

Theory of Elementary Particles

homework I (April 09)

- At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like “II-1, II-3, IV-2”).
- Pick up any problems that are suitable for your study. **You are not expected to work on all of them!**
- Format: Reports do not have to be written neatly; hand-writing is perfectly O.K. Do not waste your time!
- Keep your own copy, if you need one. Reports will not be returned.

1. Follow-up [A]

Fill non-trivial gaps in derivations, calculations etc. during the lecture. If you encounter a gap that cannot be filled, state clearly what is yet to be proved or understood.

2. Non-relativistic QED Lagrangian, Fine Structure of Atomic Spectrum [C]

Let us derive QED Lagrangian in the non-relativistic limit.

(a) Confirm that the following choice of gamma matrices,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \tau^i \\ -\tau^i & \end{pmatrix}, \quad (1)$$

satisfies the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbf{1}_{4 \times 4}. \quad (2)$$

Here, $\eta^{\mu\nu} = \text{diag}(+, -, -, -)$.

(b) Now let us begin with the following QED Lagrangian,

$$\mathcal{L} = \bar{\Psi} [i\gamma^\mu (\partial_\mu - ieA_\mu) - m] \Psi, \quad (3)$$

$$= -\Psi^\dagger \begin{bmatrix} m + (-i\partial_t - e\varphi) & \vec{\tau} \cdot (-i\vec{\partial} + e\vec{A}) \\ \vec{\tau} \cdot (-i\vec{\partial} + e\vec{A}) & -m + (-i\partial_t - e\varphi) \end{bmatrix} \Psi. \quad (4)$$

Here,

$$\Psi = \begin{pmatrix} \psi^{(0)} \\ \bar{\psi}^{(0)} \end{pmatrix} \quad (5)$$

is a 4-component spinor field, and $\bar{\Psi} = \Psi^\dagger \gamma^0$, $A_\mu = (\varphi, -\vec{A})$, and $e > 0$. It is known that the upper 2 components of Ψ , $\psi^{(0)}$, is almost an electron field, and the lower 2

components, $\psi^{c(0)}$, a positron field, when the gamma matrices are chosen as in (1). In the following, we will go through a series of field redefinition to make this distinction even more clearer. As a first step, define a new 4-component spinor field $\Psi^{(1)}$ by

$$\Psi^{(1)} \equiv \exp \left[\begin{pmatrix} \frac{\vec{\tau} \cdot (-i\vec{\partial} + e\vec{A})}{2m} \\ -\frac{\vec{\tau} \cdot (-i\vec{\partial} + e\vec{A})}{2m} \end{pmatrix} \right] \Psi, \quad (6)$$

and rewrite the Lagrangian (4) in terms of $\Psi^{(1)}$, rather than Ψ . Keep the terms that are $\mathcal{O}(m^{+1})$, $\mathcal{O}(m^0)$ and $\mathcal{O}(m^{-1})$ for now, and ignore the rest. Now you have seen a classic result of spin- \vec{B} coupling.

(c) In order to get rid of the off-diagonal term that still exists at $\mathcal{O}(m^{-1})$, define

$$\Psi^{(2)} \equiv \exp \left[\frac{-i}{4m^2} \begin{pmatrix} \vec{\tau} \cdot e\vec{E} & \vec{\tau} \cdot e\vec{E} \\ \vec{\tau} \cdot e\vec{E} & \vec{\tau} \cdot e\vec{E} \end{pmatrix} \right] \Psi^{(1)}; \quad (7)$$

here, $\vec{E} = -\partial_t \vec{A} - \vec{\partial}\varphi$. By rewriting the Lagrangian in terms of $\Psi^{(2)}$, show that

$$\begin{aligned} \mathcal{L} = & -\Psi^{(2)\dagger} \left[\begin{pmatrix} m & \\ & -m \end{pmatrix} - (i\partial_t + e\varphi) \right. \\ & \left. + \frac{1}{2m} \left\{ (-i\vec{\partial} + e\vec{A})^2 \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix} + e\vec{B} \cdot \begin{pmatrix} \vec{\tau} & \\ & -\vec{\tau} \end{pmatrix} \right\} \right. \\ & \left. + \frac{e\vec{\partial} \cdot \vec{E}}{8m^2} + \frac{e}{4m^2} (\vec{E} \times (-i\vec{\partial} + e\vec{A})) \begin{pmatrix} \vec{\tau} & \\ & -\vec{\tau} \end{pmatrix} + \dots \right] \Psi^{(2)}. \end{aligned} \quad (8)$$

Here, $\mathcal{O}(1/m^3)$ terms as well as off-diagonal terms of $\mathcal{O}(1/m^2)$ are omitted. Because of the absence of off-diagonal terms of $\mathcal{O}(1/m)$, the upper 2 components of $\Psi^{(2)}$, $\psi^{(2)}$, is more purely an electron field than $\psi^{(0)}$ is. [remark: although there are off-diagonal terms $\psi^{(2)}\psi^{c(2)}$ whose coefficients are of $\mathcal{O}(1/m^2)$, they can be absorbed by further field redefinition. After the redefinition, diagonal terms of $\mathcal{O}(1/m^3)$ will appear. In that sense, the off-diagonal $\mathcal{O}(1/m^2)$ terms are smaller effects than the diagonal $\mathcal{O}(1/m^2)$ effects.] Because $\vec{E} = \frac{e}{4\pi} \frac{\vec{r}}{r^3}$ around a proton, the last term gives rise to a spin-orbit coupling of the form $\frac{\alpha}{4m^2 r^3} \vec{L} \cdot \vec{\tau}$.

