$\S 3$ Math supplement

$\S 3.1$ Singularity and its resolution

Consider a geometry given by

$$X = \{ (z_1, \cdots, z_n) \mid f(z's) = 0 \} \subset \mathbb{C}^n.$$

(def) X is singular at a point $(z_1, \cdots, z_n) \in X$.

iff. $f(z_1, \cdots, z_n) = 0$ and $\frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_n} = 0$.

If $X = \{ (z_1, \cdots, z_n) \mid f_a(z's) = 0 \text{ for } a = 1, 2, \cdots, k \} \subset \mathbb{C}^n$ $(k < n)$

then X is singular at $(z_1, \cdots, z_n) \in X$

iff. $f_a(z's) = 0$ for all $a = 1 \sim k$ and

$$\operatorname{rank}\left[\left(\frac{\partial f_a(z's)}{\partial z_i} \right)_{k \times n \text{ matrix}} \right] < k.$$

eg.

$\{(x, y) \mid y^2 = x^2\} \subset \mathbb{C}^2$ singular @ $(x, y) = (0, 0)$

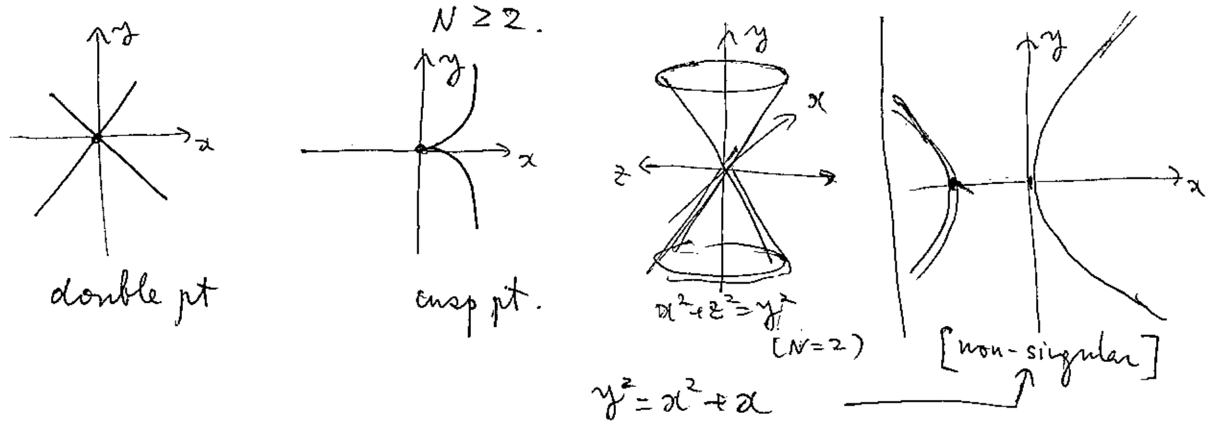
$\{(x, y) \mid y^2 = x^3\} \subset \mathbb{C}^2$ " @ $(x, y) = (0, 0)$

$\{(x, y) \mid y^2 = x^n\} \subset \mathbb{C}^2$ " @ $(x, y) = (0, 0)$.

$n \geq 2$.

$\{(x, y, z) \mid y^2 = x^2 + z^N\} \subset \mathbb{C}^3$ singular @ $(x, y, z) = (0, 0, 0)$

$N \geq 2$.

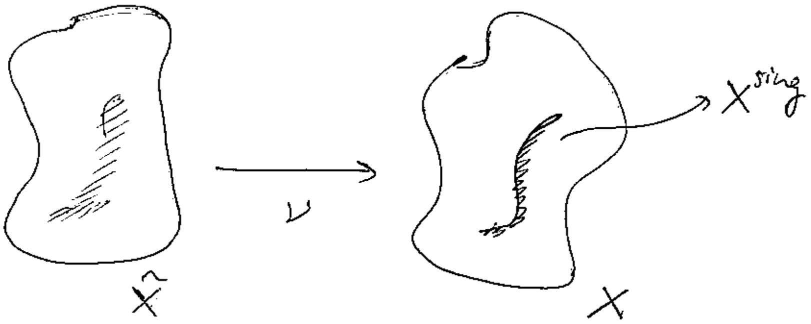


<u>resolution of singularity</u>

(phys) a math procedure to "dig out" vanishingly small cycles buried at singularities.

