Theory of Elementary Particles

homework IV (May 07)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**
- Pick up any problems that are suitable for your study. **You are not expected to work on all of them!**
- Format: Reports do not have to be written neatly; hand-writing is perfectly O.K. Do not waste your time!
- Keep your own copy, if you need one. Reports will not be returned.

1. **Follow-up** [A]

   Fill non-trivial gaps in derivations, calculations etc. during the lecture. If you encounter a gap that cannot be filled, state clearly what is yet to be proved or understood.

2. **Sorting out Order by Order** [B]

   At 1-loop level in QED (without field or coupling renormalization), electron self-energy is given by

$$\begin{aligned} -i\Sigma^{(1)}(p,M) &= \frac{-ie^2}{16\pi^2}\int_0^1 dx\,[-2(1-x)\not{p}+4M]\ln\left(\frac{(1-x)\Lambda^2+xM^2-x(1-x)p^2}{xM^2-x(1-x)p^2}\right), \\ &=: -i\left[A^{(1)}(p^2,M^2)\not{p}+B^{(1)}(p^2,M^2)\right] \end{aligned} \quad (1)$$

   where higher covariant derivative regularization is used for the (Feynman gauge) photon propagator. When the self-energy (sum of all the 1 particle irreducible diagrams) is $-i\Sigma(p,M) - i\left[A(p^2,M^2)\not{p}+B(p^2,M^2)\right]$, the electron propagator is

$$\frac{i}{\not{p}-M-\Sigma(p,M)} = \frac{i}{(1-A)\not{p}-(M+B)} = \frac{i\ [(1-A)\not{p}+(M+B)]}{(1-A)^2p^2-(M+B)^2}, \quad (2)$$

   and the physical mass-square (the pole in p^2 plane in the spectral representation) is defined as a solution of

$$(1-A(p^2,M^2))^2p^2-(M+B(p^2,M^2))^2=0. \quad (3)$$

   (a) Find the physical mass-square m^2 up to the level of $\mathcal{O}(e^2)$ (and ignore $\mathcal{O}(e^4)$ corrections); in doing so, use $A^{(1)}(p^2,M^2)$ and $B^{(1)}(p^2,M^2)$ for $A(p^2,M^2)$ and $B(p^2,M^2)$, and substitute $p^2=m^2=m^2_{(0)}+e^2m^2_{(1)}+\mathcal{O}(e^4)$. The result should be (if I am not wrong...)

$$m^2 = M^2+\left[2M^2A^{(1)}(M^2,M^2)+2MB^{(1)}(M^2,M^2)\right]+\mathcal{O}(e^4). \quad (4)$$

(b) Show that

$$M^2 - \left\{m^2 - \left[2m^2 A^{(1)}(m^2, m^2) + 2mB^{(1)}(m^2, m^2)\right]\right\} \quad (5)$$

is of order $\mathcal{O}(e^4)$. [note that the arguments of $A^{(1)}$ and $B^{(1)}(m^2, m^2)$ are m^2, not M^2. This means that

$$\begin{aligned} M - m &= -\left[mA^{(1)}(m^2, m^2) + B^{(1)}(m^2, m^2)\right] + \mathcal{O}(e^4), & (6) \\ &= -\left[MA^{(1)}(M^2, M^2) + B^{(1)}(M^2, M^2)\right] + \mathcal{O}(e^4). & (7) \end{aligned}$$

3. **Quantum Correction I: fermion propagator** [B]

(a) Show that the denominator of the propagator of the renormalized fermion field $[\not{p} - m - \Sigma]$ can be written as follows at 1-loop level in the on-shell renormalized perturbation theory:

$$\left[(1 + C(m^2)) - \Delta A(p^2, m^2)\right] \not{p} - \left[(1 + C(m^2))m + \Delta B(p^2, m^2)\right]. \quad (8)$$

(b) Confirm that $p^2 = m^2$ is the zero of the denominator (that is, the pole of the propagator), and the residue is 1 at the pole, as expected.

(c) Expand $\Delta A(p^2, m^2)$, $\Delta B(p^2, m^2)$ and $C(m^2)$ in m^2/Λ^2 and $|p^2|/\Lambda^2$, assuming that $\Lambda^2 \gg m^2, |p^2|$. Note that they remain finite in the large Λ^2 limit! Note also that the propagator is not simply the one of the tree level $i/[\not{p} - m]$ any more, but is quantum-corrected!!

