$\boxed{\S 4.\ \text{F-theory}}$

does not have a formulation... microscopically.

§ 4.1 Relation to IIB and M-theory

Type IIB : $SL(2;\mathbb{Z})$ symmetry

acting on $\begin{pmatrix} B^{(2)} \\ C^{(2)} \end{pmatrix}$. $\begin{pmatrix} F1 \\ D1 \end{pmatrix}$ $\begin{pmatrix} NS5 \\ D5 \end{pmatrix}$

electric magnetic charge.

$\begin{pmatrix} \tau \\ 1 \end{pmatrix}$. $\tau' = \dfrac{\alpha\tau+\beta}{\gamma\tau+\delta}$ $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2;\mathbb{Z})$

For IIB w/ const. τ.

$SL(2;\mathbb{Z})$ change $\tau \to \tau'$.

Around a D7-brane...

magnetic charge of $C^{(0)}$

$$\cancel{dC^{(0)}} \quad d(G^{(1)}) = \delta^{(1)}_{(D7)}$$

$\Delta C^{(0)} = 1$ around D7

IIB / ($CY_3/\mathbb{Z}_2$ orientifold) various branch cuts.

$SL(2;\mathbb{Z})$ monodromy.

☆ M-theory$/T^2$ — IIB/S^1 How to implement D7's here?

T^2 complex torus. $\Longrightarrow$ elliptic curve.

elliptic fns.

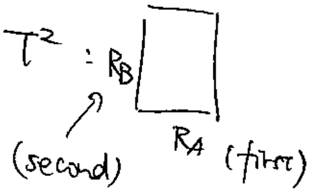
$\wp(u)$, $\wp'(u)$ always satisfy $y^2 = 4x^3 - g_2 x - g_3$

for some $g_2, g_3 \in \mathbb{C}$.

$\boxed{\mathbb{P}^2 \supset \left(Y^2 Z = X^3 + f X Z^2 + g Z^3\right)}$

$$\begin{cases} \dfrac{1}{g_s^{\mathrm{IIB}}} = \left(\dfrac{R_B}{R_A}\right) \\ S^1_{\mathrm{IIB}} \text{ radius} : \dfrac{l_{11}^3}{(R_A R_B)} \end{cases}$$

T^2 :

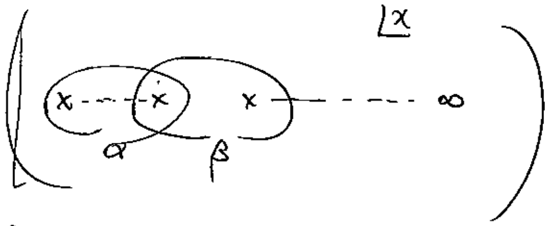


$\underline{\underline{SL(2;\mathbb{Z}) \text{ on } T^2.}}$

$\underline{\tau} = \int \dfrac{dx}{y}$ on a 1-cycle on T^2.

~~$y^2 = x(x-u)(x-1)$~~

$y^2 = (x^2 - z)(x-1)$



$\alpha \longrightarrow \alpha$

$\beta \longrightarrow \beta + \alpha$ (1-cycle.)

$$\tau = \frac{\int_\beta dx/y}{\int_\alpha dx/y} \qquad \tau \to (\tau + 1)$$

<u>elliptic curve vary</u> over $\underline{z}$ coordinate.

<u>Seiberg-Witten curve.</u>

D=4 N=2 SUSY. SU(2) gauge theory.

$\Uparrow$

O3 + a pair of D3-brane.

Nf pairs of D7 $\Rightarrow$ Nf flavour

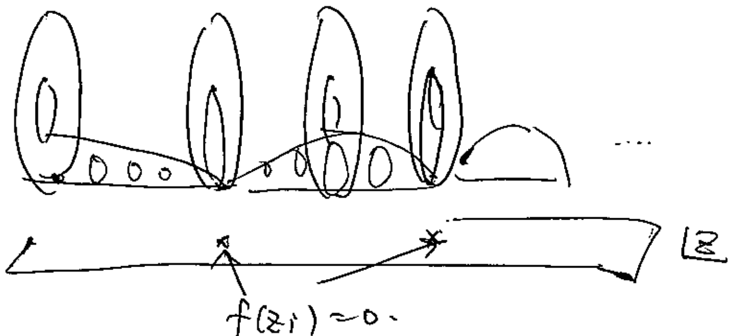
w/ SO(2Nf) sym.

elliptic curve

$\downarrow$ monopole + dyon singularity $\Longleftarrow$ O7 in F-theory/M-theory.