(math) for X with $\left(\underset{\neq \phi}{X^{sing}} \subset X \right)$, find $\tilde{X}$ and a (non-singular) holomorphic map $\nu : \hat{X} \longrightarrow X$ s.t.

$\nu|_{\text{non-sing}} : \nu^{-1}(X \setminus X^{sing}) \longrightarrow (X \setminus X^{sing})$ is an isomorphism.



<u>How to do that? : blow-up</u>

(1) $\boxed{\text{Blow-up of } \mathbb{C}^n \text{ centered at } (z_1, \cdots, z_n) = (0, \cdots, 0) =: \widetilde{\mathbb{C}}^n}$

$\widetilde{\mathbb{C}}^n := \left\{ (z_1, \cdots, z_n), [\zeta_1 : \cdots : \zeta_n] \in \mathbb{C}^n \times \mathbb{P}^{n-1} \;\middle|\; z_i \zeta_j = z_j \zeta_i \right\}$

$\pi : \widetilde{\mathbb{C}}^n \longrightarrow \mathbb{C}^n$ simply by forgeting $[\zeta_1 : \cdots : \zeta_n]$

- for a point $(z_1, \cdots, z_n) \in \mathbb{C}^n$ that is not $(0, \cdots, 0)$.

  e.g. $z_1 \neq 0 \longrightarrow \zeta_j = \zeta_1 \left(\frac{z_j}{z_1} \right) \rightarrow$ a specified pt in $\mathbb{P}^{n-1}$.

  1-to-1 via π in $\widetilde{\mathbb{C}}^n \longleftrightarrow \mathbb{C}^n$.

- for the point. $(z_1, \cdots, z_n) = (0, \cdots, 0)$

  $\pi^{-1}((\vec{0})) = \mathbb{P}^{n-1}$

Cover $\widetilde{\mathbb{C}}^n$ by n-open patches.

$\mathcal{U}_i \subset \widetilde{\mathbb{C}}^n$

$\hookrightarrow := \{ \zeta_i \neq 0 \text{ subspace of } \widetilde{\mathbb{C}}^n \}$

$\longrightarrow \underline{\bigcup_{i=1 \sim n} \mathcal{U}_i = \widetilde{\mathbb{C}}^n}$

In $\mathcal{U}_i$

$z_i \zeta_j = z_j \zeta_i \Rightarrow$ take $\left\{ \left(z_i, \left(\frac{\zeta_j}{\zeta_i} \right) \right) \right\}_{j \neq i}$

$z_j = z_i \left(\frac{\zeta_j}{\zeta_i} \right)$

as a set of local coordinates

each ~~one of~~ $\mathcal{U}_i$ ... $\simeq \mathbb{C}^n$

$\widetilde{\mathbb{C}}^n$ non-singular.

$\pi|_{\mathcal{U}_i} : \widetilde{\mathbb{C}}^n \longrightarrow \mathbb{C}^n$

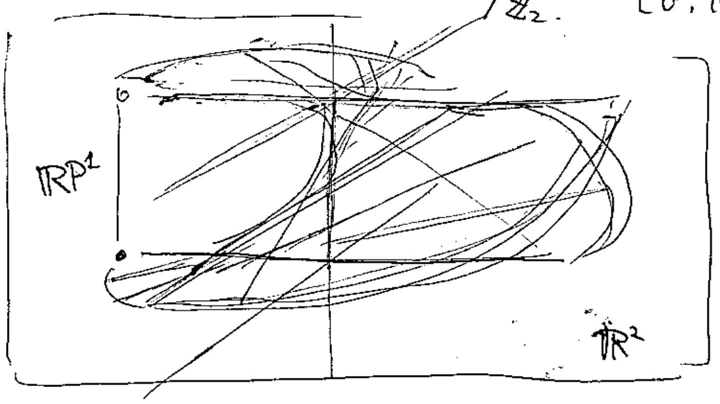
$\left(z_i, \left(\frac{\zeta_j}{\zeta_i} \right) \right) \longmapsto (z_i, z_{j \neq i}) = \left(z_i, z_i \cdot \left(\frac{\zeta_j}{\zeta_i} \right)_{j \neq i} \right)$

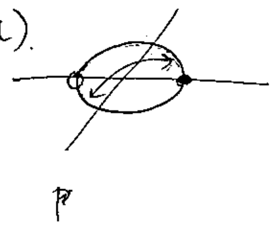
hol fcns of local coordinates of π.

