§ 4.3 Het - F duality in 6D

adiabatically fibre 「the Het - F duality @ 8D」

↑ intuition.

$$\left[\mathrm{Het}\Big/\begin{pmatrix}\text{ell. fib} \\ \text{K3}\end{pmatrix} \;—\; F\Big/\begin{pmatrix}\text{ell. fib } CY_3 \\ \text{that is also K3-fibred}\end{pmatrix} \right] \stackrel{?}{=}$$

$\pi_Z : (Z_2 = K3) \longrightarrow \underline{(B_1 = \mathbb{P}^1)}$

fibre. T^2.

$\pi_X : X_3 \longrightarrow B_2'$

$\pi' : B_2' \longrightarrow \underline{(B_1 = \mathbb{P}^1)}$

$(\pi' \circ \pi_X) : X_3 \longrightarrow (B_1 = \mathbb{P}^1)$

fibre = ell. K3

「Het-F duality @ 8D」

more data

$\boxed{\text{Het :}}$

$$dH = \mathrm{tr}(R \wedge R) - \mathrm{tr}_1'(F \wedge F) - \mathrm{tr}_2'(F \wedge F).$$

Bianchi id.

$$\Rightarrow \quad 0 = \int_{Z_2 = K3} dH = \int \mathrm{tr}(R \wedge R) - \int \mathrm{tr}_1'(F \wedge F) + \mathrm{tr}_2'(F \wedge F)$$

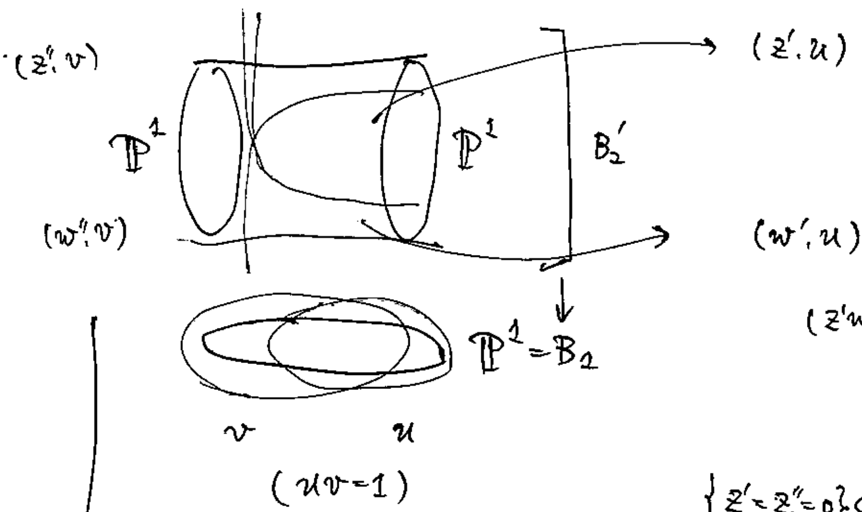
$$\Rightarrow \quad \left(24 = \int_{Z_2 = K3} c_2(TZ_2)\right) = -\int_{K3} \left(\frac{\overset{(vis)}{ch_{2,R}}(F \wedge F)}{2T_R} + \frac{\overset{(hid)}{ch_{2,R}}(F \wedge F)}{2T_R} \right)$$

$$= \quad (12-n) \quad + \quad (12+n)$$

instanton #.

$\boxed{\text{F-theory}}$

$\pi' : B_2' \longrightarrow B_1$ $\quad \mathbb{P}^1$ fibration over $\mathbb{P}^1$



$$\begin{cases} z'' = u^n z' \\ w'' = u^{-n} w' \end{cases}$$

$(z'w' = 1)$.

$B_2' = \mathbb{F}_n$ (Hirzebruch surface)

$\{z' = z'' = 0\} \Leftrightarrow C_{vis}$ $\quad \{w' = w'' = 0\} \Leftrightarrow C_{hid}$.

claim

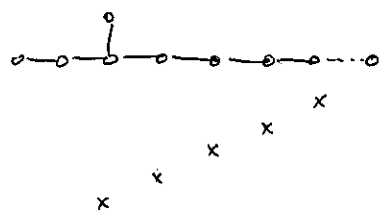
Het/ell. K3 instanton distrib. $(12-n) + (12+n)$ vis $\quad$ hid.	$\Longleftrightarrow$ dual	F/ell. CY_3. X_3 (also K3-fib). base $B_2' = \mathbb{F}_n$.

moduli counting on both sides.

