NEEDLET TOOLS IN COSMOLOGY AND ASTROPHYSICS

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JOINT WORK WITH

- **Baldi, Kerkyacharian and Picard (2006,2007,2008) Asymptotic Properties, Spectrum Estimation, Subsampling, Density Estimation**
- **Pietrobon et al. (2006,2007,2008): ISW, Correlation CMB-LSS, (Cross-)Spectrum Estimation, Detection of Asymmetries**
- **Lan (2008a,b): Needlets Bispectrum, Uncorrelation for Mexican Needlets**
- **Rudjord, Hansen, Liguori and Matarrese (2009a,b): Non-Gaussianity,Bispectrum, Asymmetries in f_{NL}**
- **Geller, Geller and Hansen, Geller and Lan (2008a,b, 2009): Spin Needlets: Localization and Uncorrelation, Statistical Analysis of Polarization, Asymptotics**

Isotropic Random Fields On A Sphere

$$T(\theta,\varphi)\ ,\ \ 0\leq\theta\leq\pi\quad 0\leq\varphi<2\pi$$

$$ET(\theta,\varphi)=0\ ,\ \ ET^2(\theta,\varphi)<\infty$$

SPHERICAL HARMONICS

$$Y_{lm}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\varphi}\ \text{, for } m\geq 0\ ,$$

$$Y_{lm}(\theta,\varphi)=(-1)^m Y^*_{l,-m}(\theta,\varphi)\ \text{, for } m<0\ ,$$

SPECTRAL REPRESENTATIONS

SPECTRAL REPRESENTATIONS

$$T(\theta,\varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\varphi),$$

$$a_{lm} = \int_{-\pi}^{\pi} \int_{0}^{\pi} T(\theta,\varphi) Y^*_{lm}(\theta,\varphi) \sin\theta d\theta d\varphi \ .$$

$$E a_{lm} \overline{a_{l'm'}} = \delta_l^{l'} \delta_m^{m'} C_l \ , \ \text{for all } m, l \ .$$

WHY WAVELETS?

- LOCALIZATION IN REAL AND HARMONIC DOMAIN
- HANDLING FOREGROUNDS AND MASKED REGIONS
- SEARCHING FOR FEATURES/ASYMMETRIES (COLD SPOT)
- ANGULAR POWER SPECTRUM ESTIMATION
- BISPECTRUM, TESTING FOR NON-GAUSSIANITY
- DENOISING - COMPONENT SEPARATION
- POLARIZATION - SPIN FIELDS
- DENSITY ESTIMATION - SOURCE DETECTION - COSMIC RAYS

Motivations for Needlets

- No Tangent Plane Approximation
- Quasi-Exponential Localization in Pixel Space
- Bounded Support on Multipoles
- Tight Frame - Exact Reconstruction Formula
- Uncorrelation/Independence in Harmonic and Pixel Space
- Rigorous High Frequency Asymptotic Theory
- Computationally Convenient
- Adaptation to the Mask and Local Features
- Extension to Spin/Polarization Data
- Extensions to Cosmic Rays - Weak Lensing

Needlets Definition (NPW2006)

This construction is due to Narcowich, Petrushev and Ward (2006a,b), see also Geller and Mayeli (2007,2008a,b).

$$\psi_{jk}(\hat{\gamma}) = \sqrt{\lambda_{jk}} \sum_{\ell} b(\frac{\ell}{B^j}) \sum_{m=-\ell}^{\ell} \overline{Y}_{\ell m}(\hat{\gamma}) Y_{\ell m}(\xi_{jk}); \qquad (1)$$

where ξ_{jk} can be takes as HealPix points,

$$\int_{S^2} Y_{lm}(x)\overline{Y}_{l'm'}(x)dx = \sum_{\xi_{jk}\in\mathcal{X}_{[2B^j+1]}} \lambda_{jk} Y_{lm}(\xi_{jk})\overline{Y}_{l'm'}(\xi_{jk}) \ .$$

$\lambda_{jk} \approx$ pixel area

The function $b(.)$

1. $b^2(.)$ has support in $[\frac{1}{B}, B]$, and hence $b(\frac{\ell}{B^j})$ has support in $\ell \in [B^{j-1}, B^{j+1}]$
2. the function $b(.)$ is infinitely differentiable in $(0, \infty)$.
3. we have

$$\sum_{j=1}^{\infty} b^2(\frac{\ell}{B^j}) \equiv 1 \text{ for all } \ell > B. \tag{2}$$

(partitions of unity)
We need $B > 1$, for instance $B = 2$

THE SHAPE OF $b(\frac{\cdot}{Bj})$

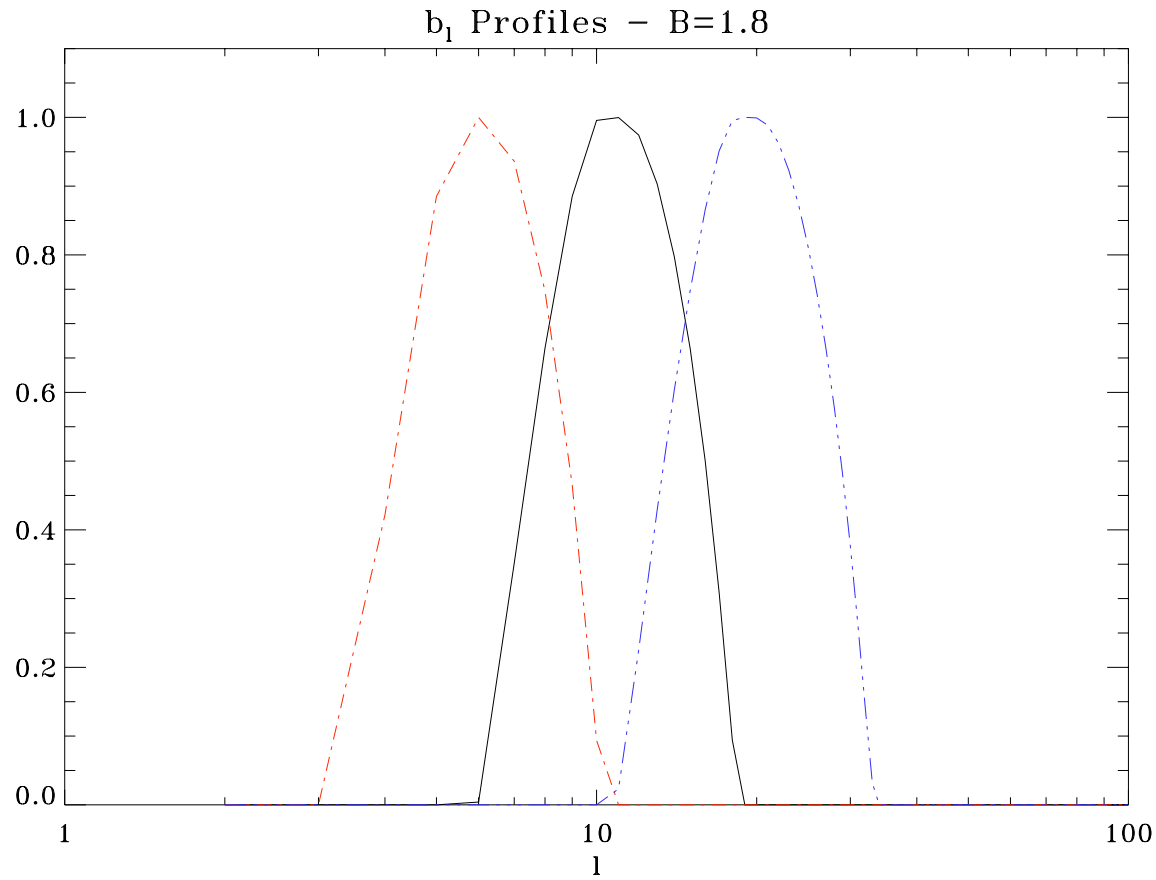


Figure 1: Partition of unity

Localization property

Localization property
For any M there exists a constant c_M s.t.,for every $\xi \in \mathbb{S}^2$:

$$|\psi_{jk}(\xi)| \leq \frac{c_M B^j}{(1 + B^j \arccos\langle \xi_{jk}, \xi \rangle)^M} \ .$$

(Quasi-Exponential localization) Recall that
$\arccos\langle \xi_{jk}, \xi \rangle \to d(\xi, \xi_{jk})$, geodesic distance on the sphere.