4. **Summing up Geometric Series for Photon Propagator** [B]

Photon propagator is

$$\frac{-i}{q^2 + i\epsilon}\left[\eta_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}\right], \quad (9)$$

where ξ is a gauge parameter, and $\xi = 1$ [$\xi = 0$] corresponds to the Feynman gauge [Landau gauge], respectively. When the photon "self-energy" (sum of 1 particle irreducible diagrams: better known as vacuum polarization in this case; see homework V (or VI)) is given by

$$i\left(q^2 \eta_{\mu\nu} - q_\mu q_\nu\right)\Pi(q^2) \quad (10)$$

for some function $\Pi(q^2)$ of q^2, the quantum corrected photon propagator is of the form

$$\begin{aligned} &\frac{-i}{q^2 + i\epsilon}\left[\eta_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}\right] \\ + &\frac{-i}{q^2 + i\epsilon}\left[\eta_{\mu\kappa} + (\xi - 1)\frac{q_\mu q_\kappa}{q^2}\right] i(q^2\eta^{\kappa\lambda} - q^\kappa q^\lambda)\Pi(q^2)\frac{-i}{q^2 + i\epsilon}\left[\eta_{\lambda\nu} + (\xi - 1)\frac{q_\lambda q_\nu}{q^2}\right] + \cdots. \end{aligned}$$

Sum up this geometric series to show that it is the same as

$$\frac{-i}{(q^2 + i\epsilon)(1 - \Pi(q^2))}\left[\eta_{\mu\nu} - \frac{q_\mu q_n u}{q^2}\right] + \xi\frac{-iq_\mu q_\nu}{q^2 q^2}. \quad (11)$$

5. **1-Loop Calculation III: Photon Vacuum Polarization in Pauli–Villars** [C]

Photon 1-loop "self-energy" (or vacuum polarization) in QED

$$\int d^4x d^4y e^{iq'\cdot x} e^{-iq\cdot y} \langle 0| T\left\{ \left(ie\overline{\Psi}_I \gamma^\nu \Psi_I\right)(x) \left(ie\overline{\Psi}_I\gamma^\mu\Psi_I\right)(y)\right\}|0\rangle =: (2\pi)^4\delta^4(q'-q)\, i\mathcal{M}^{\mu\nu} \quad (12)$$

corresponds to the Feynman diagram in Figure 1 (a). Let us calculate this by using the Pauli–Villars regularization, and show that $i\mathcal{M}^{\mu\nu}$ is indeed of the form (10). To do this,

(a) show that, for a Dirac fermion with mass M,

$$i\mathcal{M}^{\mu\nu}(q^2, M^2) = (-4ie^2)\int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4} \frac{\left[\frac{1}{2}(k_E^2)\eta^{\mu\nu}\right] + \left[x(1-x)(q^2\eta^{\mu\nu} - 2q^\mu q^\nu)\right] + \left[M^2\eta^{\mu\nu}\right]}{\left[k_E^2 + M^2 - x(1-x)q^2\right]^2} \quad (13)$$

after Wick rotation. k_E^2 indicates that the 4-dim Euclidean metric is used in determining $k \cdot k$.

(b) Carry out angle and radial integration of 4-dimensional d^4k_E space; as a pre-regulator, introduce a cut-off in the range of integration, $k_E^2 \leq \Lambda_0^2$. Note that this integral in the momentum cut-off regularization $i\mathcal{M}^{\mu\nu}_{\text{mom. cut}}(q^2, M^2; \Lambda_0^2)$ does not have a form of (10) at all.

(c) The photon 1-loop "self-energy" (vacuum polarization) in the Pauli–Villars regularization is given by

$$i\mathcal{M}^{\mu\nu}_{\text{P.V}}(p^2, M^2) = \lim_{\Lambda_0^2\to\infty}\left[\sum_{j=0}^{3} \gamma_j \mathcal{M}^{\mu\nu}_{\text{mom. cut}}(q^2, M_j^2; \Lambda_0^2)\right], \quad (14)$$

just like in homework III-3. $\gamma_0 = +1$ and $M_0^2 = M_2$ by definition. We should take $\gamma_{1,2} = -1$ and $\gamma_3 = +1$, and $M_0^2 + M_3^2 = M_1^2 + M_2^2$ so that the integral remains finite, when the pre-regulator (momentum cutoff) is removed ($\Lambda_0^2 \to \infty$). Show that

$$i\mathcal{M}^{\mu\nu}_{\text{P.V.}}(p^2, M^2) = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\prod_j \left[M_j^2 - x(1-x)q^2\right]^{\gamma_j}\right). \quad (15)$$

