Theory of Elementary Particles

homework III (April 23)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**
- Pick up any problems that are suitable for your study. **You are not expected to work on all of them!**
- Format: Reports do not have to be written neatly; hand-writing is perfectly O.K. Do not waste your time!
- Keep your own copy, if you need one. Reports will not be returned.

1. **Follow-up** [A]

   Fill non-trivial gaps in derivations, calculations etc. during the lecture. If you encounter a gap that cannot be filled, state clearly what is yet to be proved or understood.

2. **Complex Phase of Scattering Amplitudes** [B]

   Show, by counting the number of $(\pm i)$'s appearing in propagators, interaction verticies and momentum loop integrals (after Wick rotation), that those complex phases do not contribute to the complex phase of scattering amplitudes $\mathcal{M}$.

   Note that an extra (i) is in the relation between the connected part of the S-matrix and the scattering amplitude $\mathcal{M}$:

$$S^c = (2\pi)^4 \delta^4(\sum_i p_i)\; i\mathcal{M}. \tag{1}$$

3. **1-Loop Calculation I, Pauli–Villars Regularization, Unitarity** [C]

   Let us consider a theory where a complex scalar field φ and a 4-component (Dirac) fermion Ψ has an interaction (called Yukawa interaction);

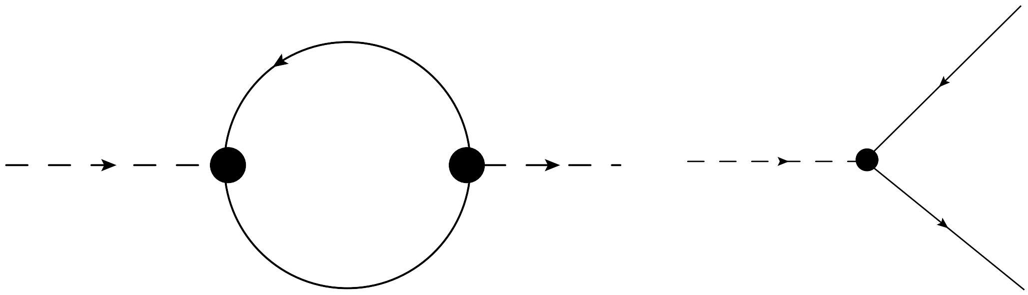
$$\mathcal{L}_{\rm kin} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - M_\varphi^2 |\varphi|^2 + \overline{\Psi}\,(i\gamma^\mu\partial_\mu - m)\,\Psi, \tag{2}$$

$$\mathcal{L}_{\rm int} = \lambda\varphi\overline{\Psi}\left(\frac{1-\gamma_5}{2}\right)\Psi + \lambda^*\varphi^*\overline{\Psi}\left(\frac{1+\gamma_5}{2}\right)\Psi. \tag{3}$$

   (a) Compute the 1-loop contribution to the scalar self-energy $-i\Sigma(p^2, m^2) = i\mathcal{M}(p^2, m^2)$, that is, Figure 1 (a), and show that it is

$$\mathcal{M} = \frac{2|\lambda|^2}{16\pi^2}\int_0^1 dx \int_0^\infty dK_E\, \frac{K_E(K_E + x(1-x)p^2)}{[K_E + m^2 - x(1-x)p^2]^2}; \tag{4}$$

   here, p^μ is the momentum of the scalar field coming from the left, and K_E corresopnds to the invariant momentum square $(k'\cdot k')$ in the Euclidean signature of shifted momentum



(a) (b)

Figure 1: Scalar self-energy 1-loop diagram (a) and scalar decay diagram (b).

k'. [Did you remember to include the extra (-1) factor for a fermion loop?] Confirm that this integral is approximately $\propto \int dK_E$ (unlike $\int dK_E K_E^{-1}$ or $\int dK_E 1/K_E^2$) for $K_E \gg m^2, |p^2|$.

(b) (**momentum cut-off regularization**) When the divergent integral $\mathcal{M}$ above is made well-defined (finite) by replacing the integral over $K_E \in [0, \infty]$ with a finite range integral over $K_E \in [0, \Lambda_0^2]$, the 1-loop scalar self-energy is denoted by $-i\Sigma^{\text{mom. cutoff}} = i\mathcal{M}^{\text{mom. cutoff}}$. Determine $\mathcal{M}^{\text{mom. cutoff}}$ by carring out the integration.

(c) (**Pauli–Villars regularization**) As an alternative to the momentum cut-off regularization, one can make the 1-loop divergent integral $\mathcal{M}$ well-defined, by introducing other spiecies of "fermions" Ψ_j $(j = 1, 2\cdots)$ that have exactly the same interaction with φ as $\Psi_0 := \Psi$. Those "fermions" are assumed to have mass M_j and signature of the 1-loop diagram (+ for ordinary bosons and − for ordinary fermions) that are either the same $(\gamma_j = +1)$ as or opposite $(\gamma_j = -1)$ from that of Ψ_0 for each j. This regularization is called Pauli–Villars regularization. To see how this work, let us first consider introducing just $\Psi_{j=1}$ whose mass is M_1 and the signature opposite $(\gamma_1 = -1)$. Show that the K_E integral of

$$\mathcal{M}(p^2, m^2) - \mathcal{M}(p^2, M_1^2) = \sum_{j=0}^{1} \gamma_j \mathcal{M}(p^2, M_j^2) \tag{5}$$

is still approximately $\propto dK_E$ for $m^2, |p^2| \ll K_E \ll M_1^2$, but the integral becomes $\propto dK_E M_1^2/K_E$ approximately. This means that the Pauli–Villars regularization cannot render the divergent 1-loop integral $\mathcal{M}$ finite, if we are to introduce only one spiecies of "fermion" $\Psi_{j=1}$.