(d) (if you are not tired yet...) Although we have carried out $1/m$ -expansion of the QED Lagrangian so far, we should be aware that the expansion parameter $1/m$ is not a dimensionless constant. The $1/m$ corrections always come in the form of a dimensionless combination (E/m) for some energy scale E , and the $1/m$ expansion becomes useful only when this ratio (E/m) is small. Thanks to the fact that the Bohr radius is $r_B = 1/(m\alpha)$ in natural unit, and the fact $\alpha \simeq 1/137 \ll 1$, application to condensed matter physics, where $|\vec{p}| = |-i\vec{\partial}| \lesssim 1/r_B$ (Brillouin zone), and to atomic physics $|\vec{p}| \sim 1/r_B$ satisfy this condition. $|\vec{p}|/m \sim \alpha$, and α actually plays the role of expansion parameter.

The hydrogen binding energy comes from $e\varphi \sim \alpha/r_B \sim m\alpha^2$ and $p^2/m \sim 1/(mr_B^2) \sim m\alpha^2$. Show that the last two terms of (8) give rise to the correction of $\mathcal{O}(m\alpha^4)$ to the binding energy by naive analysis as above. In order to determine the $\mathcal{O}(m\alpha^4)$ correction to the binding energy of a hydrogen atom, however, one will notice that the p^4/m^3 term also needs to be calculated. Because this term is not generated by the field redefinition that absorbs the $\mathcal{O}(1/m^2)$ off-diagonal terms in (8), one only needs to calculate the p^4/m^3 term in the Lagrangian (8) written in terms of $\Psi^{(2)}$. Show that it is

$$\Delta\mathcal{L} = \Psi^{(2)\dagger} \left[\frac{(\vec{p}^2)^2}{8m^3} \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix} \right] \Psi^{(2)}. \quad (9)$$

(e) (not a homework problem: just for your fun) Therefore, the fine structure of a hydrogen atom can be calculated by treating

$$\Delta H = -\frac{(\vec{p}^2)^2}{8m^3} + \frac{e(\vec{\partial} \cdot \vec{E})}{8m^2} + \frac{e}{4m^2} \vec{\tau} \cdot (\vec{E} \times \vec{p}) \quad (10)$$

as additional part of the Hamiltonian of Schrödinger equation, and evaluating the matrix element of this ΔH using the wavefunctions labeled by quantum numbers (n, l, m) . Here, $\vec{p} = -i\vec{\partial}$, $e\varphi(r) = \alpha/r$, and $\vec{E} = -\vec{\partial}\varphi$. See [Landau–Lifshitz series vol 4, Quantum Electrodynamics section 33–34]. The binding energy at order $\mathcal{O}(m\alpha^4)$ becomes

$$E_{n,j,l,s} \simeq m \left(-\frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^3(j+1/2)} + \frac{3\alpha^4}{8n^4} \right). \quad (11)$$

The degeneracy among the states with the same value of n is partially lifted, although the states with the same n and the same j (that is, $(l, s) = (j - 1/2, 1/2)$ state and $(l, s) = (j + 1/2, 1/2)$ state) are still degenerate at this $\mathcal{O}(m\alpha^4)$ level.

The same result can be obtained also by solving Dirac equation with $\varphi(r) = e/(4\pi r)$, instead of the Schrödinger equation. The result is this:¹

$$E = m \left(1 + \left(\frac{\alpha}{n - (j + 1/2) + \sqrt{(j + 1/2)^2 - \alpha^2}} \right)^2 \right)^{-1/2}. \quad (12)$$

Expanding this expression as a power series of α , (11) is obtained. Note that the degeneracy between the $l = j \mp 1/2$ states is not lifted at all order in α in this expression. This remaining degeneracy is lifted by a QED 1-loop effect (Lamb shift),² which is not taken into account by simply solving the Dirac equation.

¹See Landau–Lifshitz vol. 4 (QED) section 36, or K. Nishijima, “*Relativistic Quantum Mechanics*”, (Baifu-Kan)

²See Bjorken–Drell section 8.7 (or Landau–Lifshitz vol 4 (QED) section 123).

