Theory of Elementary Particles

homework VI (May 21)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**
- Pick up any problems that are suitable for your study. **You are not expected to work on all of them!**
- Format: Reports do not have to be written neatly; hand-writing is perfectly O.K. Do not waste your time!
- Keep your own copy, if you need one. Reports will not be returned.

1. **Follow-up** [A]

   Fill non-trivial gaps in derivations, calculations etc. during the lecture. If you encounter a gap that cannot be filled, state clearly what is yet to be proved or understood.

2. **A Consequence of QED Ward Identity** [B]

   Wavefunction renormalization constant Z_2 of a Dirac fermion with a pole mass $p^2 = m^2$ in QED is given by

$$1 + \delta_{Z2} =: Z_2 = \left. \frac{(1-A)}{(1-A)^2 + 2(A-1)p^2 \frac{\partial A}{\partial p^2} - 2(M+B)\frac{\partial B}{\partial p^2}} \right|_{p^2=m^2}, \tag{1}$$

where $A(p^2, M^2)$ and $B(p^2, M^2)$ characterize the fermion self-energy

$$-i\Sigma(p, M) := -i \left[A(p^2, M^2)\not{p} + B(p^2, M^2) \right]. \tag{2}$$

At 1-loop ($\mathcal{O}(e^2)$) level, the fermion self-energy (Figure 1 (a)) is given by

$$\begin{aligned} -i\Sigma^{(1)}(p, M) &= \frac{-ie^2}{16\pi^2} \int_0^1 dx \left[-2(1-x)\not{p} + 4M \right] \ln \left(\frac{(1-x)\Lambda^2 + xM^2 - x(1-x)p^2}{xM^2 - x(1-x)p^2} \right), \\ &=: -i \left[A^{(1)}(p^2, M^2)\, \not{p} + B^{(1)}(p^2, M^2) \right] \end{aligned} \tag{3}$$

in the higher covariant derivative regularization for the photon propagator, and the wave-function renormalization constant becomes

$$\delta^{(1)}_{Z2} = \left. \left[A^{(1)} + 2M^2 \frac{\partial A^{(1)}}{\partial p^2} + 2M \frac{\partial B^{(1)}}{\partial p^2} \right] \right|_{p^2=M^2} \tag{4}$$

at this $\mathcal{O}(e^2)$ level.

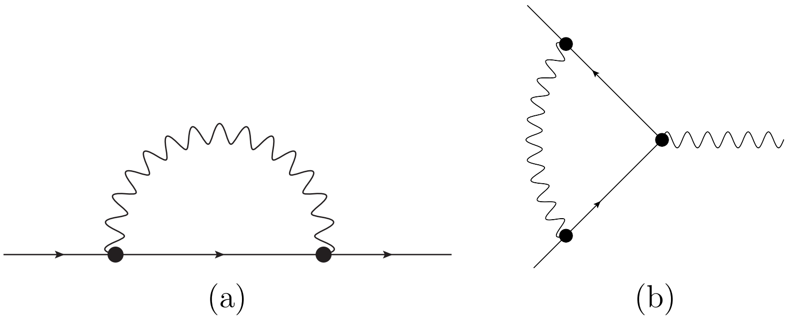

Figure 1: Fermion self-energy and fermion-photon vertex corrections at 1-loop.

On the other hand, fermion–fermion–photon vetex $ie\Gamma^\mu$—including quantum corrections—is known to be cast into the form

$$ie\Gamma^\mu = ie\left[V_1\gamma^\mu - \frac{V_2}{4m}\left[\gamma^\mu, \gamma^\nu\right] q_\nu\right] + (***) \times (\not{p} - m) + \left(\not{p}' - m\right) \times (***)\,; \qquad (5)$$

here, we assume that the momentum of the fermion coming from below in Figure 1 (b) is p, that of the fermion going out to the above p', and the photon comes from the right with momentum $q = p' - p$. As a result of calculation in homework III-4, one will find, in higher covariant derivative regularization, that

$$\begin{aligned} V_1^{(1)} &= \frac{e^2}{8\pi^2}\int dxdy \left\{ \ln\left(\frac{(1-x-y)\Lambda^2 + (x+y)^2M^2 - xyq^2}{(x+y)^2M^2 - xyq^2}\right)\right. \qquad (6) \\ &\quad + \left[\left\{1 - 4(1-x-y) + (1-x-y)^2\right\}M^2 + (1-x)(1-y)q^2\right] \times \\ &\quad \left.\left[\frac{1}{(x+y)^2M^2 - xyq^2} - \frac{1}{(1-x-y)\Lambda^2 + (x+y)^2M^2 - xyq^2}\right]\right\}, \\ V_2^{(1)} &= \frac{e^2}{16\pi^2}\int dxdy\, (1-x-y)(x+y)4M^2 \times \qquad (7) \\ &\quad \left[\frac{1}{(x+y)^2M^2 - xyq^2} - \frac{1}{(1-x-y)\Lambda^2 + (x+y)^2M^2 - xyq^2}\right] \end{aligned}$$

Here, $dxdy$ integral should be carried out in a trianglular region determined by $0 \leq x, y \leq 1$, $x + y \leq 1$.