π being $\stackrel{\text{def}}{=}$ holomorphic.

eg. $\underline{\underline{\widetilde{\mathbb{C}}^2}}$

$\mathbb{RP}^1 = S^1 / \mathbb{Z}_2$. $[0, \pi)$.





② proper transform

$X \subset \mathbb{C}^n$, $(z_1, \cdots, z_n) \in X$.

- $\pi^{-1}(X) \subset \tilde{\mathbb{C}}^n$
- closure of $\pi^{-1}(X \setminus \{0\}) \subset \tilde{\mathbb{C}}^n$

} not the same.

$\angle$ (def) proper transform of X.
in the blow up of the ambient space $\mathbb{C}^n$ centered @ $(0, \cdots, 0) \in \mathbb{C}^n$.

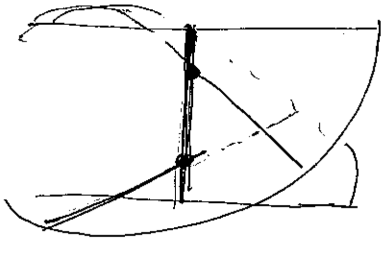
$\bar{X} \subset \tilde{\mathbb{C}}^n$.

$\pi|_{\bar{X}} : \bar{X} \longrightarrow X$.

$\pi : \tilde{\mathbb{C}}^n \longrightarrow \mathbb{C}^n$.

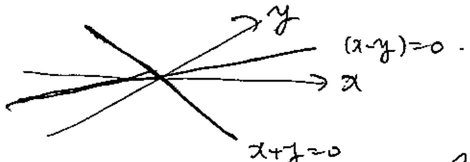
(def) except. locus : inv. image of $(0, \cdots, 0)$ in $\tilde{\mathbb{C}}^n$ or $\bar{X}$

e.g. proper transform of $\underset{X}{\underset{\parallel}{\{(x,y) \mid y^2 = x^2\}}} \subset \mathbb{C}^2$.



$\pi^{-1}(X)$: 3 components (irr.)

$\bar{X}$: 2 irr. components.



$\bar{X}$: non-singular

$\mathcal{U}_1 : \{(x,y), [\xi . \eta] \mid \xi \neq 0\}$

$\Rightarrow (x, \frac{\eta}{\xi}) = (x_1, y_1)$

$\mathcal{U}_2 : \{(x,y), [\xi . \eta] \mid \eta \neq 0\}$

$\Rightarrow (\frac{\xi}{\eta}, y) = (x_2, y_2)$

$\pi : \mathcal{U}_1 \longrightarrow \mathbb{C}^2$

$(x_1, y_1) \longmapsto (x, y) = (x_1, x_1 y_1)$

$\pi : \mathcal{U}_2 \longrightarrow \mathbb{C}^2$

$(x_2, y_2) \longmapsto (x, y) = (x_2 y_2, y_2)$

In $\mathcal{U}_1$: $\pi^{-1}(X)$: given by $0 = \pi^*(y^2 - x^2) = (x_1)^2 (y_1^2 - 1)$.

~~the~~ except. locus in $\mathcal{U}_1$: $(x_1=0,\ \forall y_1)$ $\longrightarrow \simeq \mathbb{P}^1$.

$\bar{X}$: $(y_1)^2-1=0.$ $\forall x_1$

similarly in $\mathcal{U}_2$

$\bar{X}$: $1-(x_2)^2=0$ $\forall y_2$.

$\pi^{-1}(\bar{X})$: $(y_2)^2(1-(x_2)^2)=0$.

③ repeat processes like this multiple times, if necessary.

Exercise : $X=\{(x,y)\mid y^2=x^3\}\subset\mathbb{C}^2$.