(d) (not a problem) If we take the Pauli–Villars regulator masses M_1^2, M_2^2 and M_3^2 much larger than the original Dirac fermion mass M^2 and momentum flow q^2, the last logarithmic factor is approximately

$$\ln\left(\frac{M^2 - x(1-x)q^2}{\overline{M}^2}\right), \qquad \overline{M}^2 := M_1^2 M_2^2 / M_3^2. \quad (16)$$

In the Pauli–Villars regularization, $i\mathcal{M}^{\mu\nu}$ is in the form of (10) as expected from the gauge invariance of QED, and (at 1-loop,)

$$\Pi^{(1)}(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\frac{M^2 - x(1-x)q^2}{\overline{M}^2}\right). \quad (17)$$

6. **Mass Correction of Non-relativistic Fermion (Heavy Quark Effective Theory)** [C]

Consider a non-relativisitic fermion with a (-1) unit of electric charge (just like an electron).

(a) Show (understand) that the 1-particle irreducible diagram (Figure 1 (b)) for mass correction and wavefunction renormalization is given at the leading order in $1/M$ expansion by

$$i\mathcal{M} = -i\Sigma = \int \frac{d\omega}{(2\pi)} \int \frac{d^3\vec{k}}{(2\pi)^3} \; (ie)\frac{i}{\omega^0+\omega}(ie)\frac{-i}{\omega^2 - |\vec{k}|^2 + i\epsilon}, \tag{18}$$

where the spacial momentum $\vec{p}$ is set to $\vec{0}$, and $\omega^0 = p^0 - M$ is the energy flow of the external fermion field.

Note that only the $A_0 = \varphi$ component of photon contributes at this leve of fermion mass non-relativistic expansion ($1/M$ expansion). [c.f. homework I-2 and II-4] This mass / wavefunction correction from QED becomes that from non-Abelian gauge theories by replacing $(ie)^2$ with $(-ig\rho_R(t^a))(-ig\rho_R(t^a)) = -g^2 C_2(R)\mathbf{1}$. In the case of QCD and a quark, $C_2(R) = 4/3$.

(b) It is necessary to regularize this integral, or otherwise the self-energy correction is not well-defined. So, we use the higher covariant derivativee regularization for the photon propagator, which is to modify the photon propagator in the following way:

$$\frac{-i}{\omega^2 - \vec{k}^2 + i\epsilon} = \frac{-i}{k^2 + i\epsilon} \longrightarrow \frac{-i}{k^2 - k^4/\Lambda^2} \to \frac{i\Lambda^2}{(k^2+i\epsilon)(k^2 - \Lambda^2 + i\epsilon)}. \tag{19}$$

Here, we have in mind a situation characterized by $\omega^0 \ll \Lambda \ll M$. Do the Wick rotation, carry out $d^3\vec{k}$ integration and $d\omega$ integration. One will find that

$$-i\Sigma = \frac{i}{16\pi^2}\int_0^1 dx \frac{e^2\Lambda}{\sqrt{x}} \; \frac{2i}{\sqrt{1-A^2}} \ln\left[1 - \frac{i\sqrt{1-A^2}}{A}\right]\Bigg|_{A=\frac{\omega^0}{\sqrt{x}\Lambda}} \tag{20}$$

(c) Expand the self-energy $\Sigma(\omega^0;\Lambda)$ in ω^0/Λ, and keep only the terms that are in a non-negative power of the regulator energy scale Λ. Show, if the range of dx integration is limited to $[(\mu/\Lambda)]$, that

$$\Sigma(E;\Lambda,\mu) = -\left(\frac{e^2}{8\pi}\right)\Lambda + \left(\frac{e^2}{8\pi^2}\right)\; \omega^0 \; \ln\left(\frac{\Lambda^2}{\mu^2}\right). \tag{21}$$

This corresponds to the decomposision of the fermion self-energy $\Sigma(p^\mu;\Lambda) = B + A\not{p}$. The mass correction is linearly divergent in the regulator energy scale $\Lambda \ll M$, while the wavefunction renormalization is logarithmically divergent.