Het.

$$\begin{aligned} E_8 \supset\ & SU(2) \times E_7 \\ & SU(3) \times E_6 \\ & SU(4) \times SO(10) \\ & SU(5) \times SU(5) \\ & SU(6) \times SU(3) \times SU(2) \end{aligned}$$

Extended Dynkin diagram of E_8.



$\mathrm{Res}^{E_8}_{SU(2)\times E_7}(248) = (\text{adj.}, 1) + (1, \text{adj}) + (2, 56)$ $\quad 3 + 133 + 112 = 248$.

$\mathrm{Res}^{E_8}_{SU(3)\times E_6}(248) = (\text{adj.}, 1) + (1., \text{adj}) + [(3.27) + \text{h.c.}]$ $\quad 8 + 78 + 162 = 248$

$\mathrm{Res}^{E_8}_{SU(4)\times SO(10)}(248) = (\text{adj.}, 1) + (1., \text{adj.}) + [(4, 16) + \text{h.c.}] + (\wedge^2 4, 10)$ $\quad 15 + 45 + 128 + 60 = 248$

$\mathrm{Res}^{E_8}_{SU(5)\times SU(5)}(248) = (\text{adj.}, 1) + (1., \text{adj.}) + [(5, \wedge^2 5) + (\wedge^2 5, \bar{5})] + \text{h.c.}$

$\vdots$

$24 + 24 + (50 + 50) \times 2 = 248$.

A class of Het vacua.

$$\text{w/} \quad \overbrace{E_7 \times \langle SU(2)\rangle}^{(12-n)} \times \overbrace{\langle E_8 \rangle}^{(12+n)}$$

E_7 unbroken

$$\Rightarrow \#(\text{hypermultiplets}) = \begin{cases} \text{boson.} \\ \frac{1}{2} h^1 (K3 ; \rho(V)) = -\frac{1}{2}\chi(K3 ; \rho(V)) \\ \text{fermion} \\ -\frac{1}{2}\,\mathrm{index}_\rho \not{D} \end{cases}$$

$$\chi(\cancel{K3}\,CY ; \rho(V)) = \int_{Z=CY} ch_\rho(V)\, td(TZ)$$

$$= \int_Z ch_\rho(V)\, e^{\frac{1}{2}c_1(TZ)} \hat{A}(TZ)$$

$Z = CY$

$$\Leftrightarrow \int_Z ch_\rho(V)\,\hat{A}(TZ) = \mathrm{index}_\rho \not{D}$$

for $K3 = Z_2$

$$\chi(\rho(V)) = rk[\rho(V)] \int_{K3} \left(\frac{c_2 + \overset{\to 0}{c_1^2}}{12}\right) + \int_{K3} c_1(\rho(V)) \left(\frac{\overset{\to 0}{c_1}}{2}\right) + \int_{K3} ch_{2,\rho}(V)$$

$$= 2\, rk[\rho(V)] + 2T_\rho \left(\frac{\int ch_{2,\rho}(V)}{2T_\rho}\right)$$

$$= 2\, rk[\rho(V)] - 2T_\rho \times (12-n)$$

$$\boxed{\#(\text{hyper}) = \cancel{2}\, T_\rho \times (12-n) - rk(\rho)}$$

E_8-instanton (hidden) $\quad \#(\text{hyper}) = 30 \times (12+n) - 248.$

$$\begin{cases} E_7^{vis} - \text{singl.} \\ \langle SU(2)\rangle - \text{instanton} \quad \#(\text{hyper}) = 2 \times (12-n) - 3 \\ E_7^{vis} - 56 \\ \langle SU(2)\rangle - \underline{\underline{2}} \quad \#(\text{hyper}) = \frac{1}{2} \times (12-n) - 2 = \frac{1}{2}(8-n) \end{cases}$$

$$\begin{cases} E_6^{vis} - \cancel{\text{singl}}\, 27 \\ \langle SU(3)\rangle - \cancel{\text{instanton}}\ \underline{\underline{3}} \quad \#(\text{hyper}) = \left\{\frac{1}{2} \times (12-n) - 3\right\} \times 2 = (6-n) \\ E_6^{vis} - \cancel{27}\ \text{singl} \\ \langle SU(3)\rangle - \text{instanton} \quad \#(\text{hyper}) = 3 \times (12-n) - 8 \end{cases}$$