This singular X becomes non-singular $\bar{X}$ in $\tilde{\mathbb{C}}^2$:

Exercise : $X=\{(x,y)\mid y^2=x^n\}\subset\mathbb{C}^2$ w/ $n\geq 4$

$\rightarrow$ still $\bar{X}$ singular in $\tilde{\mathbb{C}}^2$ w/ $n\rightarrow(n-2)$

$X=\{(x,y,z)\mid y^2=x^2+z^N\}\subset\mathbb{C}^3$ ((A_{N-1} singularity))

$\bar{X}\cap\mathcal{U}_3=\{(x_3,y_3,z_3)\mid y_3^2=x_3^2+z_3^{N-2}\}\subset\mathcal{U}_3\subset\tilde{\mathbb{C}}^3$

$\bar{\bar{X}}\cap\mathcal{U}_{33}=\{(x_{33},y_{33},z_{33})\mid y_{33}^2=x_{33}^2+z_{33}^{N-4}\}$

In $\mathcal{U}_3$

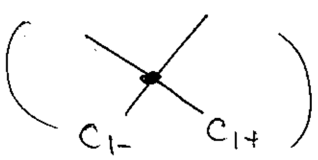
$\pi^{-1}(X)$: $z_3^2(-y_3^2+x_3^2+z_3^{N-2})=0$.

except. locus in $\tilde{\mathbb{C}}^3$ $=\mathbb{P}^2$.

in $\bar{X}\big|_{\mathcal{U}_3}=\{-y_3^2+x_3^2+z_3^{N-2}=0\}\cap\{z_3=0\}$

$\Longrightarrow$ $(y_3\pm x_3)=0$ in $\mathbb{P}^2$

$(\eta\pm\xi=0)$ $\mathbb{P}^1 * \mathbb{P}^1$

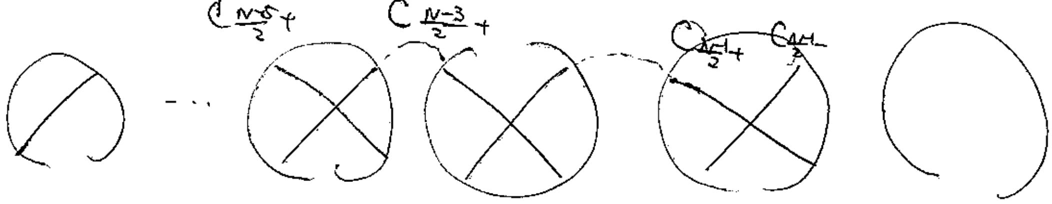


$C_{1\pm}$: except. locus in $\bar{X}$

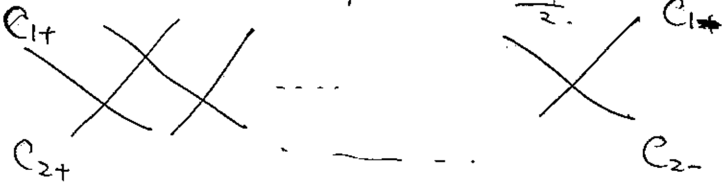
vanishing cycle

repeating this process

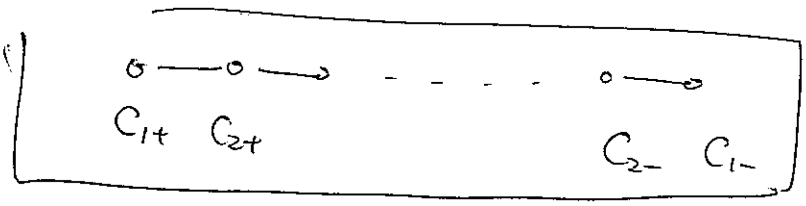
if N : is odd



curve w/ intersection pts $\underbrace{U_{33\cdots3}}_{\frac{N-1}{2}}$



or



Dynkin diagram of A_{N-1}.

$(N-1)$ 2-cycles as except. locus

by resolving A_{N-1} singularity.

Intersection form $(-C_{A_{N-1}})$

☆ multiplicity in algebraic geometry.

motivation: $ax^2+bx+c=0$ when $b^2-4ac=0$.