7. **Magnetic and Electric Dipole Moment** [B]

From the QED Lagrangian

$$\mathcal{L}_{\rm QED} = \overline{\Psi}\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - M\right]\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{22}$$

one finds that there is a magnetic field–spin coupling in the Hamiltonian:

$$\Delta H = +\frac{e}{M}\vec{B}\cdot\vec{s}_e + \frac{e}{M}\vec{B}\cdot\vec{s}_{\bar{e}}; \qquad \vec{s}_e = \psi^\dagger\frac{\vec{\tau}}{2}\psi, \quad \vec{s}_{\bar{e}} = \psi^{c\dagger}\frac{\vec{\tau}}{2}\psi^c. \tag{23}$$

Here, 2-component spinor fields ψ and ψ^c correspond to $\psi^{(n)}$ and $\psi^{c(n)}$ with $n \geq 1$ in the homework problem I-2. See the homework problem I-2 for more information. It is conventional that the electron magnetic moment $\vec{m}$ is characterized by $\Delta H = -\vec{m}\cdot\vec{B}$, and its relation to the electron angular momentum by a g-factor as in $\vec{m} = -(e/2m_e)g\vec{j}$. Thus, the tree-level QED gives rise to the celebrated result $g = 2$.

(a) Consider a theory whose Lagrangian is given by

$$\mathcal{L}' = \mathcal{L}_{\rm QED} + \frac{ie}{8m_e}f_2\,\overline{\Psi}\,[\gamma^\mu,\gamma^\nu]\,\Psi\,F_{\mu\nu}, \tag{24}$$

where f_2 is a dimensionless parameter. Show that there is an extra term in the Hamiltonian

$$\Delta H = \frac{e}{m_e}f_2\vec{B}\cdot(\vec{s}_e + \vec{s}_{\bar{e}}). \tag{25}$$

This means that $g = 2 + 2f_2$ for the electron field. For this purpose, it is sufficient to use $\Psi^{(0)}$ and $\psi^{(0)}$ and the convention of gamma matrices in homework I-2. One can also use

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu), \quad F_{12} = F^{12} = -B^3, \quad F^{30} = -F_{30} = E^3. \tag{26}$$

(b) Consider next a theory whose Lagrangian has yet another term

$$\mathcal{L}'' = \mathcal{L}_{\rm QED} + \frac{e}{8m_e}g_2\,\overline{\Psi}\,[\gamma^\mu,\gamma^\nu]\,\gamma_5\Psi\,F_{\mu\nu}, \tag{27}$$

where[1] $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$, and g_2 is another dimensionless constant. Show that there is an extra term in the Hamiltonian of this theory:

$$\Delta H = -\frac{e}{m_e}g_2\,\vec{E}\cdot(\vec{s}_e + \vec{s}_{\bar{e}}). \tag{30}$$

This means that electron has an electric dipole moment $\vec{d} = +(e/m_e)g_2\vec{s}$.

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$$\gamma_5 = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \ \text{if} \ \gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \ \gamma^i = \begin{pmatrix} & \tau^i \\ -\tau^i & \end{pmatrix}, \tag{28}$$

$$\gamma_5 = \begin{pmatrix} -\mathbf{1} & \\ & \mathbf{1} \end{pmatrix}, \ \text{if} \ \gamma^0 = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \ \gamma^i = \begin{pmatrix} & \tau^i \\ -\tau^i & \end{pmatrix}, \tag{29}$$

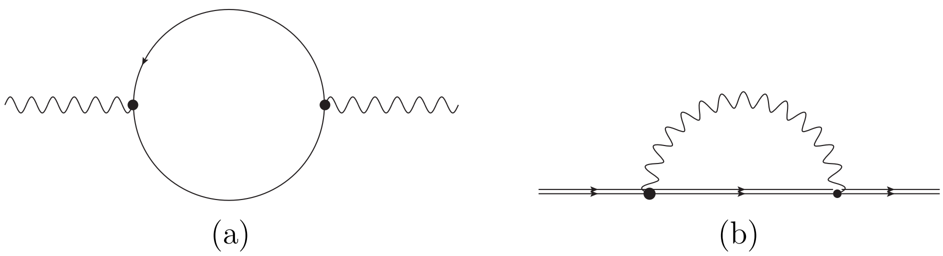

Figure 1: Self-energy graph of photon (a) and heavy fermion (b).