Needlet shape

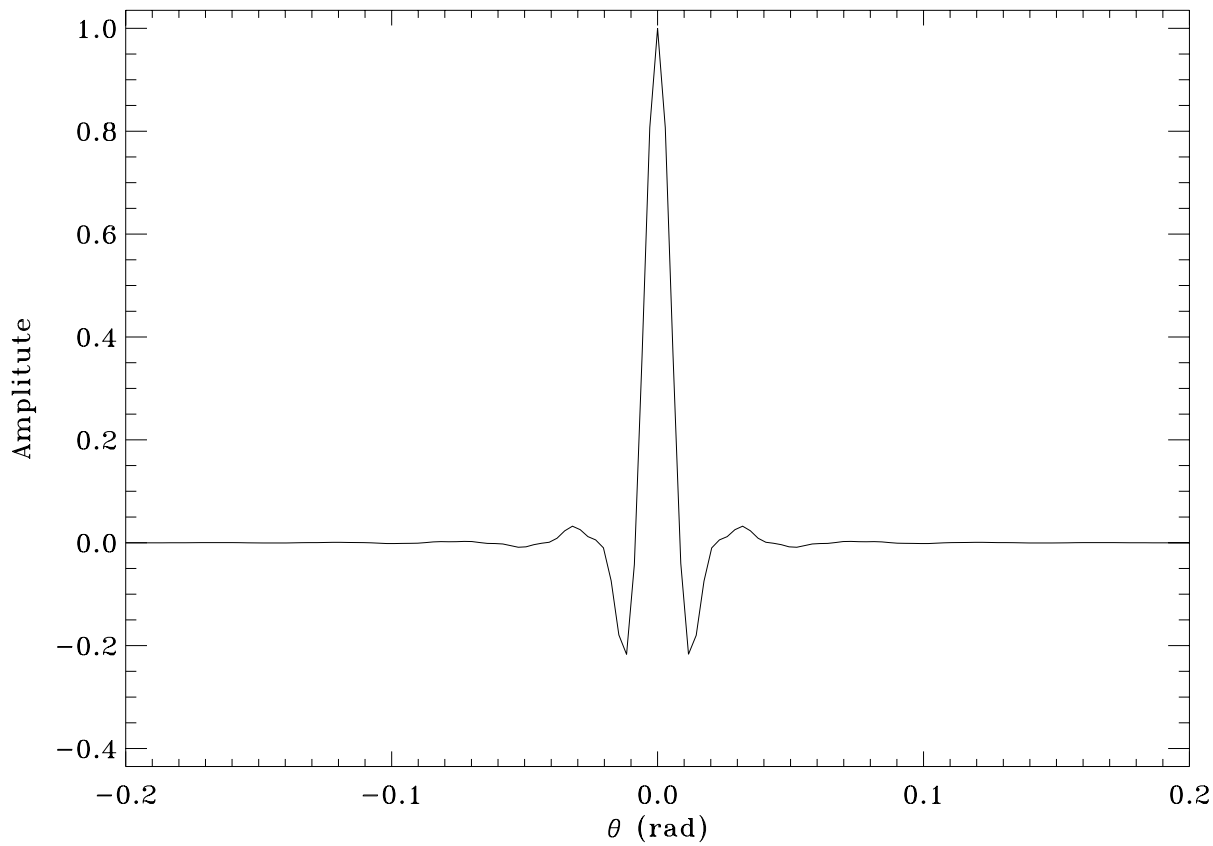


Figure 2: Needlets

THE ROLE OF j

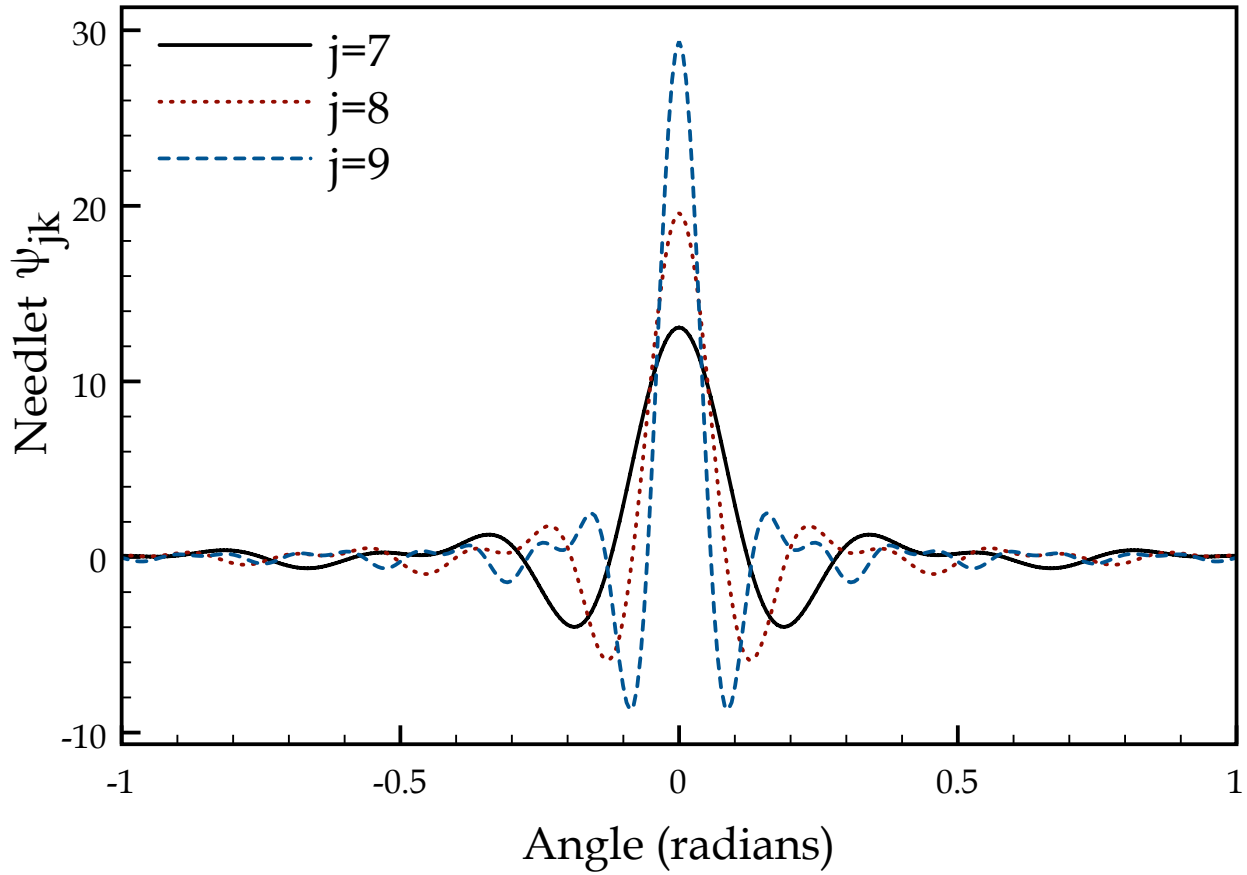


Figure 3: Needlets

Localization in the harmonic domain

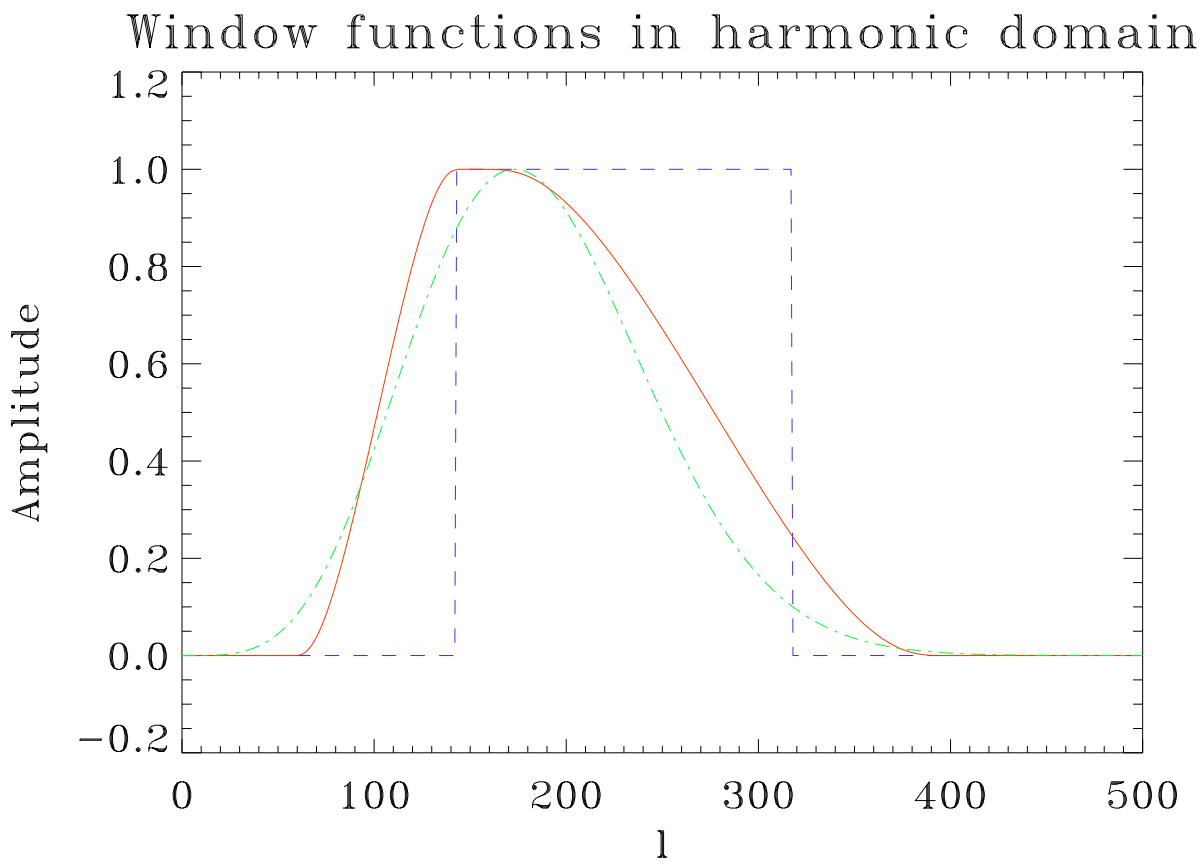


Figure 4: Needlets (red), tophat and SMHW (green)

A comparison

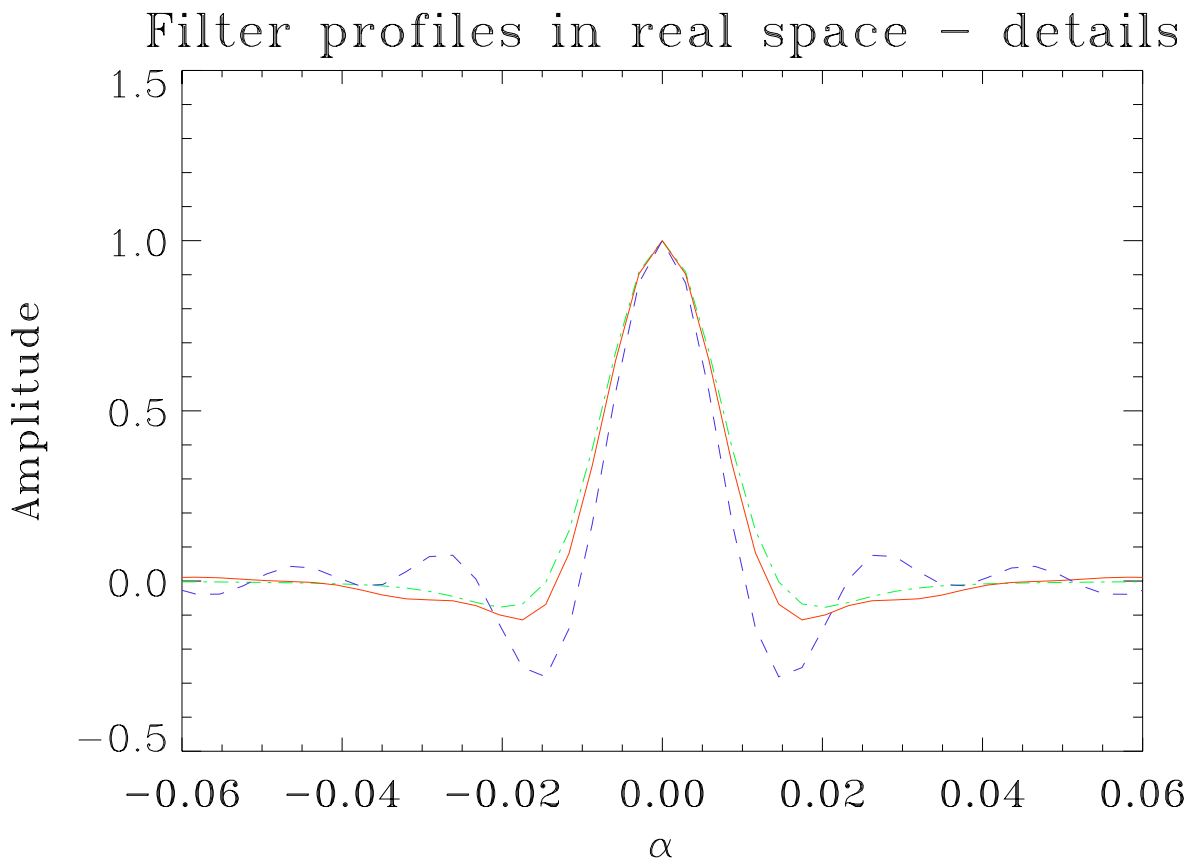


Figure 5: Needlets (red), tophat and SMHW (green)