Coulomb phase moduli space (D3-brane position)

$y^2 = x^3 + x^2 + f(z)$

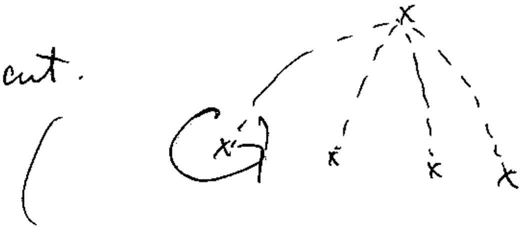


$\begin{cases} D7: \text{ elliptic curve degenerate.} \\ F1: \text{ M2 wrapped on the } S^1 \end{cases}$

Introduce a basis, branch cut.

$H_1(T^2;\mathbb{Z})$

α, β



$(p_i\alpha + q_i\beta) \in H_1(T^2;\mathbb{Z})$ shrinks at pt i

$\Longrightarrow$ (p_i, q_i) 7-brane.

$SL(2;\mathbb{Z})$ monodromy

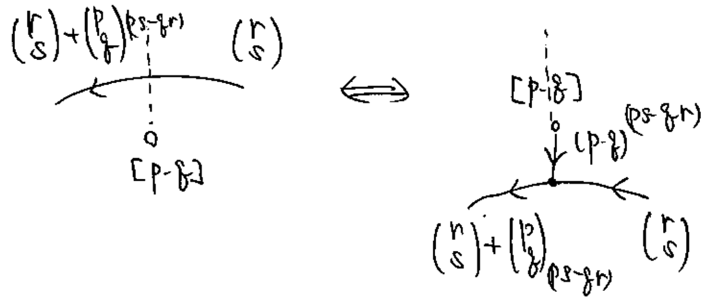
$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q' & -p' \\ -q & p \end{pmatrix} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}$$

M2 on

$(p\alpha + q\beta) \Rightarrow (p,q)$ string ending on (p,q) 7-brane.

elliptic fibre: $\begin{cases} \text{IIB: like principale bdle.} \\ \text{M: physical object (part of space-time)} \end{cases}$

$$\begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix} = \mathbb{1} + \begin{pmatrix} p \\ q \end{pmatrix} \otimes (-q, p)$$



string junction.

7-brane...

$\pi_X : X_n \longrightarrow B_{n-1}$. elliptic fibration.

in Weierstrass form.

$$y^2 = x^3 + f x + g$$

$$\Delta \equiv (4f^3 + 27 g^2) \quad \text{discriminant.}$$

(just like $(b^2 - 4ac)$ of $(ax^2 + bx + c)$

$\boxed{\Delta \equiv \prod_{i<j} [(e_i - e_j)^2]}$)

single 7-brane

$$y^2 = x^3 + x^2 + z \qquad @\ z = 0.$$

multiple D7-branes.

$$y^2 = x^3 + x^2 + \prod_i (z - z_i)$$

close together $\Rightarrow$ $\underline{y^2 = x^3 + x^2 + z^N}$ + (small perturb.)

A_{N-1} singularity $\Leftrightarrow$ $N \times$ D7-brane.

D_N-type, E_r-type } singularity in X_n. $\Rightarrow$ 7-brane with D_N or $E_{6,7,8}$ gauge group on 7-brane.

☆ non-Abelian gauge sym $\Longleftrightarrow$ singular X_n in the Weierstrass form.

☆ single D7 : $y^2 = x^3 + x^2 + z$. non-singular.

$\Delta \propto (z)$, though.

✓ $U(1)$ vector : not on 7-brane but in between branes.