3. Fermi Surface and Hole Excitation [B]

Consider a Lagrangian (and corresponding Hamiltonian) of non-relativistic electron.

$$\mathcal{L} = \psi^\dagger \left[i\partial_t + \frac{1}{2m} \vec{\partial} \cdot \vec{\partial} \right] \psi, \quad (13)$$

$$H = \int d^3x \psi^\dagger \left[-\frac{1}{2m} \vec{\partial} \cdot \vec{\partial} \right] \psi. \quad (14)$$

The two-component spinor field ψ here corresponds to the upper 2 components $\psi^{(n)}$ ($n = 1, 2, \dots$) of the non-relativistic QED Lagrangian (in problem I-2), with yet another field redefinition $\psi^{(n)}(\vec{x}, t) = e^{-imt} \psi(\vec{x}, t)$. With the creation and annihilation operators of states with a given momentum, the field operators are written as

$$\psi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \sum_r \xi_r a_{\vec{p},r} e^{-iE_{\vec{p}}t + i\vec{p}\cdot\vec{x}}, \quad \{a_{\vec{p},r}, a_{\vec{q},s}^\dagger\} = \delta_{r,s} (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \quad (15)$$

Here, $E_{\vec{p}} = |\vec{p}|^2/(2m)$.

(a) Show that

$$H' \equiv H - \epsilon_F \int d^3x \psi^\dagger \psi = \int \frac{d^3p}{(2\pi)^3} \sum_r (E_{\vec{p}} - \epsilon_F) a_{\vec{p},r}^\dagger a_{\vec{p},r} + \text{const.} \quad (16)$$

(b) Show that $\psi^\dagger \psi$ is the $\mu = 0$ component of the Noether current corresponding to the electron-number symmetry (phase rotation of ψ) in (13). [remark: Thus, $H' = H - \epsilon_F N_e$, where N_e is the electron number. ϵ_F is regarded as the chemical potential (Fermi energy).]

(c) Define $b_{\vec{p},r} \equiv a_{-\vec{p},r}^\dagger$ for $|\vec{p}|$ below the Fermi momentum p_F . Rewrite H' in terms of $a_{\vec{p},r}$ with $|\vec{p}| \geq p_F$ and $b_{\vec{p},r}$ with $|\vec{p}| < p_F$, and show that the vacuum state (lowest-energy state) $|0\rangle'$ of this Hamiltonian H' is the one annihilated by all of $a_{\vec{p},r}$ with $|\vec{p}| \geq p_F$ and $b_{\vec{p},r}$ with $|\vec{p}| < p_F$. It will be easy to see that a hole excitation state, $b_{\vec{p}}^\dagger |0\rangle'$, is where a state with $-\vec{p}$ below the Fermi surface remains unfilled.

(d) (not a homework problem) Note that this system can be described by a Lagrangian

$$\mathcal{L}' = \psi^\dagger \left[i\partial_t + \frac{1}{2m} \vec{\partial} \cdot \vec{\partial} + \epsilon_F \right] \psi. \quad (17)$$

4. Time-ordered Exponential for a System with Explicit Time Dependence [B]

In the interaction picture, the difference between the time evolution due to the full Hamiltonian H and the evolution due to the free-part Hamiltonian H_0 is characterized by an operator

$$U(t_1, t_2; t_*) \equiv e^{iH_0(t_1-t_*)} e^{-iH(t_1-t_2)} e^{-iH_0(t_2-t_*)}. \quad (18)$$

This expression is appropriate, however, only when H and H_0 do not have explicit time dependence. But, one may be interested in studying a quantum field theory system under time-varying external field (expanding universe also works as a time-varying external field).

- (a) Find an expression for $U(t_1, t_2; t_*)$ that is appropriate even when H and H_0 have explicit time dependence.
- (b) Show that such $U(t_1, t_2; t_*)$ is still given by a time-ordered exponential of $V_I(t)$. (The definition of $V_I(t)$ should also be stated clearly.)

5. Decay Rate, Cross Section and S-matrix [C]

If you have not learned how the decay rate of a particle and cross section of two particles are related to S-matrix, take your time to do so by reading a textbook on QFT. ... and then, show that the expression for decay rate is always of mass-dimension +1, and the one for cross section of two particles always of mass-dimension -2, no matter how many particles are contained in the final states.

6. Quantum Mechanics and Quantum Field Theory [B]

When Hamiltonian of a system contains explicit space coordinates, that is, when space-translation symmetry is violated, one particle in-coming plane wave state no longer remains as it is in time evolution.

$$|n, \vec{p}\rangle^{\text{in}} \neq |n, \vec{p}\rangle^{\text{out}}. \quad (19)$$

This is the case in a quantum field theory given by

$$\mathcal{L} = \psi^\dagger [i\partial_t - \mathcal{H}(\vec{x})] \psi = \psi^\dagger \left[i\partial_t + \frac{1}{2m} \vec{\partial} \cdot \vec{\partial} - U(r) \right] \psi. \quad (20)$$

The potential scattering calculated by using Quantum Mechanics with the Hamiltonian \mathcal{H} is the same as calculation of S-matrix from one particle state to another one particle state—if this statement is obvious, this problem is not for you. If this is not obvious to you, then explain why this relation holds, and show how the cross section of this potential scattering is given in terms of the S-matrix.

7. spectral representation and in/out state normalization

—→ homework of the next week