Localization in the real domain

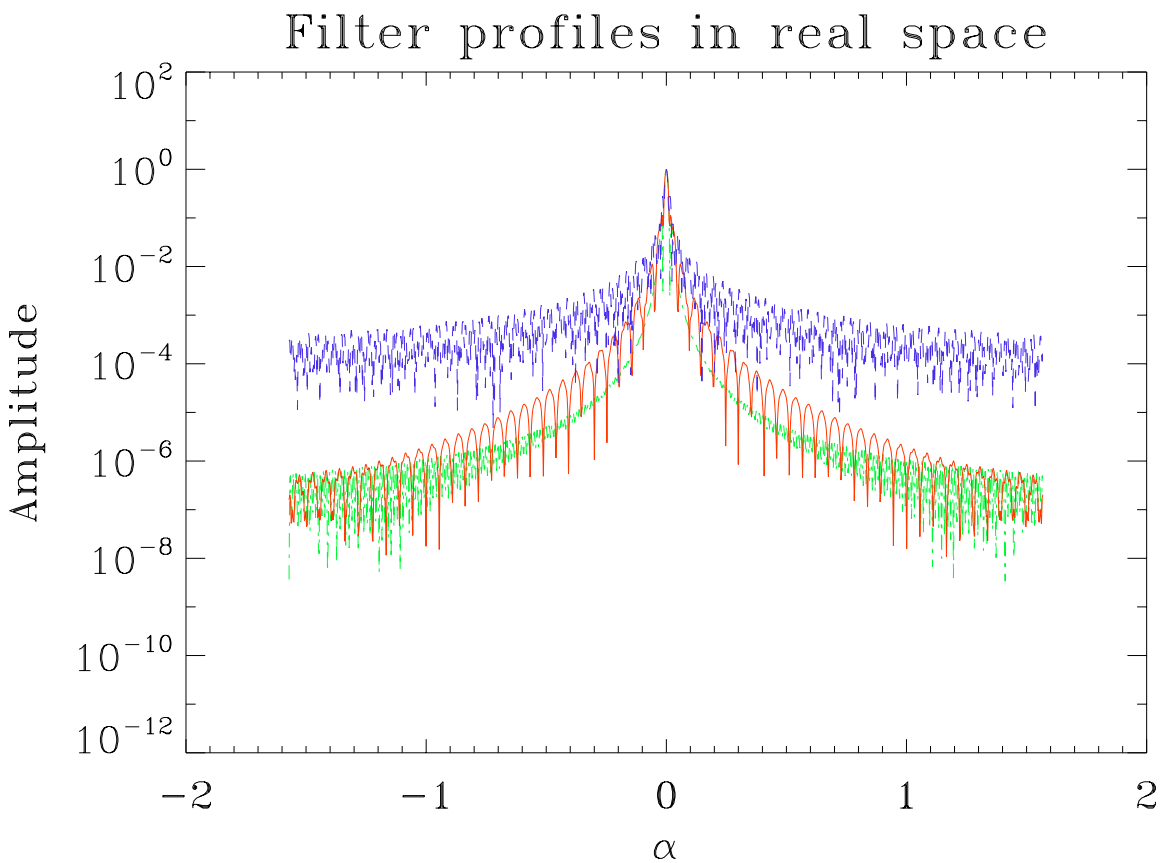


Figure 6: Correlation across j

Needlet coefficients

The (random) needlets coefficients are given by

$$\beta_{jk} = \int_{\mathbb{S}^2} f(x)\overline{\psi_{jk}}(x)dx = \sqrt{\lambda_{jk}} \sum_{l} b\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} a_{lm} Y_{lm}\left(\xi_{jk}\right) \ . \tag{3}$$

Tight Frame Property

Needlets represent a *tight frame.* , i.e. a countable set of functions $\{e_j\}$ s. t., for all $f \in L^2(S^2)$, we have

$$\sum_j < f, e_j >^2 \equiv \int_{S^2} f(\hat{\gamma})^2 d\Omega,$$

In our framework, the norm-preserving property becomes

$$\sum_{j,k} \beta_{jk}^2 \equiv \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi} \widehat{C}_\ell \,, \tag{4}$$

where

$$\widehat{C}_\ell = \frac{1}{2\ell+1} \sum_m |a_{\ell m}|^2$$

Reconstruction property

The needlet transform is defined by

$$\int_{\mathbb{S}^2} f(x)\overline{\psi_{jk}}(x)dx = \beta_{jk} \ ,$$

and the exact reconstruction property holds, i.e.

$$f(x) = \sum_{jk} \beta_{jk}\psi_{jk}(x) \ ,$$

the equality holding in the L^2 sense.

Asymptotic Uncorrelation

Under some regularity conditions on C_l, uncorrelation inequality (Baldi, Kerkyacharian, Marinucci and Picard (2006)):

Lemma 1 *We have*

$$\left|Corr(\beta_{j,k}, \beta_{j,k'})\right| \leq \frac{C_M}{(1 + B^j d(\xi_{j,k}, \xi_{j,k'}))^M} \tag{5}$$

where $d(\xi_{j,k}, \xi_{j,k'}) = \arccos(\langle \xi_{j,k}, \xi_{j,k'} \rangle)$.

The needlet coefficients at any finite distance are asymptotically uncorrelated.

IMPORTANT NOTICE: this is NOT due to localization, compare Lan&Marinucci (2008), Mayeli (2008).

Correlation in real domain

on line processing :