Just like the wavefunction renormalization constant Z_2 characterizes partial information of self-energy diagrams, a parameter $Z_1 := 1/V_1(q^2 = 0)$ is used to capture partial information of vertex corrections $ie\Gamma^\mu$. At 1-loop,

$$\delta_{Z1}^{(1)} = (Z_1 - 1)^{\text{1-loop}} = \left[\frac{1}{1 + V_1^{(1)}(q^2 = 0)} - 1\right]^{\text{1-loop}} = -V_1^{(1)}(q^2 = 0). \qquad (8)$$

Problem: It is known from Ward identity in QED that $Z_1 = Z_2$ at all order in perturbation theory. Verify this relation at 1-loop level. [that is, show that $\delta_{Z2}^{(1)} = \delta_{Z1}^{(1)}$.]

See [Peskin–Schröder] section 7.1, if necessary. It is also good to know that Mathematica is sometimes quite useful.

3. **Anomalous Magnetic Moment in Renormalized Perturbation Theory** [B]

   It will be easy to note that $V_2^{(1)}(q^2)$ in (7) remains finite, when we take a $\Lambda \to \infty$ limit. As explained in the class, counter terms give rise only to the the $\propto ie\gamma^\mu$ component as in the first term of (5), not to the $\propto ie[\gamma^\mu, \gamma^\nu]q_\nu$ component at 1-loop level. Thus, $V_2^{(1)}(q^2)$ in (7) itself becomes the final 1-loop result in renormalized perturbation theory (after replacing M by the renormalized mass parameter m_e).

   (a) Confirm that the $ie[\gamma^\mu, \gamma^\nu]q_\nu$ component in the fermion-fermion-photon matrix element (amplitude) (5) is also obtained as a tree-level result, if there is an etxtra term in the Lagrangian,

   $$\Delta\mathcal{L} = +\frac{ie_r}{8m_e}\overline{\Psi}\left[\gamma^\mu, \gamma^\nu\right]\Psi\, V_2 F_{\mu\nu}, \tag{9}$$

   and ignore the 1-loop contribution we discussed above. Here, we take V_2 here to be the same as those in (5, 7).

   (b) Therefore, the 1-loop correction due to the $V_2^{(1)}$ term in (5) plays the same role as the additional term in the homework problem IV-7 eq. (24), with an identification $f_2 = V_2^{(1)}$. Because the homework problem IV-7 shows that the extra contribution to the anomalous magnetic moment is $\Delta g = 2f_2$, the 1-loop contribution from QED is $\Delta g \simeq 2V_2^{(1)}(q^2 = 0)$. Evaluate $V_2^{(1)}(q^2 = 0)$ to show that

   $$\Delta g^{(1)} = \frac{\alpha_e}{\pi}, \qquad \left(\text{where } \alpha_e = \frac{e_r^2}{(4\pi)}\right) \tag{10}$$

4. **Vacuum Polarization and Linear Response** [C]

   Let $J^\mu = \overline{\Psi}_e\gamma^\mu\Psi_e$ be the QED current of a Dirac fermion field corresponding to electron. We call $\Pi(q^2)$ in

   $$(ie)^2\int d^4x d^4y e^{+iq'\cdot x}e^{-iq\cdot y}\langle 0|T\{J^\mu(x)J^\nu(y)\}|0\rangle = (2\pi)^4\delta^4(q'-q)\times i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2) \tag{11}$$

   vacuum polarization. Why is that? Let us see why in the following.

   From the QED Lagrangian

   $$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu_{\text{EM}} + \overline{\Psi}_e\left[i\gamma^\mu\partial_\mu - m\right]\Psi_e, \tag{12}$$

follows the Maxwell equation[1]

$$\partial_\nu F^{\nu\mu} = j^\mu_{\rm EM}, \qquad\qquad j^\mu_{\rm EM} := -eJ^\mu_e = -e\overline{\Psi}_e\gamma^\mu\Psi_e. \tag{14}$$

Here, we adopt a convention $e > 0$, and Ψ_e is the 4-component Dirac spinor field for the electron.