$\hookrightarrow (x-x_*)^2=0$ double root.

not set theoretically but in terms of algebra.

generalization

$\{-y+ax^2+bx+c=0\}$ & $\{y=0\}$ in $\mathbb{C}^2$. intersection.

set a local coord. in $\{y=0\}$ ($\to$ i.e. x)

and $(-y+ax^2+bx+c)\big|_{y=0}$ $\to$ multiplicity as above.

$\Rightarrow$ $\boxed{\{f_m(x,y)=0\} \cap \{f_n(x,y)=0\} \Rightarrow m\times n \text{ points in } \mathbb{P}^2}$.

if multiplicity info included.

$$\pi^{-1}(X = \{y^2 - x^2 = 0\} \subset \mathbb{C}^2) \Rightarrow (x_1)^2\{1-(y_1)^2\} = 0$$

multiplicity

$$\underline{2\{x_1=0\} + \{y_1-1=0\} + \{y_1+1=0\}}$$

except locus of the resolution of A_{N-1} singularity

$$(C_{1+} + C_{2+} + \cdots \quad + C_{1-} + C_{2-} + \cdots)$$

all coeff. $= 1$.

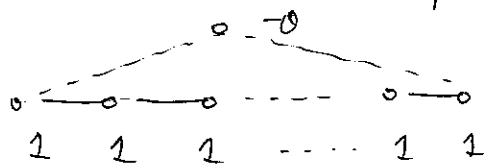
$\cancel{C_{+0} = \sum_{i=1}^{r} n_i}$

$$\boxed{\theta = \sum_{i=1}^{r} n_i \alpha_i}$$

max root

simple roots $n_i \in \mathbb{N}$

A_{N-1}

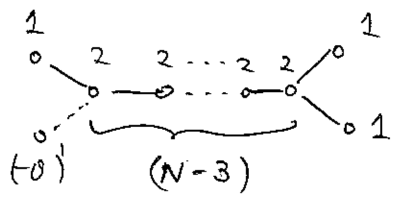


$\theta = L_1 - L_N$

$L_i - L_{i+1}$

$i = 1 \sim N-1$.

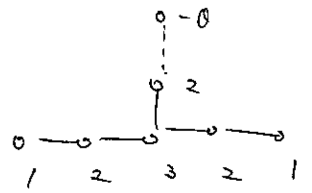
D_N



$\cancel{L_+ = \frac{1}{2}(L_1 + \cdots + L_N)}$ $\cancel{L_+ = (L_1+L_N)}$

$\cancel{L_- = \frac{1}{2}(L_1 - L_2)}$ $\cancel{L_- = (L_1-L_2)}$

E_6

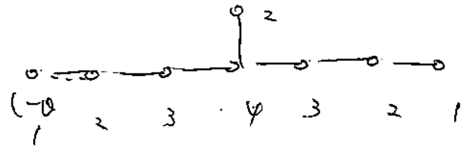


simple roots $\cancel{L_i - L_{i+1}}$

$$\left(\begin{array}{ll} L_{N-1} - L_N, & i = 2 \sim N-1 \\ L_{N-1} + L_N, & L_i - L_{i+1} \end{array} \right) \quad i = 2 \sim N-2.$$

$$\boxed{\theta = (L_1 + L_2)}$$

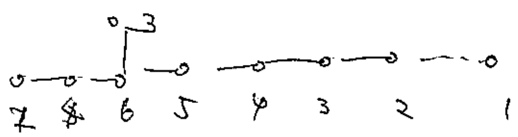
E_7



$$\boxed{\sum_{i=0}^{r} n_i = T_G}$$

dual Coxeter #

E_8



A_{N-1}	:	N
D_N	:	$(2N)-2$
E_6	:	12
E_7	:	18
E_8	:	30

exceptional locus of A-D-E singularity resolution.

$$\Rightarrow \boxed{\sum_{i=1}^{n} n_i C_i}$$

A_{N-1} singularity : $y^2 = x^2 + z^N$

D_n singularity : $y^2 = x^2 z + z^{n-1}$

E_6 singularity : $y^2 = x^3 + z^4$

E_7 singularity : $y^2 = x^3 + x z^3$

E_8 singularity : $y^2 = x^3 + z^5$