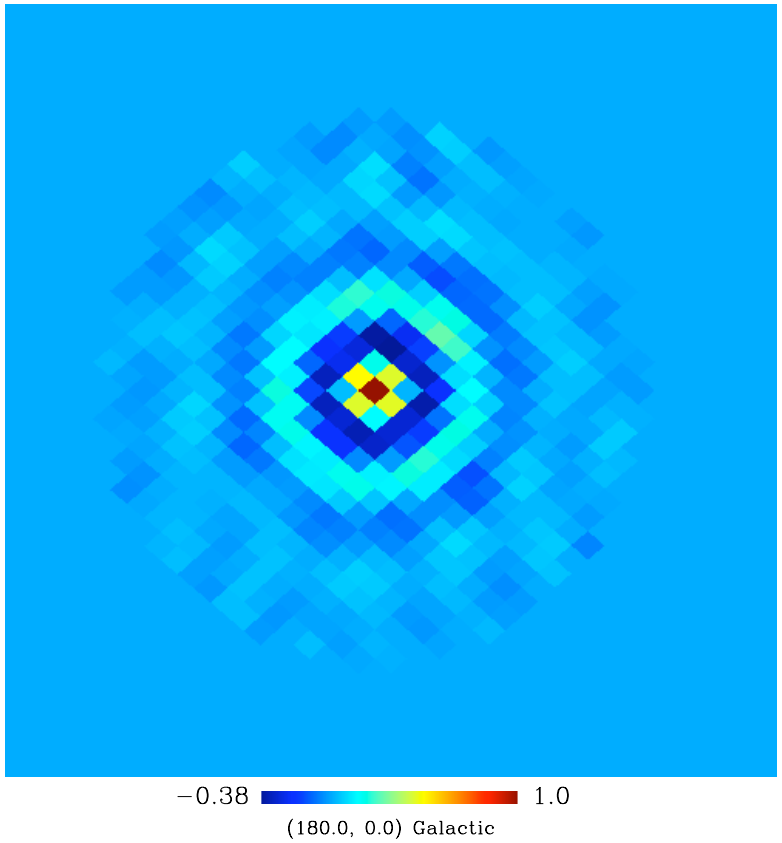


Figure 7: Correlation across pixels

Correlation in frequency domain

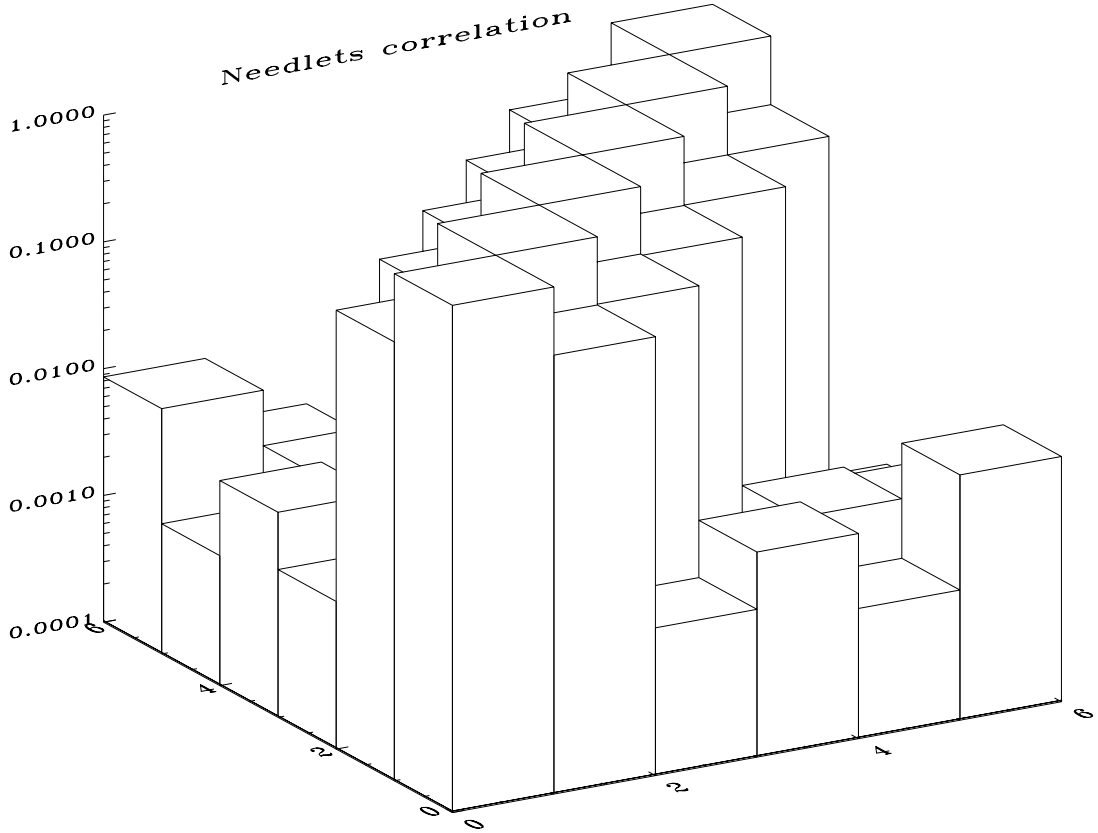


Figure 8: Correlation across j

Masked regions

It is interesting to stress that

$$\sqrt{\frac{\mathrm{E}[(\widetilde{\beta}^G_{jk} - \widetilde{\beta}_{jk})^2]}{\mathrm{E}[\widetilde{\beta}^2_{jk}]}} \leq c_1 B^{j(\alpha-1)} \frac{C_M B^j}{(1+B^j\varepsilon)^M} 2\pi\sqrt{2V^*} \leq C'_M B^{2j(\alpha-M)}$$

For M large enough, it is not difficult to show that, up to different normalizing constants, the behaviour of the needlet coefficients at high frequencies is not affected asymptotically by the presence of sky cuts.

Effect of the mask

Pixels with coefficients varying more than 5 percent -CMB

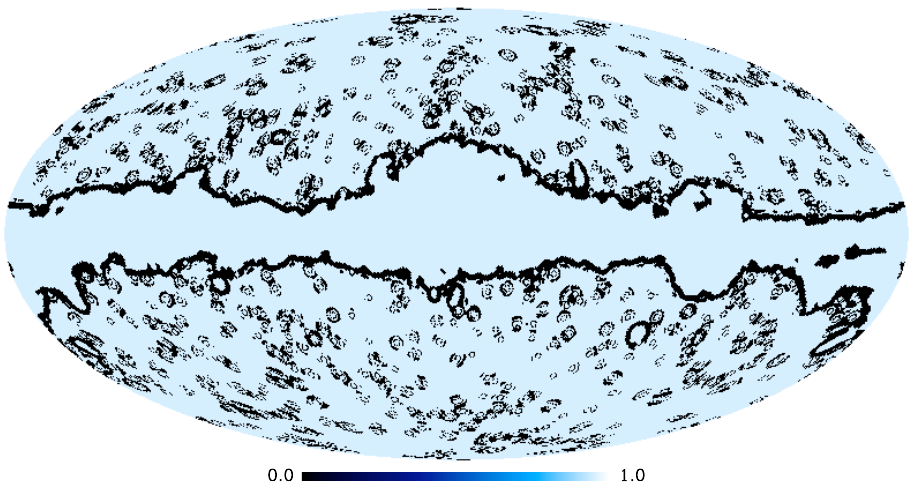


Figure 9: Effect of masks

Effect of the mask

Pixels with coefficients varying more than 5 percent - LSS

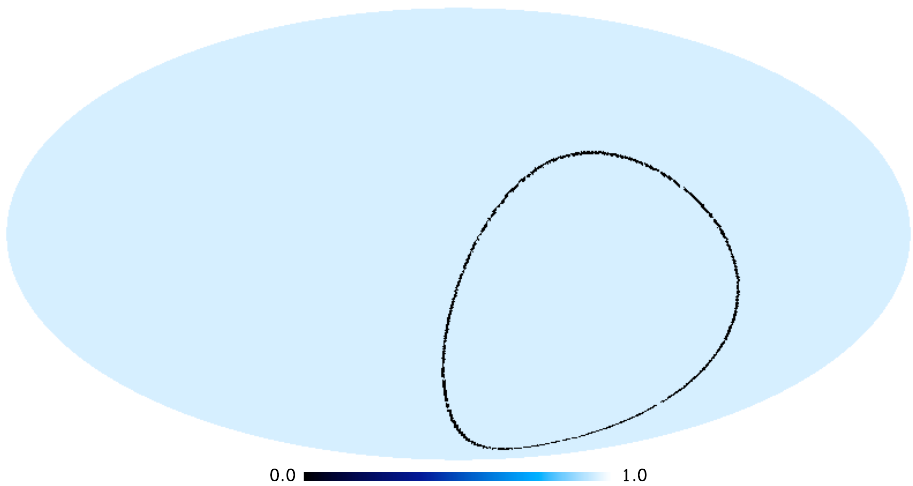


Figure 10: Effect of masks

APPLICATIONS

- **CROSS-CORRELATION CMB-LSS**
- **(CROSS-)POWER SPECTRUM ESTIMATION**
- **TESTING FOR ISOTROPY**
- **NON-GAUSSIANITY, f_{NL}, NEEDLETS BISPECTRUM**
- **POINT SOURCES**

FURTHER EXTENSIONS:

- **DIRECTIONAL DATA - COSMIC RAYS**
- **SPIN DATA - POLARIZATION**

Polynomials statistics of coefficients

(Baldi, Kerkyacharian, Marinucci and Picard (2006,2007-2009a,b)) Polynomial functionals of the (normalized) wavelets coefficients

$$h_{u,N_j} := \frac{1}{N_j} \sum_{k=1}^{N_j^2} \sum_{q=1}^{Q} w_{uq} H_q(\widehat{\beta}_{j,k}) \ , \qquad \widehat{\beta}_{j,k} := \frac{\beta_{j,k}}{\sqrt{\mathrm{E}\beta_{j,k}^2}} \ . \tag{6}$$

where $u = 1, 2, \ldots, U$.

$$\Omega_j := \left\{ \mathrm{E}[h_{u,N_j} h_{v,N_j}] \right\}_{u,v=1,\ldots,U} \ . \tag{7}$$

Central Limit Theorem

Theorem 2 *As* $N_j \to \infty$ *we have*

$$\Omega_j^{-1/2}\big(h_{1N_j}, \ldots, h_{UN_j}\big)' \underset{j\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} N(0, I_U) \ .$$

where I_U *denotes the identity matrix of dimension* U*.*

Statistical Applications - ISW effect

Search for Integrated (Late) Sachs-Wolfe effect: At large scales, we expect a non-negligible cross-correlation between CMB and Large Scale Structure Data.
We use WMAP ILC (T) and NVSS (N) catalogue of Radio Galaxies

$$\widehat{C}^{TN}_{1N_j} = \frac{1}{N_j} \sum_{k=1}^{N_j} \widehat{\beta}^T_{j,k} \widehat{\beta}^N_{j,k}$$

where

$$\mathrm{E}[\widehat{\beta}^T_{j,k} \widehat{\beta}^N_{j,k}] = \sum_{B^{j-1} \leq l \leq B^{j+1}} b^2(\tfrac{l}{B^j}) C_l^{TN} \frac{2l+1}{4\pi}$$

See Pietrobon, Balbi and Marinucci (2006).

