_e \) matter falls inwards, but has not reached the density for electron capture, so locally it looks like the \( Y_e \) increases by itself. After bounce, the shock and the consequent neutrino interaction influence the \( Y_e \) distribution. The high temperature allows rapid neutrino emission, which creates a trough of \( Y_e \) from 30 - 100 km. Ripples of \( Y_e \) appears due to the finite partitioning of neutrino energy band. As the shock propagates outwards, at 100, 200 and then 300 ms, we can see the trough widens. Furthermore, the neutrino, which diffuses outwards outside the neutrinosphere, smooths out the \( Y_e \) fluctuations created by acoustic waves right after bounce.

In the left panel of Figure 40 we plot the neutrino energy spectra of the same model similar to Figure 38. The number flux is taken at 300 km from the neutron star core. The number reaching the Earth can be scaled accordingly. There is no data for the initial model because no matter has reached nuclear density. At bounce, one can see the \( \nu_e \) has already a spectrum comparable with the thermal spectrum. But the \( \nu_e \) spectrum is still extremely low. At 25 ms
The neutrino signal from an accretion induced collapse of a white dwarf into a neutron star is also plotted for comparison. The accretion induced collapse assumes a simple collapse of a Chandrasekhar mass isothermal white dwarf due to an initial reduction of $Y_e$. It can be seen qualitatively the two models have similar pattern. At the beginning, the a strong pulse of $\nu_e$ is emitted. But as the neutrinosphere of different energy bands start to form. The neutrino emission drops. After a few pulsations, it reaches an equilibrium value about $2 \times 10^{52}$ erg s$^{-1}$. One minor difference is that the ONeMg case shows more oscillations than the cold AIC case. The $\nu_e$ shows a similar behaviour. It has a much lower luminosity. Consistent to the literature, the first peak appears later than the $\nu_e$ peak, about 20 ms after. The ONeMg model has about 50 % higher $\nu_e$ flux than the cold AIC model.

At last we plot at Figure 41 the neutrino number flux profile at 100, 200 and 300 ms after bounce for both $\nu_e$ (solid line) and $\nu_\mu$ (dashed line). For low energy bands (3 MeV - 8 MeV), $\nu_e$ is the dominant species. They are mostly created just outside the NS, surface. No neutrino absorption can be seen and most neutrinos are produced within the innermost 100 km. On the contrary, $\nu_\mu$ is completely not produced in the NS, and is gradually produced in the shock-heated matter outside the NS. Its number emission is at least one order of magnitude lower. However, as neutrino energy increases, the drop of $\nu_e$ number flux is faster than the drop of $\nu_\mu$. It is because the creation of $\nu_\mu$ has a lower energy constraints than $\nu_e$. Notice that to create $\nu_e$, the electron should have a chemical potential not only for the creation of itself, but also the mass difference between $n$ and $p$ ($\sim 1.2$ MeV). At 20 - 100 km, the density has already drops below $10^{12}$ g cm$^{-3}$. This means the nucleons is no longer degenerate and thus it has a much lower
chemical potential than those in the core. So, this leaves a strong cutoff in the high energy $\nu_e$. On the other hand, the production of $\bar{\nu}_e$ is aided by the energy difference for the same origin. So, its drop in number flux is less steep than $\nu_e$.

For higher neutrino energy, more features can be observed. At 14, 23 and 39 MeV, both $\nu_e$ and $\bar{\nu}_e$ show a first increasing function up to 80 km and then slightly drop until 100 km. The change of $\nu_e$ is larger than that of $\nu_e$, showing that more $\nu_e$ is absorbed. As a result, this explains the local bump of $Y_e$ in the right panel of Figure 39.

For even higher neutrino energy (65, 108 and 180 MeV), the drops of $\nu_e$ becomes so rapid that it becomes irrelevant to the neutrino transport and the global neutrino flux. $\bar{\nu}_e$ also shows a similar feature but with lower strength. But they are also unimportant to the global neutrino population.

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