Theory of Elementary Particles

homework II (April 16)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like “II-1, II-3, IV-2”).**
- Pick up any problems that are suitable for your study. **You are not expected to work on all of them!**
- Format: Reports do not have to be written neatly; hand-writing is perfectly O.K. Do not waste your time!
- Keep your own copy, if you need one. Reports will not be returned.

1. **Follow-up** [A]

   Fill non-trivial gaps in derivations, calculations etc. during the lecture. If you encounter a gap that cannot be filled, state clearly what is yet to be proved or understood.

2. **In/out State Normalization and Källen–Lehmann Spectral Representation** [B]

   A coefficient Z appearing in the Källen–Lehmann spectral representation,[1]

$$\int d^4(y-x)\ \langle\Omega|T\left\{\phi(y)\phi(x)\right\}|\Omega\rangle e^{ik\cdot(y-x)} = \frac{i\,Z}{k^2-m^2+i\epsilon} + (\text{other singularities}), \qquad (1)$$

   also sets the normalization of in/out-states,

$$|\vec{p}\rangle^{\text{in}} = \frac{1}{\sqrt{Z}\langle\Omega|0\rangle_{T_-}} e^{-iH(t_*-T_-)} e^{iH_0(t_*-T_-)}\ a^\dagger_{\vec{p}}|0\rangle\ \sqrt{2E_{\vec{p}}}, \qquad (2)$$

$$|\vec{p}\rangle^{\text{out}} = \frac{1}{\sqrt{Z}\langle\Omega|0\rangle_{T_+}} e^{-iH(t_*-T_+)} e^{iH_0(t_*-T_+)}\ a^\dagger_{\vec{p}}|0\rangle\ \sqrt{2E_{\vec{p}}}. \qquad (3)$$

   In the following, let us see why this is the case.

   (a) First, show i) that an integral

$$\int_{-\infty}^{T_-} dx^0 \int d^3x\ \phi_I^*(x)|0\rangle\ e^{-iq\cdot x} \qquad (4)$$

   converges if q^0 is (real positive)$\times(1+i\epsilon)$, and ii) that a one-particle state in the free theory

$$|\vec{p}\rangle^{\text{free}} = a^\dagger_{\vec{p}}|0\rangle\sqrt{2E_{\vec{p}}} \qquad (5)$$

   is proportional to the residue of a pole $i/(q^0 - E_{\vec{q}})$ in the complex q^0 plane. Under an understanding that q^0 is always chosen to be real positive (plus a little positive imaginary part), one can also say that this is a residue of a pole at $i/(q^2-m^2)$.

[1] To learn more, see [PS] section 7.2 or [W-I] Chap. 10.

(b) Second, by using the result above, show that the inner product of

$$\left[\frac{1}{\langle\Omega|0\rangle_{T_+}} e^{-iH(t_*-T_+)} e^{iH_0(t_*-T_+)}\ a^\dagger_{\vec{p}}|0\rangle\ \sqrt{2E_{\vec{p}}}\right]^\dagger = \left[\frac{\sqrt{2E_{\vec{p}}}}{\langle 0|\Omega\rangle_{T_+}}\ \langle 0|a_{\vec{q}}\ e^{iH_0(T_+-t_*)} e^{-iH(T_+-t_*)}\right]$$

and

$$\left[\frac{1}{\langle\Omega|0\rangle_{T_-}} e^{-iH(t_*-T_-)} e^{iH_0(t_*-T_-)}\ a^\dagger_{\vec{q}}|0\rangle\ \sqrt{2E_{\vec{q}}}\right]$$

is the same as the residue of

$$\int d^4y \int d^4x\ e^{ip\cdot y} e^{-iq\cdot x}\ \langle\Omega|T\left\{\phi(y)\phi^*(x)\right\}|\Omega\rangle \tag{6}$$

at the singularity

$$\frac{i}{(p^2-m^2)}\frac{i}{(q^2-m^2)}. \tag{7}$$

As before, assume that p^0 and q^0 are both real and positive with an infinitesimally small positive imaginary part. It is O.K. to assume for now that interactions switch off before T_- and after T_+.

In order to show this, one can use

$$\langle 0(T_+)|\Omega(T_+)\rangle\langle\Omega(T_-)|0(T_-)\rangle = \langle 0|T\left\{\exp\left(-i\int dt' V_I(t')\right)\right\}|0\rangle, \tag{8}$$

which is equivalent to $\langle\Omega|\Omega\rangle = 1$.