Results

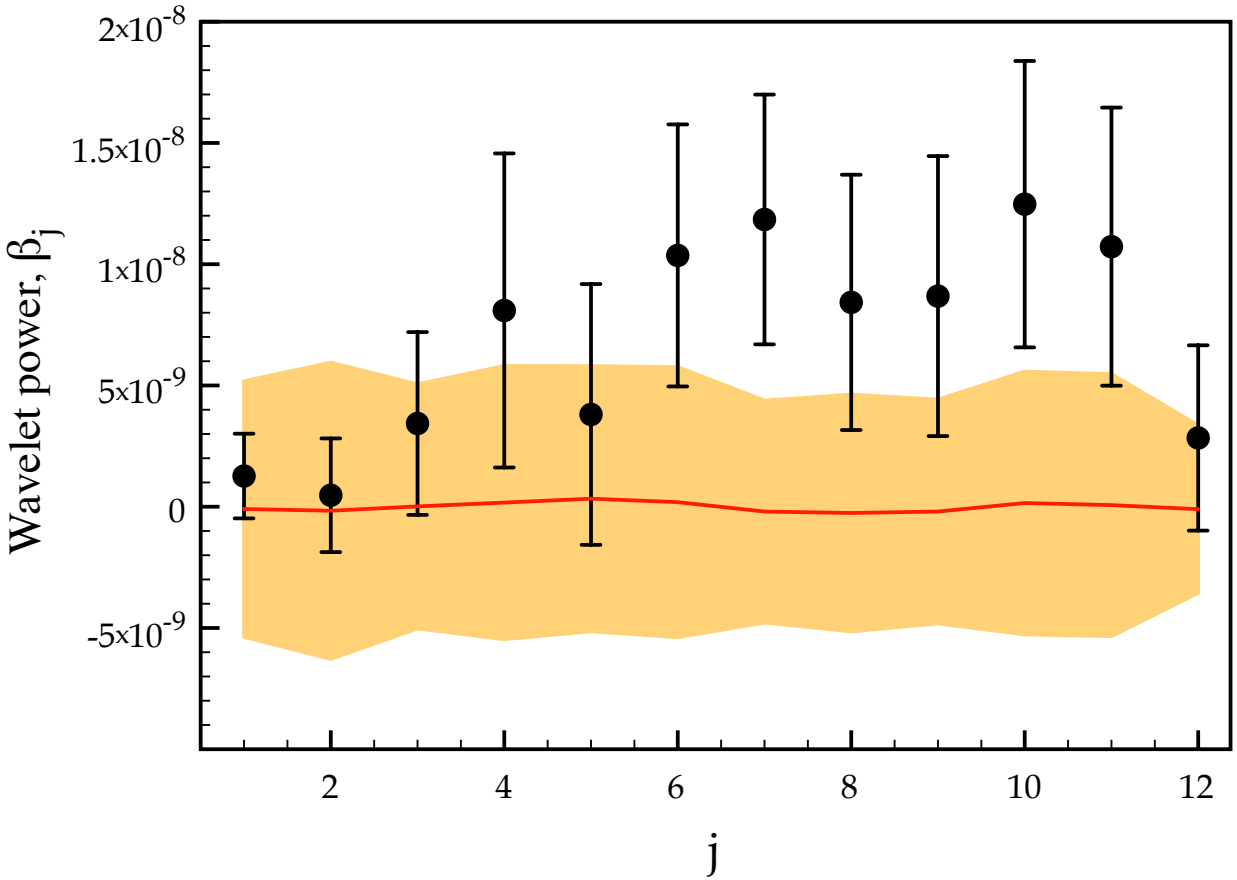


Figure 11: Needlets Cross-Spectrum

Results

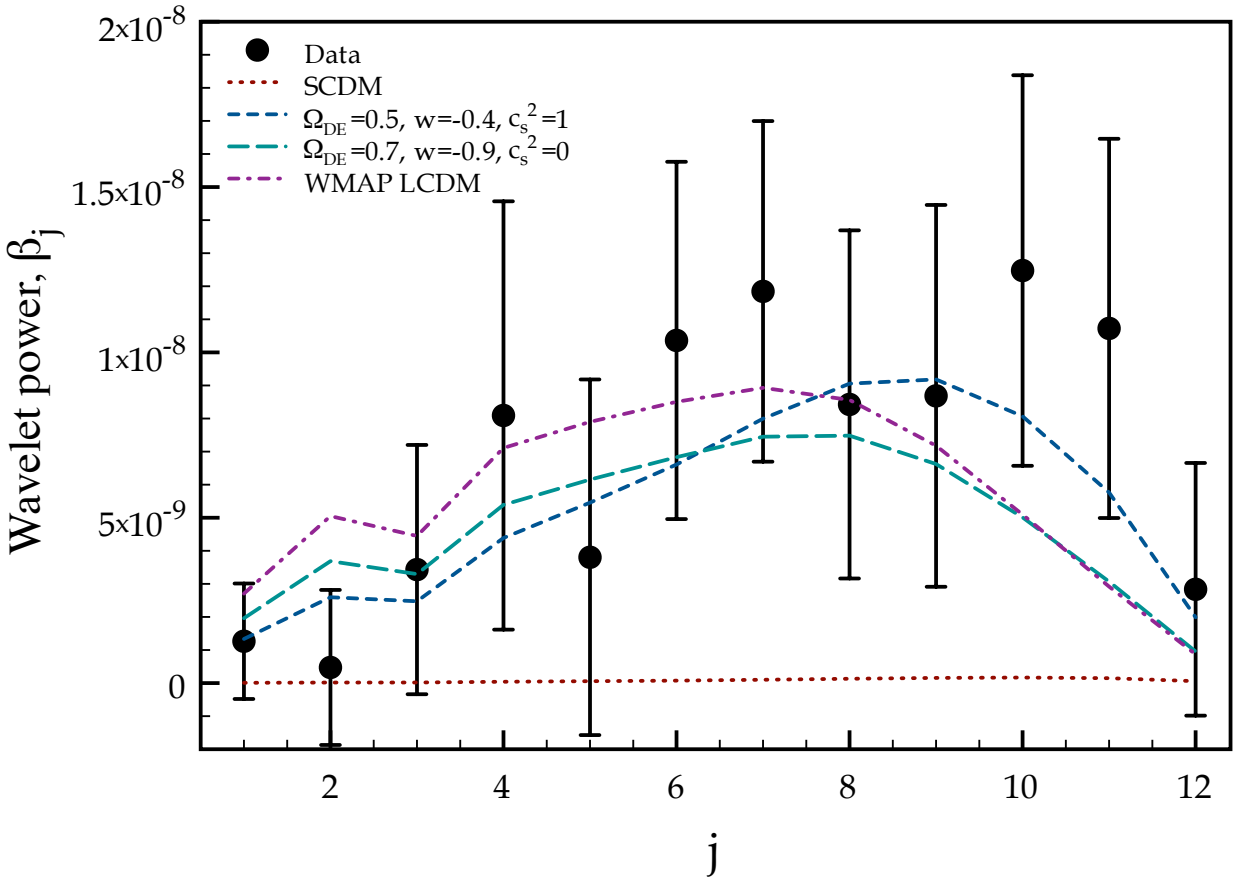


Figure 12: Needlets Cross-Spectrum

- SIGNIFICANT DETECTION - CHISQUARE FIT
- PARAMETER FIT FOR DARK ENERGY
- SIMULATED AND ANALYTIC CONFIDENCE INTERVALS
- CONTROL ON MASK AND FREQUENCIES
- PEAK AT 2-10 DEGREES
- WMAP WORKING GROUP

Spectral Estimator

Consider the estimator

$$\Gamma_j = \frac{1}{N_j} \sum_{k=1}^{N_j} \{\widehat{\beta}^2_{j,k}\}$$

where

$$\mathrm{E}\widehat{\beta}^2_{j,k} = \sum_{B^{j-1} \leq l \leq B^{j+1}} b^2(\tfrac{l}{B^j}) C_l \, \frac{2l+1}{4\pi}.$$

See Pietrobon, Balbi and Marinucci (2006), Baldi, Kerkyacharian, Marinucci and Picard (2006,2007), Fay et al. (2008), Fay and Guilloux (2008), Pietrobon et al. (2008).

Spectral Estimator

Due to localization and uncorrelation properties, the previous estimator can be evaluated on subsets of the sky, and used to search for features/anysotropies (Pietrobon et al. (2008)). Statistical significance can be evaluated analytically and from simulations.

Needlet coefficients - features

Searching for significant spots and asymmetries.