$\vdots$

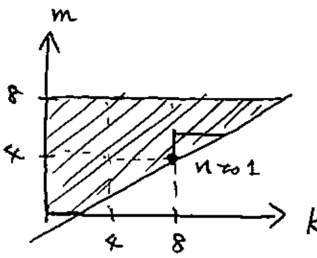
Corresponding class of vacua <u>in F-theory</u>

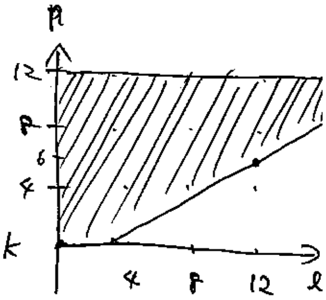
elliptic fibred CY_3 on $F_n = B_2'$

~~$y^{12} = (x')^3$~~

$y^2 = x^3 + f x + g$

f:





$$\begin{cases} f(z', u) = \sum_{m,k} f_{m,k}\, u^k (z')^m \\ g(z', u) = \sum_{p,l} g_{p,l}\, u^l (z')^p \end{cases}$$

$0 \le m \le 8, \quad 0 \le k \le n(m-4)+8$

$\underbrace{0 \le p \le 12, \quad 0 \le l \le n(p-6)+12.}$

in the (z', u) patch of $F_n = B_2'$.

def. eq of X_3 in the 3 other patches : determined accordingly.

This condition : $CY_3 \cdot X_3$.

$$X_2 = CY_2 = K3 \iff \begin{cases} f \in \Gamma(B_1 = \mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(8)) \\ g \in \Gamma(B_1 = \mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(12)). \end{cases}$$

$$\begin{pmatrix} \mathcal{O}_{\mathbb{P}^1}(8) = (T\mathbb{P}^1)^{\otimes 4} \\ \mathcal{O}_{\mathbb{P}^1}(12) = (T\mathbb{P}^1)^{\otimes 6} \end{pmatrix}$$

now

$$\begin{cases} f \in \Gamma(B_2' = F_n; (\Lambda^2 T F_n)^{\otimes 4}) \\ g \in \Gamma(B_2' = F_n; (\Lambda^2 T F_n)^{\otimes 6}). \end{cases}$$

X_3 : CY_3 condition

$E_7 \times \langle SU(2) \rangle \times \langle E_8 \rangle$

coeff. $\begin{cases} f_m(u) z^m & \text{for } m \ge 3 \\ g_p(u) z^m & \text{for } p \ge 5 \end{cases}$ $\neq 0$.

$\#(f_{m,k} \neq 0) = (9-n) + 9 + (36+10n)$

$\#(g_{p,l} \neq 0) = (13-n) + 13 + (78+21n)$

(sum) $\quad (22-2n) + 22 + (114+31n)$

$- \begin{pmatrix} x, y, z', u \\ \text{redef} \end{pmatrix} \text{D.O.F} \quad (6+n)$

$-1 + (21-2n) + 20 + (112+30n)$

one more hyper

$(\ C^{(3)} \text{ on } (h^{2,1}+\underline{\underline{1}})\ ) \Rightarrow \boxed{(21-2n) + 20 + (112+30n)}$

~~E_7~~ for generic $u = u_*$ (fixed)

$$y^2 = x^3 + x \cdot \{z^3 (f_3(u_*)) + z^4 f_4(u_*) + \cdots\} + \{z^5 g_5(u_*) + \cdots\}$$
$$\simeq x^3 + x z^3$$

E_7-type singularity.