(c) Finally, show that the resiude of Eq. (6) at (7) is

$$(2\pi)^3\delta^3(\vec{p}-\vec{q})(2E_{\vec{p}})\ Z, \tag{9}$$

by using Eq. (1). This completes the proof of the normalization (2, 3).

(d) (not a report problem) Normalization of multi-particle in-states / out-states is set as follows. Firsct think of a theory where particles in a group 1 interact withing the group, and those in another group 2 also do so within group 2, but not with particles in the other group. Then the in-state witn one particle (spiecies n_1) in the group 1 and one particle (spiecies n_2) in the group 2 should be set as follows:

$$|\{n_1,\vec{p}_1;n_2,\vec{p}_2\}\rangle^{\text{in}} = \frac{1}{\sqrt{Z_1}\sqrt{Z_2}\left[\langle\Omega_1|\otimes\langle\Omega_2|0\rangle\right]_{T_-}} e^{-iH(t_*-T_-)} e^{iH_0(t*-T_-)}|\{n_1,\vec{p}_1;n_2,\vec{p}_2\}\rangle^{\text{free}}. \tag{10}$$

In theories where all the particles interact, therefore,

$$|\{n_i,\vec{p}_i\}_{i\in I}\rangle^{\text{in}} = \frac{1}{\langle\Omega|0\rangle_{T_-}\prod_{i\in I}\sqrt{Z_i}} e^{-iH(t_*-T_-)} e^{iH_0(t*-T_-)}|\{n_i,\vec{p}_i\}_{i\in I}\rangle^{\text{free}}. \tag{11}$$

3. **Time Evolution and Retarded Propagator** [B]

When a Hamiltonian contains explicit time translational symmetry, vacuum state $|\Omega\rangle$ does not remain to be the lowest energy state. This means that

$$|\Omega\rangle^{\rm in} \neq |\Omega\rangle^{\rm out}. \tag{12}$$

We encounter such a situation, for example, when we work on a condensed matter system with time-varying external field, or on field fluctuations in expanding universe.

(a) Typical questions one might wish to ask in such a system will be time evolution of observables in a state that started out as a vacuum at an earlier time. That is to study t-dependence of

$$^{\rm in}\langle\Omega|\mathcal{O}(\vec{x},t)|\Omega\rangle^{\rm in}. \tag{13}$$

Find an equivalent expression that is given in terms of the interaction picture operator $\mathcal{O}_I(\vec{x},t)$ and $U(t_1,t_2;t_*)$ with appropriately chosen (t_1,t_2,t_*).

(b) If this time evolution problem is to be studied in perturbation theory, then the expression obtained above should be expanded in a power series of $V_I(t')$. Write down the first few terms of this expansion, and confirm that the expression only involves commutators of $V_I(t')$'s and $\mathcal{O}_I(\vec{x},t)$ within $\langle 0|$ and $|0\rangle$.

(c) Show that the propagator useful for this problem,

$$[\phi(\vec{x},t),\phi^*(\vec{y},t')]\times\Theta(t-t'), \quad \text{is given by} \quad \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip^0(t-t')+i\vec{p}\cdot(\vec{x}-\vec{y})}}{p^2-m^2+ip^0\epsilon}. \tag{14}$$

This is the propagator in the retarded boundary condition; we should say that is a reasonable choice, when we want to solve a time evolution problem. Note also that both particle and anti-particle contribute to this propagator from t' to t.

4. **Non-relativistic Propagator in Feynman Boundary Condition** [B]

(a) Using a 2-component fermion field (for non-relativistic electron) in the interaction picture

$$\psi_I(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\sum_r a_{r,\vec{p}}\,\xi_r\,e^{i\vec{p}\cdot\vec{x}-iE_{\vec{p}}t} \tag{15}$$

for a free (bilinear) Hamiltonian

$$\mathcal{H}_0 = \psi^\dagger\left[-\frac{\vec{\partial}\cdot\vec{\partial}}{2m}\right]\psi, \tag{16}$$

calculate the propagator (Green function)

$$\int d^3(\vec{x}-\vec{y})\,d(t-t')\,e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}e^{i\omega(t-t')}\,\langle 0|T\left\{\psi_I(\vec{x},t)\psi_I^\dagger(\vec{y},t')\right\}|0\rangle. \tag{17}$$

(b) Explain the relation between this propagator and

$$\frac{i\,[\not{p}+m]}{p^2-m^2+i\epsilon} \tag{18}$$

for 4-component spinor fields. [cf. homework problem I-2]

5. **Tree Level Calculation I, Forward–Backward Asymmetry** [B]

Scattering amplitude $\mathcal{M}$ is defined by

$$S^c_{\{m,p\};\{n,q\}} = (2\pi)^4\delta^4(\sum_i p_i - \sum_j q_j)\; i\mathcal{M}, \tag{19}$$

by factoring out the 4-momentum conservation, which always exists for any scattering process in Lorentz invariant theories. Here, S^c is the connected part of the S-matrix (cluster decomposition). $\mathcal{M}$ is also called (invariant) matrix element.