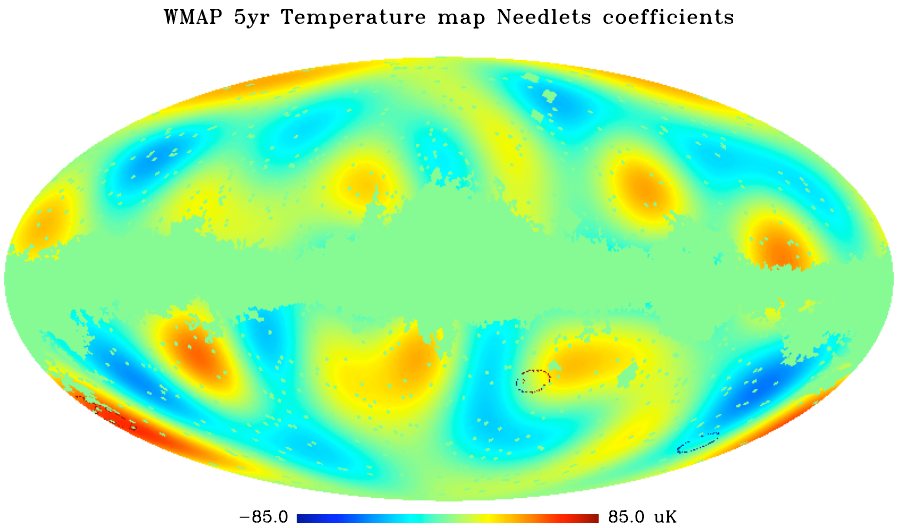


Figure 13: j=3, B=1.8

Needlet coefficients - features

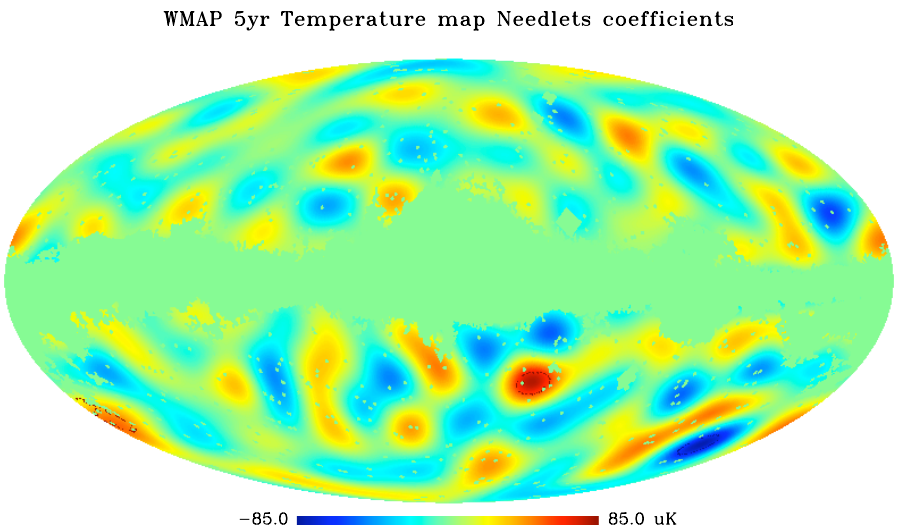


Figure 14: j=4, B=1.8

Needlet coefficients - features

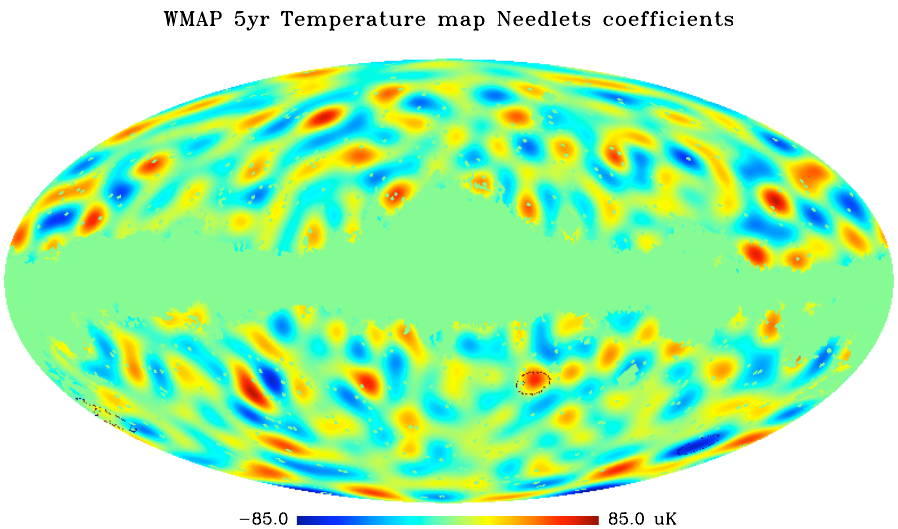


Figure 15: j=5, B=1.8

SIGNIFICANT VALUES ARE BETWEEN $\ell = 6$ and $\ell = 33$

Asymmetries in the angular power spectrum

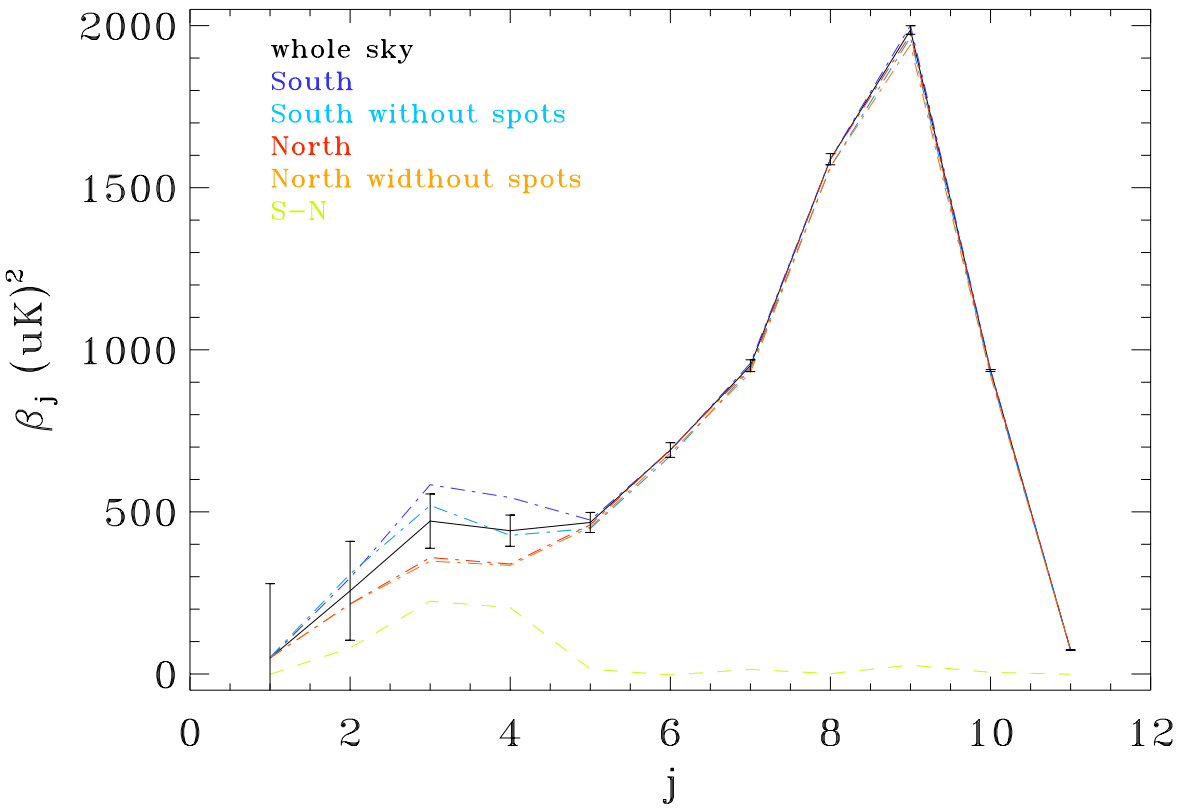


Figure 16: Angular power spectrum estimator

Tests for non-Gaussianity

(Lan and Marinucci (2008), Rudjord et al. (2009a,b), Pietrobon et al. (2009a,b))

Standard Non-Gaussian model

$$T^{NG} = T + f_{NL}[T^2 - ET^2] \tag{8}$$

The nonlinearity parameter f_{NL} is one of the key objects of CMB data analysis.

Bispectrum

Under isotropy

$$\langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3} \ ,$$

which can be estimated by (full sky)

$$\widehat{B}_{\ell_1 \ell_2 \ell_3} = \sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \ .$$

Needlets Bispectrum

Needlets Bispectrum

$$I_{j_1 j_2 j_3}(\mathrm{d}\Omega) = \sum_{k}^{pixels \in \mathrm{d}\Omega} \frac{\beta_{j_1 k}\beta_{j_2 k}\beta_{j_3 k}}{\sigma_{j_1 k}\sigma_{j_2 k}\sigma_{j_3 k}} \tag{9}$$

we have approximately

$$\frac{4\pi}{N}\sum_{k}\beta_{j_1 k}\beta_{j_2 k}\beta_{j_3 k} =$$

$$= \sum_{\ell_1 m_1}\sum_{\ell_2 m_2}\sum_{\ell_3 m_3} b(\frac{\ell_1}{B^{j_1}})b(\frac{\ell_2}{B^{j_2}})b(\frac{\ell_3}{B^{j_3}})$$

$$\times \frac{4\pi}{N}\sum_{k} a_{\ell_1 m_1}a_{\ell_2 m_2}a_{\ell_3 m_3}Y_{\ell_1 m_1}(\widehat{\gamma}_k)Y_{\ell_2 m_2}(\widehat{\gamma}_k)Y_{\ell_3 m_3}(\widehat{\gamma}_k)$$

Needlets bispectrum

We obtain

$$\sum_{\ell_i m_i} b(\frac{\ell_1}{B^{j_1}}) b(\frac{\ell_2}{B^{j_2}}) b(\frac{\ell_3}{B^{j_3}})$$

$$\times \quad a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} h_{\ell_1 \ell_2 \ell_3}$$

$$= \sum_{\ell_1 \ell_2 \ell_3} b(\frac{\ell_1}{B^{j_1}}) b(\frac{\ell_2}{B^{j_2}}) b(\frac{\ell_3}{B^{j_3}}) h_{\ell_1 \ell_2 \ell_3} \widehat{B}_{\ell_1 \ell_2 \ell_3} \ .$$

$$h_{\ell_1 \ell_2 \ell_3} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \ ;$$

Main features

- Smoothed standard bispectrum
- Computationally extremely convenient
- Tight control of harmonic localization
- Analytic expressions for the expected bispectrum
- Search for optimal configurations
- Search for anisotropies - directional dependence

Estimates of f_{NL}

We consider Maximum Likelihood - GLS estimates

$$\widehat{f}_{NL} = \frac{\left\langle \hat{I}_{j_1 j_2 j_3} \right\rangle^T \mathbf{C}^{-1} I_{j_1 j_2 j_3}}{\left\langle \hat{I}_{j_1 j_2 j_3} \right\rangle^T \mathbf{C}^{-1} \left\langle \hat{I}_{j_1 j_2 j_3} \right\rangle}$$

where C estimates the covariance matrix

Some results on WMAP5

freq. channel	*KQ75*	*KQ85*
V + *W*	73 ± 31	78 ± 29
V	58 ± 35	55 ± 33
W	74 ± 37	72 ± 34
Q	-8 ± 39	-9 ± 37

Table 1: f_{NL} estimates and 1σ error bars. These estimates are found using the full CMB sky.

ℓ_{max}	n_j	V+W	V	W	Q
324	25	64 ($\pm$71)	77 ($\pm$73)	63 ($\pm$75)	26 ($\pm$74)
390	26	44 ($\pm$61)	81 ($\pm$64)	35 ($\pm$66)	25 ($\pm$66)
471	27	44 ($\pm$55)	71 ($\pm$60)	40 ($\pm$62)	31 ($\pm$62)
567	28	42 ($\pm$52)	56 ($\pm$56)	55 ($\pm$58)	43 ($\pm$56)
683	29	73 ($\pm$49)	71 ($\pm$52)	72 ($\pm$54)	23 ($\pm$50)
823	30	81 ($\pm$43)	80 ($\pm$46)	72 ($\pm$49)	24 ($\pm$46)
1000	31	89 ($\pm$39)	105 ($\pm$42)	54 ($\pm$45)	9 ($\pm$44)

Table 2: The estimated values for f_{NL} for different number n_j of needlet scales.

Directional dependence

	V+W	*V+W*	*Q*	*V*	*W*
ring	*KQ75*	*KQ85*	*KQ75*	*KQ75*	*KQ75*
1	91 ± 95	93 ± 95	47 ± 118	66 ± 106	94 ± 111
2	11 ± 68	6 ± 68	-18 ± 83	1 ± 76	28 ± 79
3	80 ± 80	43 ± 71	-149 ± 100	19 ± 90	13 ± 93
4	283 ± 183	122 ± 113	700 ± 226	462 ± 205	253 ± 213
5	117 ± 82	128 ± 70	-61 ± 103	37 ± 92	122 ± 96
6	39 ± 66	53 ± 66	-81 ± 81	35 ± 74	15 ± 78
7	158 ± 93	156 ± 93	174 ± 114	138 ± 104	201 ± 108

Table 3: The f_{NL} estimates and 1σ error-bars for equatorial rings.

POLARIZED CMB RADIATION

$$E_x(z,t) = E_{0x}\cos(\tau+\delta_x)\ ,\ E_y(z,t) = E_{0y}\cos(\tau+\delta_y)\ , \quad (10)$$

$$\tau := \omega t - kz, \nu = 2\pi\omega/k$$

Parametric equations of an ellipse, i.e.

$$\frac{E_x^2(z,t)}{E_{0x}^2}+\frac{E_y^2(z,t)}{E_{0y}^2}-2\frac{E_x(z,t)}{E_{0x}}\frac{E_y(z,t)}{E_{0y}}\cos\delta = \sin^2\delta\ ,\ \delta := \delta_y-\delta_x\ .$$

STOKES' PARAMETER

Magnitude (i.e. CMB temperature)

$$T = E_{0x}^2 + E_{0y}^2 \ ;$$

Stokes' parameters Q and U,

$$Q = E_{0x}^2 - E_{0y}^2 \ , \ U = 2E_{0x}E_{0y}\cos\delta \ . \tag{11}$$

which are "spin 2" quantities, e.g.

$$Q' + iU' = \exp(i2\gamma)(Q + iU) \ . \tag{12}$$

SPIN s SPHERICAL HARMONICS

$$
\begin{aligned}
Y_{lms}(\vartheta,\varphi) &= (-1)^m \sqrt{\tfrac{2l+1}{4\pi}} D^l_{-ms}(\varphi,\vartheta,-\psi)\exp(-is\psi) \\
&= (-1)^m \sqrt{\tfrac{2l+1}{4\pi}} \exp(im\varphi) d^l_{-ms}(\vartheta) \ ;
\end{aligned}
\tag{13}
$$

Spectral representation of spin functions:

$$
f_s(\vartheta,\varphi) = \sum_l \sum_m a_{l;ms} Y_{l;ms}(\vartheta,\varphi) \ . \tag{14}
$$

SPIN NEEDLETS

Spin needlets are defined as (Geller and Marinucci (2008))

$$\psi_{jk;s}(p) = \sqrt{\lambda_{jk}} \sum_{l} b\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} Y_{l;ms}(p)\, \overline{Y_{l;ms}}\left(\xi_{jk}\right),$$

or more rigorously

$$\psi_{jk;s}(p) = \sqrt{\lambda_{jk}} \sum_{l} b\left(\frac{}{B^j}\right) \sum_{m=-l}^{l} \left\{Y_{l;ms}(p) \otimes \overline{Y_{l;ms}}\left(\xi_{jk}\right)\right\} .$$

As before, $\left\{\lambda_{jk}, \xi_{jk}\right\}$ are cubature points and weights, $b(\cdot) \in C^{\infty}$ is nonnegative, and has a compact support in $[1/B, B]$.