(a) Using the convention of gamma matrices for the non-relativistic case,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \tau^i \\ -\tau^i & \end{pmatrix}, \tag{15}$$

and implementing the non-relativistic approximation

$$\Psi_e \to e^{-imt}\begin{pmatrix} \psi \\ \frac{\vec{\mathbf{p}}}{2m}\psi \end{pmatrix}, \tag{16}$$

confirm that

$$J^{\mu=0} \to \psi^\dagger\psi, \qquad J^{\mu=i} \to \psi^\dagger\frac{\vec{\mathbf{p}}}{m}\psi = \psi^\dagger\vec{\mathbf{v}}\psi, \tag{17}$$

approximately. Thus, the Noether current J^μ is precisely the the sourse term of the electromagnetism.

(b) Now, let us study the expectation value of this electromagnetic current under a classical background of electromagnetic field. Because the coupling between the photon field and the current can be written as

$$-A_\mu j^\mu_{\rm EM} = eA_\mu J^\mu = e\sqrt{Z_3}A^{(r)}_\mu J^\mu = e_r\frac{Z_1}{Z_2}A^{(r)}_\mu J^\mu = e_r A^{(r)}_\mu J^\mu, \tag{18}$$

it is the expectation value of $(-e_r J^\mu)$, the source term of the renormalized gauge field $A^{(r)}_\mu$, that we really want to study.

From the homework problem II-3 (b), the expectation value is given by

$$-e_r\langle J^\mu(x)\rangle \simeq (-e_r)\int d^4y\;\Theta(x^0-y^0)\;\langle 0|\left[J^\mu(x), ie_rJ^\nu(y)\right]|0\rangle A^{(r){\rm cl}}_\nu(y) \tag{19}$$

---

[1]Fundamental equations in the electromagnetism are written generally in this way:

$$\begin{cases} \textbf{div}\,\vec{\mathbf{E}} = \frac{a}{\epsilon_0}\rho, \\ \textbf{rot}\,\vec{\mathbf{B}} = \frac{\gamma}{c^2}\frac{\partial\vec{\mathbf{E}}}{\partial t} + \frac{a\mu_0}{\gamma}\vec{\mathbf{j}} \end{cases} \qquad \begin{cases} \textbf{div}\,\vec{\mathbf{B}} = 0 \\ \textbf{rot}\,\vec{\mathbf{E}} + \frac{1}{\gamma}\frac{\partial\vec{\mathbf{B}}}{\partial t} = \vec{\mathbf{0}} \end{cases} \qquad (\epsilon_0\mu_0) = \left(\frac{\gamma}{c}\right)^2, \qquad \varphi(\vec{\mathbf{x}}) = \frac{a}{4\pi\epsilon_0}\int\frac{d^3\vec{\mathbf{x}}'\rho(\vec{\mathbf{x}}')}{|\vec{\mathbf{x}}-\vec{\mathbf{x}}'|}. \tag{13}$$

In the MKSA system, $\gamma = 1$ and $a = 1$. In the cgs-esu system, $\gamma = 1$ and $(a/\epsilon_0) = 4\pi$, while the cgs-Gauss system sets $\gamma = c$ and $(a/\epsilon_0) = 4\pi$.
Here, however, we set $\gamma = c$ and $(a/\epsilon_0) = 1$. $j^\mu_{\rm EM} = (\rho, \vec{\mathbf{j}}/c)$, $A_\mu = (\varphi, -\vec{\mathbf{A}})$, and $F^{i0} = \mathbf{E}^i$, $F^{12} = -\mathbf{B}^3$.

at the leading order of the background gauge field $A^{(r)\mathrm{cl}}_\nu(y)$. Using a relation[2] between the retarded 2-point function and time-ordered 2-point function for two operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$,

$$G^{\mathrm{ret}}_{12}(q) := \int d^4(x-y)\, e^{iq\cdot(x-y)}\, \Theta(x^0-y^0)\langle\Omega|\left[\mathcal{O}_1(x),\mathcal{O}_2(y)\right]|\Omega\rangle, \tag{20}$$

$$G^{\mathrm{adv}}_{12}(q) := -\int d^4(x-y)\, e^{iq\cdot(x-y)}\, \Theta(y^0-x^0)\langle\Omega|\left[\mathcal{O}_1(x),\mathcal{O}_2(y)\right]|\Omega\rangle, \tag{21}$$

$$G^{T}_{12}(q) := \int d^4(x-y)\, e^{iq\cdot(x-y)}\, \langle\Omega|T\left\{\mathcal{O}_1(x)\mathcal{O}_2(y)\right\}|\Omega\rangle, \tag{22}$$

$$\mathrm{Im}G^{T}_{12}(q) = \mathrm{Im}G^{\mathrm{ret}}_{12}(q) = \mathrm{Im}G^{\mathrm{adv}}_{12}(q), \tag{23}$$

show that the current expectation value (which is supposed to be real valued) is given by

$$-e_r\langle J^\mu(x)\rangle \simeq -\int d^4y \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(x-y)}\, (q^2\eta^{\mu\nu}-q^\mu q^\nu)\mathrm{Re}\left[\Pi_{\mathrm{ren}}(q^2)\right]\, A^{(r)\mathrm{cl}}_\nu(y). \tag{24}$$