$f_3(u_*)$ vanishes at $(8-n)$ pts on $B_1 = \mathbb{P}^1$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \#$(half hyper in E_7-$\underline{\underline{56}}$)

$E_6 \times \langle SU(3) \rangle \times \langle E_8 \rangle$.

coeff. $\quad f_m(u)\, z^m$ for $m \geq 3$

$g_p(u)\, z^p$ for $p \geq 5$

$g_{p=4}(u) = (g(u))^2$ $\quad\quad$ extra $\cdot \left[\frac{1}{2}(12-2n)+1\right]$ coeff.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = (7-n)$

SU(3) bdle moduli $\quad 3(12-n)-8$
SU(2) $\quad$ " $\quad\quad\quad 2(12-n)-3$
$\quad\quad\quad\quad\quad\quad\quad\quad (12-n)-5 = (7-n)$

for generic $u = u_*$ (fixed)

$$y^2 = x^3 + x(z^3 f_3(u_*) + \cdots) + (z^4 (g(u_*))^2 + z^5 g_5(u_*) + \cdots)$$
$$\simeq x^3 + z^4$$

E_6-type singularity.

§ 4. 4 Het − F duality in ~~6D~~ 4D

generalization of "n".

distribution of

$$c_2(TZ_3) = -\frac{ch_{2,\rho}(V_1)}{2T_\rho} - \frac{ch_{2\rho}(V_2)}{2T_\rho}.$$

$\Rightarrow$ in 6D $\quad c_2 : \big(\text{4-form} \underset{\text{dual in K3}}{\longleftrightarrow} \text{pt}\big)_*$ $\quad H^4(K3)$ or $H_0(K3)$.

in 4D $\quad c_2 : \big(\text{4-form} \underset{\text{dual in } CY_3}{\longleftrightarrow} \text{2-cycle}\big)$ $\quad H^4(CY_3)$ or $H_2(CY_3)$

$$\boxed{\begin{array}{l}\text{Het}/\text{ell. } CY_3 \\ \pi_Z : Z_3 \longrightarrow B_2\end{array}} \xleftrightarrow{\text{dual}} \boxed{\begin{array}{l} F/\text{ell. } CY_4 \\ \pi_X : X_4 \longrightarrow B_3' \ (\text{ell.}) \\ \pi' : B_3' \longrightarrow B_2.\ (\mathbb{P}^1) \\ \pi'\circ\pi_X : X_4 \longrightarrow B_2 \ (\text{ell. K3}).\end{array}}$$

<u>Het</u>

$$-\frac{ch_{2,\rho}(V_{vis})}{2T_\rho} = \sigma\cdot\eta_{vis} + \pi_Z^*(pts)$$

$$-\frac{ch_{2,\rho}(V_{hid})}{2T_\rho} = \sigma\cdot\eta_{hid} + \pi_Z^*(pts')$$

$$c_2(TZ_3) = \sigma\cdot 12c_1(TB_2) + \pi_Z^*\big(c_2(TB_2) + 11c_1^2(TB_2)\big)$$

$\Rightarrow \boxed{12c_1(TB_2) = \eta_{vis} + \eta_{hid}} \longleftrightarrow B_3' = \mathbb{P}\big[\mathcal{O}_{B_2}(D_{vis}) \oplus \mathcal{O}_{B_2}(D_{vis} + 6c_1(TB_2) - \eta_{vis})\big]$

generalization of

$\boxed{24 = (12-n) + (12+n)} \longleftrightarrow (B_2' = \mathbb{F}_n) = \mathbb{P}\big[\ ~~\mathcal{O}_{B}~~$

as divisor relation. $\qquad \big[\mathcal{O}_{B_1}(D_{vis}) \oplus \mathcal{O}_{B_1}(D_{vis}+n)\big]$

$$\boxed{\begin{array}{l} D_{hid} \sim D_{vis} - (6K_{B_2} + \eta_{vis}) \\ \begin{cases} D_{hid}\cdot D_{vis} = 0 \\ D_{vis}\cdot D_{vis} = (6K_{B_2} + \eta_{vis}) \end{cases}\end{array}}$$ for F-theory.

$D_{hid} \sim D_{vis} + n$

$$\begin{cases} D_{vis}\cdot D_{vis} = -n \\ D_{vis}\cdot D_{hid} = 0 \end{cases}$$