SPIN NEEDLET TRANSFORM

The spin needlet transform is defined by

$$\int_{\mathbb{S}^2} f_s(p)\overline{\psi_{jk;s}}(p)dp = \beta_{jk;s} \ ,$$

and the same inversion property holds as for standard needlets, i.e.

$$f_s(p) = \sum_{jk} \beta_{jk;s}\psi_{jk;s}(p) \ ,$$

the equality holding in the L^2 sense. The coefficients of spin needlets are

$$\beta_{jk;s} = \int_{\mathbb{S}^2} f_s(p)\overline{\psi_{jk;2}}(p)dp = \sqrt{\lambda_{jk}} \sum_l b\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} a_{l;ms} Y_{l;ms}\left(\xi_{jk}\right. \tag{15}$$

LOCALIZATION

For any $M \in \mathbb{N}$ there exists a constant $c_M > 0$ s.t., for every $\xi \in \mathbb{S}^2$:

$$\left|\psi_{jk;s}(\xi)\right| \leq \frac{c_M B^j}{(1 + B^j \arccos(\langle \xi_{jk}, \xi \rangle))^M} \text{ uniformly in } (j,k)$$

Condition 3 $\{Q + iU\}(p)$ *Gaussian and isotropic with angular power spectrum*

$C_l = l^{-\alpha} g(l) > 0$, *where* $c_0^{-1} \leq g(l) \leq c_0$, $\alpha > 2$, *for all* $l \in \mathbb{N}$,

and for every $r \in \mathbb{N}$ *there exist* $c_r > 0$ *such that*

$$|\frac{d^r}{du^r} g(u)| \leq c_r u^{-r} \text{ , } u \in (|s|, \infty) \text{ .}$$

UNCORRELATION

Example:

$$C_l = \frac{F_1(l)}{l^\beta F_2(l)} \ ,$$

where $F_1(l), F_2(l) > 0$ are polynomials of degree $q_1, q_2 > 0$, $\beta + q_2 - q_1 = \alpha$.
Under Condition 3,

$$\left|Corr\left(\beta_{jk;s}, \overline{\beta_{jk';s}}\right)\right| \leq \frac{C_M}{\left\{1 + B^j d(\xi_{jk}, \xi_{jk'})\right\}^M} \ , \ M \in \mathbb{N} \ , C_M > 0 \ . \tag{16}$$

POWER SPECTRUM ESTIMATION

(Geller and M (2008), Geller, Lan and M (2009))

$$\Gamma_{j;s} := \sum_k \left|\beta_{jk;s}\right|^2 = \sum_k \lambda_{jk} \sum_l b^2\left(\frac{\sqrt{e_{ls}}}{B^j}\right) C_l \frac{(2l+1)}{4\pi}$$

Under condition (3), we have

$$\frac{\widehat{\Gamma}^*_{j;sG} - \Gamma_{j;s}}{\sqrt{Var\left\{\widehat{\Gamma}^*_{j;sG}\right\}}} \rightarrow_d N(0,1) \text{ , as } j \rightarrow \infty \text{ .}$$

APPLICATIONS

- TESTING FOR ASYMMETRIES/FEATURES
- AUTO-POWER SPECTRUM ESTIMATION
- NOISE: CROSS-POWER SPECTRUM ESTIMATION
- HAUSMAN TEST FOR NOISE BIAS

FURTHER DEVELOPMENTS

- POLARIZATION BISPECTRUM
- E AND B MODES
- GRAVITATIONAL LENSING
- COSMIC RAYS - SOURCE DETECTION
- MEXICAN NEEDLETS (Geller and Mayeli)

Directional Data

Baldi, Kerkyacharian, Marinucci and Picard (AoS 2009)
Assume we observe $X_1, ..., X_n \in S^2$.
We wish to estimate their density on the sphere $f(x)$. We know that

$$f(x) = \sum_{j,k} \beta_{jk} \psi_{jk}(x) \tag{17}$$

where

$$\beta_{jk} = \int_{S^2} f(x) \psi_{jk}(x) dx \tag{18}$$

The coefficients β_{jk} can be estimated by

$$\widehat{\beta}_{jk} = \frac{1}{n} \sum_{i=1}^{n} \psi_{jk}(X_i) \tag{19}$$

leading to the linear wavelet estimator

$$\widehat{f}(x) = \sum_{j,k} \widehat{\beta}_{jk} \psi_{jk}(x) \tag{20}$$

It can be shown, however, that (nearly) optimal estimates are obtained by thresholding, i.e.

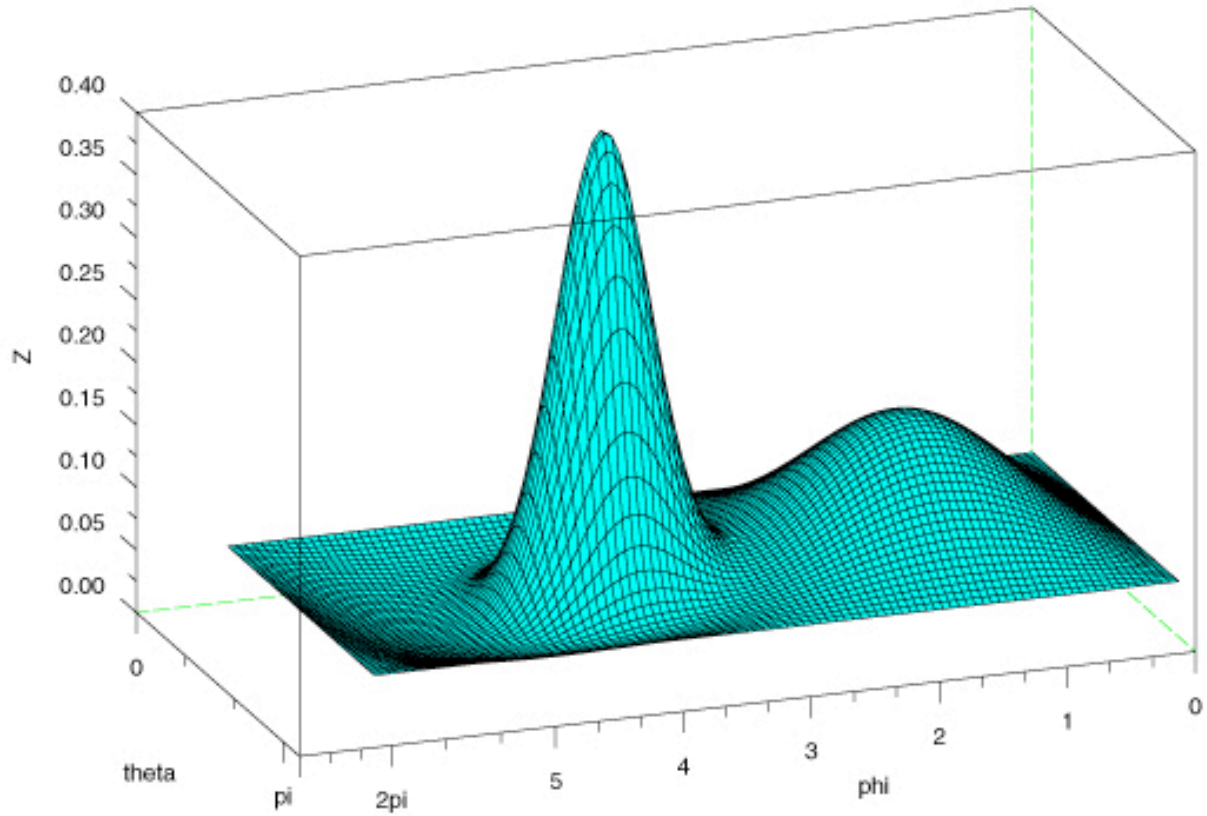
$$\widehat{f}(x) = \sum_{j,k} \widehat{\beta}^H_{jk} \psi_{jk}(x) \tag{21}$$

$$\widehat{\beta}^H_{jk} = \widehat{\beta}_{jk} I(|\widehat{\beta}_{jk}| > t_n) \tag{22}$$

Intuitively, the smallest coefficients are expected to be dominated by noise and hence discarded. We take $t_n = k_0 \sqrt{\frac{logn}{n}}$