(c) Let us now focus on charge density, $\mu=0$, under a static uniform electric field in the (positive) (x^3) direction. The classical gauge field background can be chosen as

$$A^{(r)\mathrm{cl}}_{\nu=0}(y) = -E(y^3), \qquad A^{(r)\mathrm{cl}}_{\nu\neq 0}(y) = 0. \tag{25}$$

Simplify the expression (24) for this gauge field background.

(d) Let us assume that the q^2-dependence of $\Pi_{\mathrm{ren}}(q^2)$ is negligible compared with q-dependence of other factors (such as $e^{-iq\cdot(x-y)}$) in the integrand. Now, suppose (hypothetically) that the 1-loop correction Π_{ren} switches off at $y_+ < y^3$ and $y^3 < y_-$, and that it remains its vacuum value Π_{ren} only in the interval $y^3\in[y_-,y_+]$. Thus, we are replacing $\mathrm{Re}\left[\Pi_{\mathrm{ren}}(q^2)\right]$ by $\mathrm{Re}\left[\Pi_{\mathrm{ren}}\right]\varphi(y^3)$, where $\varphi(y^3)$ is a smooth function satisfying $\varphi(y^3)=0$ in $y^3 < y_- - \epsilon$ and in $y_+ + \epsilon < y^3$, while $\varphi(y^3)=1$ in $y_- + \epsilon < y^3 < y_+ - \epsilon$. Derive the following result under this set-up:

$$-e_r\left\langle J^{\mu=0}(x)\right\rangle \simeq E\mathrm{Re}\left[\Pi_{\mathrm{ren}}\right]\frac{\partial^2}{\partial(x^3)^2}\left(x^3\varphi(x^3)\right). \tag{26}$$

Integrating this distribution of induced charges over the "boundary (region) of the vacuum" $x^3\in[x_- - \epsilon, x_- + \epsilon]$ and $x^3\in[x_+ - \epsilon, x_+ + \epsilon]$, one can further see that the surface density of the induced charge at the "boundary of the vacuum" is $\pm E\Pi_{\mathrm{ren}}$ at the boundary at $x^3 = x_\mp$.

[2]See, for example, section 7 of A. Altland and B. Simons, "*Condensed Matter Field Theory.*"

(e) (not a problem) If we are to draw analogy with the electromagnetism, the result above means that the vacuum state is polarized, with a polarization vector given by

$$\vec{\mathbf{P}} = -\Pi_{\rm ren}\vec{\mathbf{E}}. \tag{27}$$

The induced charge is related to the polarization vector by $\mathbf{div}\vec{\mathbf{P}} = -\rho_{\rm ind}$. For this reason, it is quite appropriate to call $\Pi_{\rm ren}$ vacuum polarization.

Think of a condenser (used in electronics). When the surface density of electric charge is $\pm\rho$ at the $x^3 = x_\mp$ boundaries, the electric field wihin this condenser would be given by $E = \rho$ pointing to the positive x^3 direction, if there were no polarization in the vacuum. In reality, however, the electric field should become

$$E = \rho + \rho_{\rm ind} \simeq \rho\left(1 + \Pi^{(1)}_{\rm ren} + \mathcal{O}(e^4)\right) \simeq \frac{\rho}{1 - \Pi^{(1)}_{\rm ren} + \mathcal{O}(e^4)}. \tag{28}$$

This is, in effect, to replace[3] $(a/\epsilon_0) = 1$ by $1/(1 - \Pi_{\rm ren}(q^2))$.

For more negative $\Pi_{\rm ren}$, electric field is shielded more by the vacuum polarization, and the electric field becomes weaker. The homework problem IV-5 (d) shows (at 1-loop level) that $\mathrm{Re}[\Pi^{(1)}_{\rm ren}(q^2)]$ becomes more and more positive for more spacelike q^μ. (e.g. $q^2 \sim -|\vec{q}|^2$ with large $|\vec{\mathbf{q}}|$.) Thus, electric charges are shielded the most for smaller momentum $\vec{\mathbf{q}}$.

[3] By summing up the geometric series of 1PI corrections to the photon 2-point function, we easily obtain this $1/(1 - \Pi(q^2))$ modification to the propagator, although we cannot obtain the intuitive picture of "induced charge" from that simple calculation.