✓ D7-brane moduli : part of ~~CY~~(X_n) moduli relatively to IIB

☆ CY_3 moduli + D7-brane moduli + dilaton

$\Longrightarrow CY_4$ moduli

☆ $C^{(3)}_{\text{M-theory}} \to \underbrace{C^{(2)}.dx^{10} + C'^{(2)}dx^{11}}_{\hookrightarrow SL(2;\mathbb{Z})}$

§4.2 Het$/T^2$ – F/(elliptic K3) duality.

★ elliptic K3 : construction.

— line bdle $\mathcal{O}(n)$ on $\mathbb{P}^1$.

hom. coord. $[Z, W]$ on $\mathbb{P}^1$.

patch	coord.
$U_{\check{Z}}$	$(W/Z) = w$
$U_{\check{W}}$	$(Z/W) = z$

trans. fcn of $\mathcal{O}(n)$

$$g_{\check{W}\check{Z}} = \left(Z/W\right)^n = z^n = w^{-n} \quad \text{in } U_{\check{Z}} \cap U_{\check{W}}$$

sections of $\mathcal{O}(n)$

hol-fcn $f_{\check{Z}}$ on $U_{\check{Z}}$, $f_{\check{W}}$ on $U_{\check{W}}$ s.t. $\underline{f_{\check{W}}(z) = g_{\check{W}\check{Z}}\, f_{\check{Z}}(w)}$

— Weierstrass eq.

$$\left(y^2 = x^3 + f\,x + g.\right) \text{ in } U_{\check{Z}} \text{ \& } U_{\check{W}}.$$

if we take a line bdle $\mathcal{L}$ on $\mathbb{P}^1$. (Trans fcn $g_{\check{W}\check{Z}}$)

$$\begin{cases} y_{\check{W}} = (g_{\check{W}\check{Z}})^3\, y_{\check{Z}} \\ x_{\check{W}} = (\;\;)^2\, x_{\check{Z}} \\ f_{\check{W}} = (\;\;)^4\, f_{\check{Z}} \\ g_{\check{W}} = (\;\;)^6\, g_{\check{Z}} \end{cases}$$

$f \in \Gamma(\mathbb{P}^1; \mathcal{L}^{\otimes 4})$

$g \in \Gamma(\mathbb{P}^1; \mathcal{L}^{\otimes 6})$

For this to be K3 (CY_2)

$$\boxed{c_1(\mathcal{L}) = c_1(T\mathbb{P}^1)}$$

D.O.F counting

- $T\mathbb{P}^1 = \mathcal{O}_{\mathbb{P}^1}(2)$

$$\rightarrow \begin{cases} f \in \Gamma(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(8)) \\ g \in \Gamma(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(12)). \end{cases}$$

homogeneous fcn ~~[Z,W]~~ Z,W of degree 8, 12.

$\Rightarrow$ 9 & 13 coeff. $\leadsto$ 22.

- $SL(2;\mathbb{C})$ on $\mathbb{P}^1$ $[Z,W]$

- $$\begin{matrix} x \to \lambda^2 x' \\ y \to \lambda^3 y' \end{matrix} \Rightarrow (y')^2 = (x')^3 + (x')\underbrace{\left(\frac{1}{\lambda^4} f\right)}_{f'} + \underbrace{\left(\frac{1}{\lambda^6} g\right)}_{g'}$$

$\Rightarrow 22 - 4 = 18$ D.O.F. in elliptic K3

$\left(\approx \langle C^{(0)} + i\frac{1}{g_s}\rangle,\ T^2/\mathbb{Z}_2 \text{ cpx str},\ SO(32) \text{ D7-brane position} \Downarrow 16 \right)$

like K3 metric

$$SO(3,19)/SO(3)\times SO(19).$$

now. elliptic K3 cpx str

$$\Rightarrow \boxed{O(2,18)/O(2)\times O(18)}$$ Aspinwall.

Het$/T^2$ Narain moduli

$$\boxed{O(2,18;\mathbb{R})\big/ O(2)\times O(18)}$$

intuitively

$$\boxed{\begin{cases} \text{cpx str } T^2 \\ (\omega + iB) \text{ on } T^2. \\ 16 \text{ Wilson lines.} \end{cases}}$$

$\boxed{\text{Het} - \text{F duality } 8D}$