Simulations

As an example, we try to estimate the following mixture of Gaussian density



Small threshold

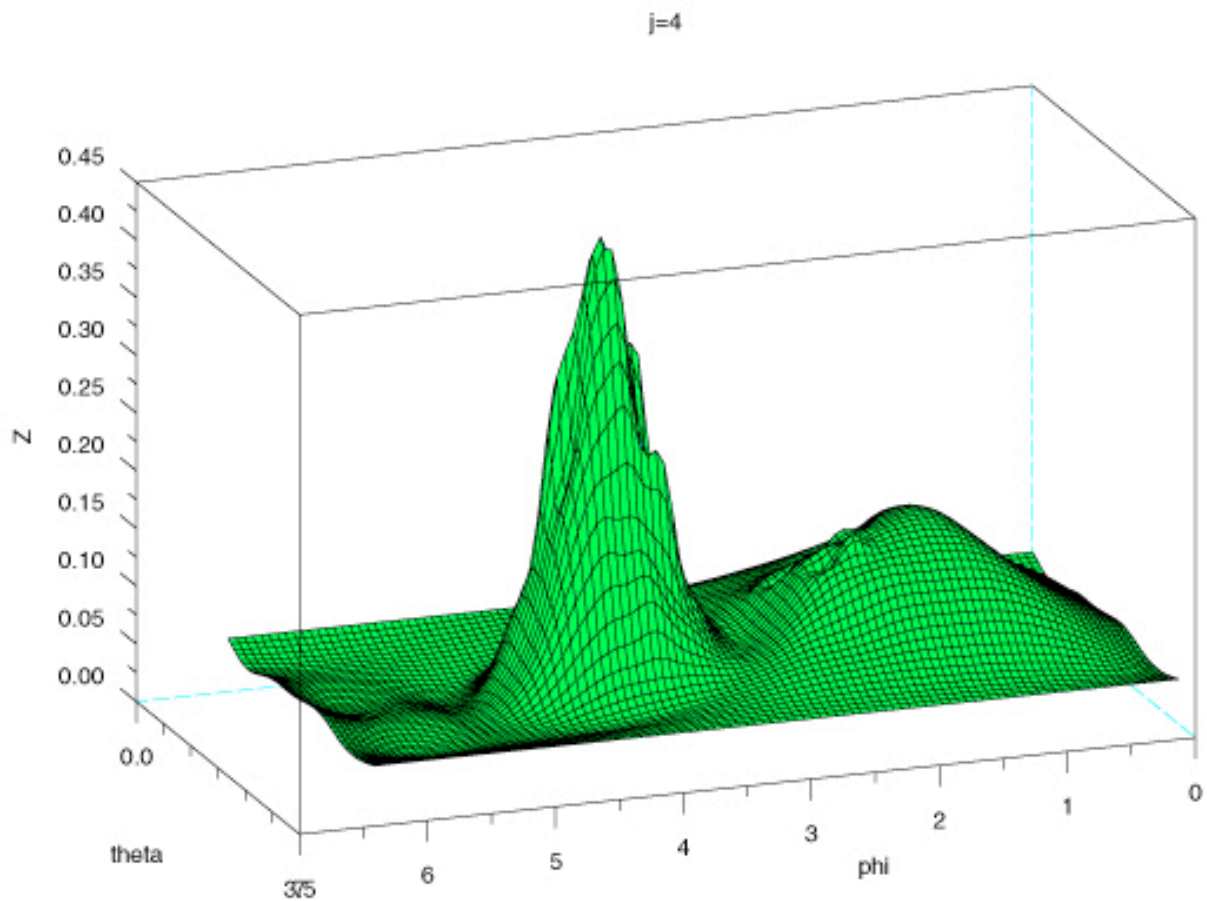


Figure 18: Threshold $k_0 = 0.25$.

Medium threshold

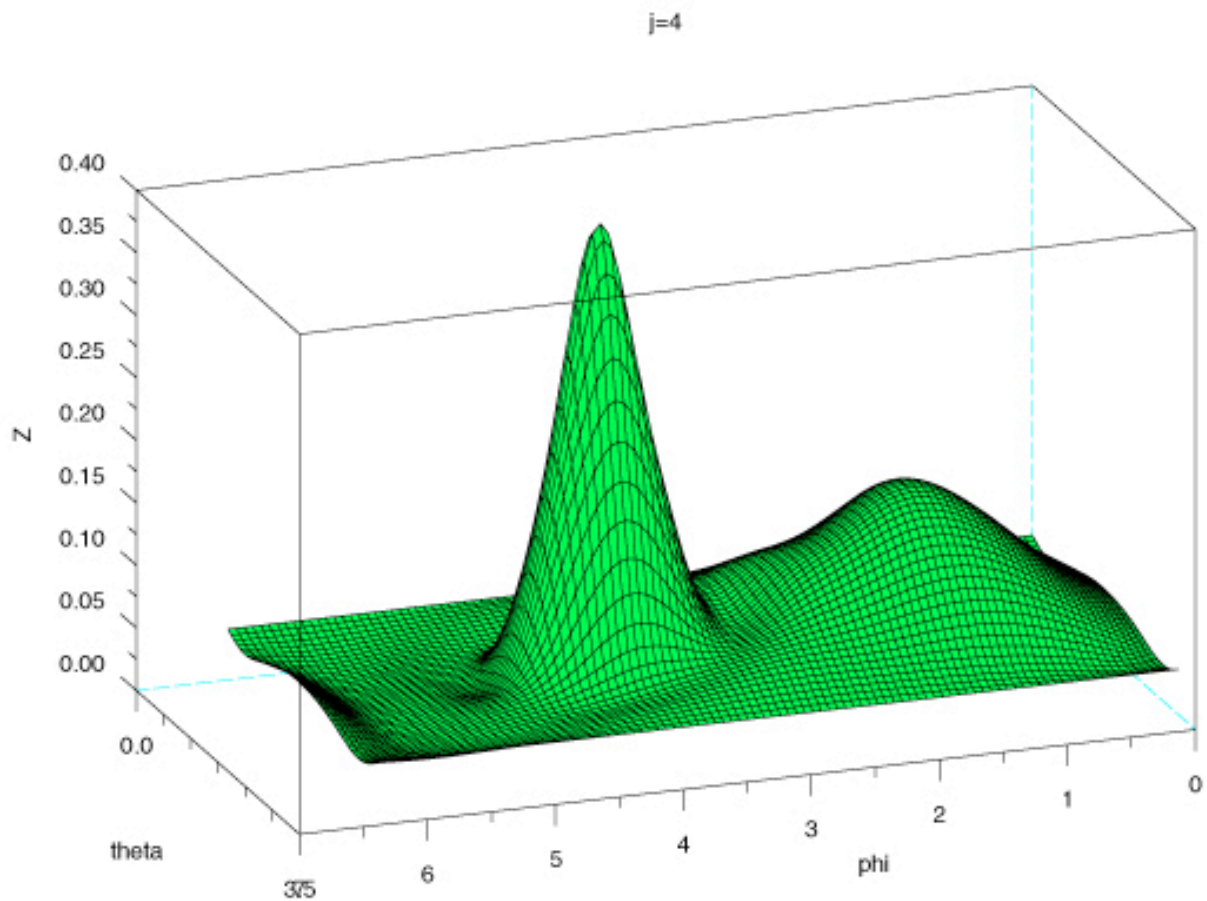


Figure 19: Threshold $k_0 = 0.45$.

High threshold

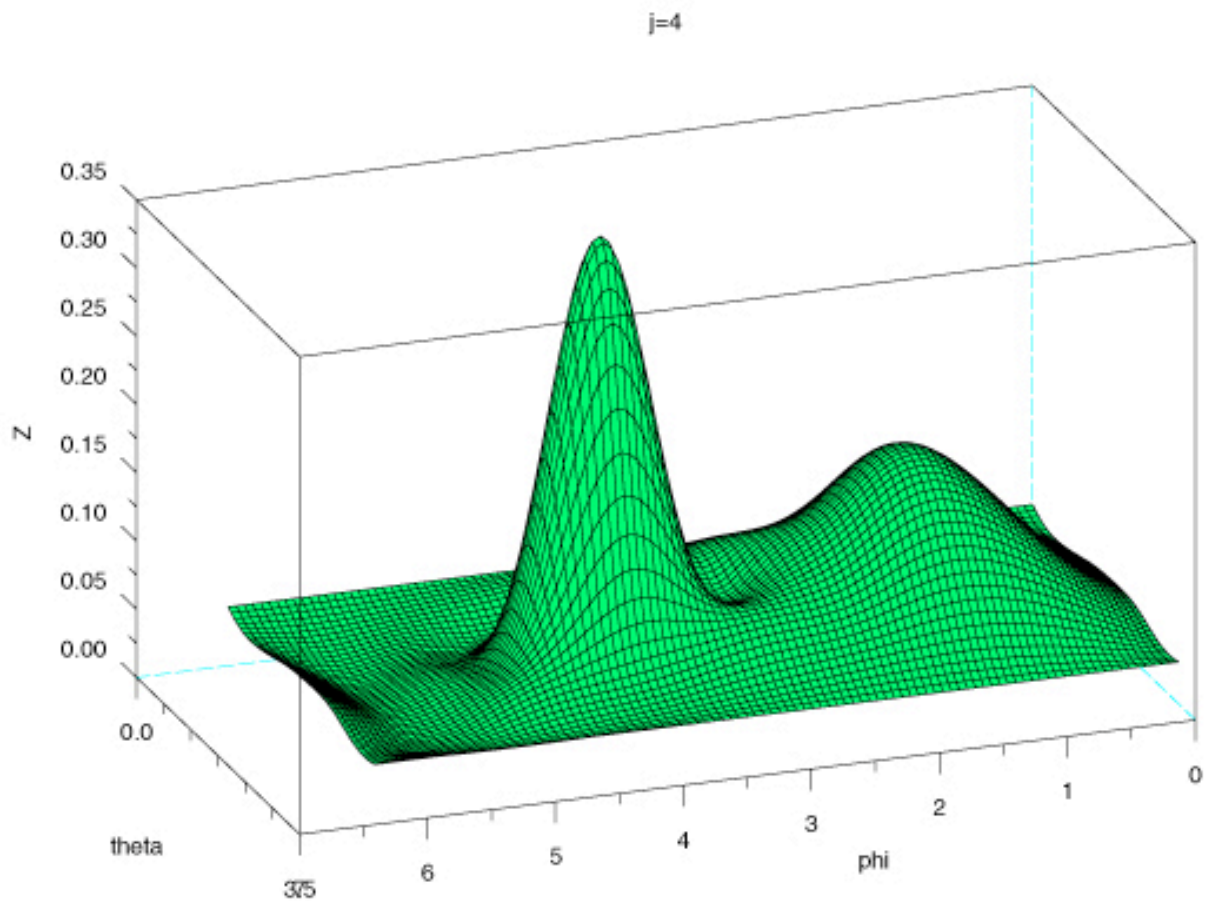


Figure 20: Threshold $k_0 = 0.7$.

The Figures provided the needlet estimators obtained for various thresholds. Thresholding estimates are especially valuable when interested in "spikes".

method	L^∞ distance
needlets $k_0 = .25$	0.073
needlets $k_0 = .45$	0.045
needlets $k_0 = .7$	0.066
Gaussian kernel $k = 10$	0.112
Gaussian kernel $k = 20$	0.064

Table 4: Estimates of the L^∞ distance with different methods.

Possible applications

- Tests for Uniformity (ARGO)
- Source detection
- Denoising
- Cross-correlation (AUGER, Fermi-Glast)