(d) This 1-loop integral for the scalar self-energy diagram can be made finite, by introducing three "fermions" $\Psi_{j=1,2,3}$. The signature of $\Psi_{j=1,2}$ are set to be opposite from that of the original fermion Ψ_0 (that is, $\gamma_{1,2} = -1$), and the signature of $\Psi_{j=3}$ to be the same as that of Ψ_0 (that is, $\gamma_3 = +1$). The 1-loop integral (including the contributions from these "fermions") become finite, if we take their masses, M_1, M_2, M_3, in such a way that the following relation is satisfied:

$$m^2 + M_3^2 = M_1^2 + M_2^2. \tag{6}$$

Compute $\mathcal{M}^{\text{P.V.}}(p^2, m^2; M_1^2, M_2^2, M_3^2)$

$$\lim_{\Lambda_0 \to \infty} \left[\sum_{j=0}^{3} \gamma_j \mathcal{M}^{\text{mom. cutoff}}(p^2, M_j^2) \right] = \lim_{\Lambda_0 \to \infty} \left[\mathcal{M}^{\text{mom. cut}}(p^2, m^2) - \mathcal{M}^{\text{mom. cut}}(p^2, M_1^2) - \cdots \right].$$

In this context of Pauli–Villars regularization, the momentum cutoff scale Λ_0 plays the role of preregulator.

(e) In the case of $4m^2 \leq p^2 \ll M_{j=1,2,3}^2$, the logarithm appearing in $\mathcal{M}^{\text{mom. cutoff}}$ and $\mathcal{M}^{\text{P.V.}}$ means that a branch cut has to be introduced along the real positive axis of the p^2 complex plane. Show that

$$\frac{1}{i} \left[\mathcal{M}(p^2 + i\epsilon, m^2) - \mathcal{M}(p^2 - i\epsilon, m^2) \right] = \frac{2\pi |\lambda|^2}{16\pi^2} \sqrt{\frac{p^2 - 4m^2}{p^2}} (p^2 - 2m^2). \tag{7}$$

Note that this result does not depend on the choice of regularization schemes.

(f) Compute the decay rate of φ (Feynman diagram Figure 1), $\Gamma(\varphi \to \Psi + \overline{\Psi})$, and confirm that $(2M_\varphi) \times \Gamma$ is the same as (7). [This is one of consequesnces of the optical theorem.] Here, we assume that $M_\varphi \geq 2m$, so that the scalar field can decay into the fermion pair.

(g) Because of this branch cut, we need to be a little more careful in phrasing how to compute the scalar self-energy 1-loop diagram. We define, for $p^2 > 4m^2$, the scalar self-energy $\Sigma(p^2, m^2)$ to be $-\mathcal{M}(p^2, m^2)$ for p^2 in the upper half complex plane, and the analytically continued one across the branch cut for p^2 in the lower half plane. Show that the propagator with 1-loop 1PI correction,

$$\frac{i}{p^2 - M_\varphi^2 - \Sigma(p^2, m^2) + i\epsilon} \tag{8}$$

has a pole at

$$p^0 \simeq M_\varphi - \frac{1}{2M_\varphi} \text{Re}\mathcal{M}(M_\varphi^2, m^2) - i\frac{\Gamma}{2} \tag{9}$$

for the $\vec{p} = \vec{0}$ case for simplicity. [This means that the propagator in the spacetime picture exhibits the time dependence $e^{-iM_\varphi t} \times e^{-\Gamma t/2}$. After taking its absolute value square of this quantum mechanical amplitude, we obtain the $e^{-\Gamma t}$ dependence of an unstable particle.]

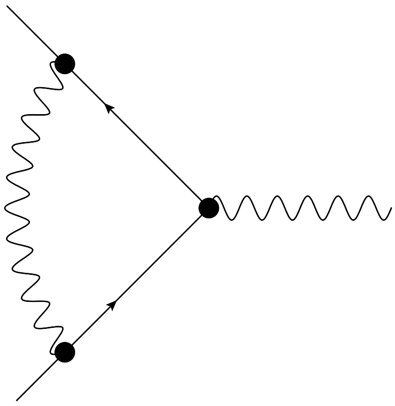


Figure 2: Fermion-photon vertex 1-loop correction.

4. **1-Loop Calculation II: Vertex Correction** [C]

   Compute the 1-loop correction (Figure 2), which should be added to the tree level contribution $ie\gamma^\mu$. Use higher covariant derivative regularization, or alternatively, follow the calculation of Peskin–Schröder section 6.3 (p.189–p.194).