(a) Compute the invariant matrix elements for the following s-channel scattering processes. Assume that all the particles are massless.

i. $|\mathcal{M}|^2$ for $[\phi+\phi^c \to \gamma^* \to \phi' + \phi'^c]$ in scalar QED; ϕ and ϕ^c, ϕ' and ϕ'^c are in charge conjugation, and ϕ and ϕ' are different scalar spiecies.

ii. $\overline{|\mathcal{M}|^2}$ for $[f+\bar{f} \to \gamma^* \to \phi+\phi^c]$. Take an average in the initial state spin configuration.

iii. $\overline{\sum|\mathcal{M}|^2}$ for $[f+\bar{f}\to\gamma^*\to f'+\bar{f}']$, like $e^-+e^+\to\mu^-+\mu^+$ scattering. Sum over the final state spin configuration, and take an average over the initial state spin configuration.

(b) (If you have enough time) In the $[f+\bar{f}\to\gamma^*\to f'+\bar{f}']$ scattering in the s-channel, but in a theory where the coupling is

$$\mathcal{L}_{\rm int} = -g_L\overline{\Psi}\gamma^\mu A_\mu\left(\frac{1-\gamma_5}{2}\right)\Psi - g_R\overline{\Psi}\gamma^\mu A_\mu\left(\frac{1+\gamma_5}{2}\right)\Psi, \tag{20}$$

compute i) $|\mathcal{M}(f_L+\bar{f}_R\to f'_L+\bar{f}'_R)|^2$ and ii) $|\mathcal{M}(f_L+\bar{f}_R\to f'_R+\bar{f}'_L)|^2$. Here, $P_L=(1-\gamma_5)/2$ and $P_R=(1+\gamma_5)/2$ only keep left-handed and right-handed spinor components, respectively.

(c) Draw a sketchy graph of those $|\mathcal{M}|^2$ (with arbitrary normalization)[2] as a function of $\cos\theta\in[-1,+1]$. For 2-body to 2-body scattering of massless particles, t/s =

[2] The purpose of this problem so far is for you to get experienced in manipulation of γ matrices, as much as in knowing the results. So, the answers are presented here.

$$\text{a}-\text{i)}\quad |\mathcal{M}|^2 = e^4\frac{(t-u)^2}{s^2},\qquad \text{ii)}\quad \overline{|\mathcal{M}|^2} = e^4\frac{s^2-(t-u)^2}{s^2},\qquad \text{iii)}\quad \overline{\sum|\mathcal{M}|^2} = 2e^4\left(\frac{u^2}{s^2}+\frac{t^2}{s^2}\right), \tag{21}$$

$$\text{b}-\text{i)}\quad |\mathcal{M}|^2 = 4(g_Lg'_L)^2\left(\frac{u^2}{s^2}\right),\qquad \text{ii)}\quad |\mathcal{M}|^2 = 4(g_Lg'_R)^2\left(\frac{t^2}{s^2}\right). \tag{22}$$

$-\sin^2(\theta/2)$ and $u/s = -\cos^2(\theta/2)$, where θ is the scattering angle in the center-of-mass frame, so that $(s+t+u) = 0$. You will see that $(u/t)^2$ is a forward scattering component, and $(t/s)^2$ a backward scattering one. So, the copmutation in b) that the forward scattering component $(u/s)^2$ and backward scattering component $(t/s)^2$ have different coefficients in $\overline{\sum |\mathcal{M}|^2}$, that is, there is forward backward asymmetry, if $g_L \neq g_R$ and $g'_L \neq g'_R$.

6. **Topology** [C]

If you have not studied algebraic topology at all yet, it is not a very bad idea to take a little time to do so. Key words are simplicial complex, homology group, and Euler number. .... and then,

(a) Confirm that the Euler numbers χ of tetrahedron, hexahedron and octahedron are all the same.

$$\chi = \#\,[\text{faces}] - \#\,[\text{edges}] + \#\,[\text{vertices}] = \sum_i (-)^i \#\,[\ i\text{-dim. objects}\ ] \tag{23}$$

(b) Compute homology groups of S^2 (surface of a balloon) and T^2 (surface of a doughnut). The Euler number of polyhedra above is the same as $\chi = \sum_i (-)^i b_i$ of S^2.