Theory of Elementary Particles

homework XI (July ??)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**
- Pick up any problems that are suitable for your study. **You are not expected to work on all of them!**
- Format: Reports do not have to be written neatly; hand-writing is perfectly O.K. Do not waste your time!
- Keep your own copy, if you need one. Reports will not be returned.

1. **Follow-up** [A]

   Fill non-trivial gaps in derivations, calculations etc. during the lecture. If you encounter a gap that cannot be filled, state clearly what is yet to be proved or understood.

2. **Phase Space of a 3-Particle Finate State, Dalitz Plot** [B]

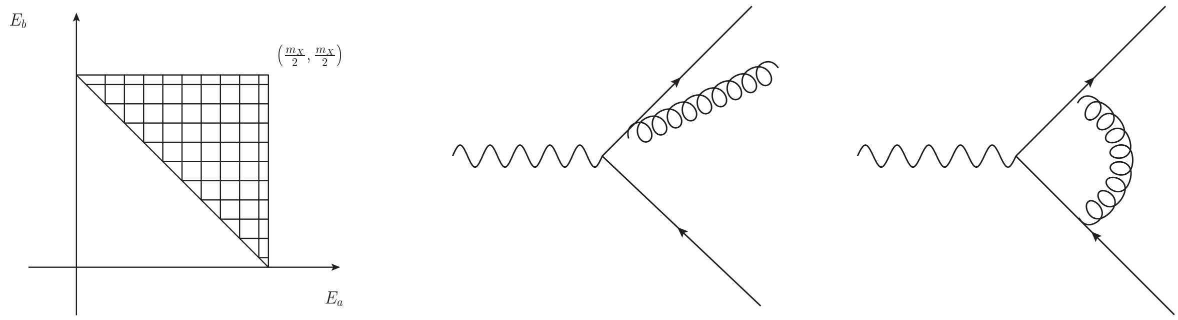
   Consider a process ending up with a 3-particle state, where the particles a, b and c are all different species. The phase space integral of this final state is

$$\int \frac{d^3\vec{p}_a}{(2\pi)^3}\frac{1}{2E_{\vec{p}_a}} \int \frac{d^3\vec{p}_b}{(2\pi)^3}\frac{1}{2E_{\vec{p}_b}} \int \frac{d^3\vec{p}_c}{(2\pi)^3}\frac{1}{2E_{\vec{p}_c}} (2\pi)^4\delta^4(p_{\rm in} - p_a - p_b - p_c)\,|\mathcal{M}|^2, \tag{1}$$

   where $\mathcal{M}$ is a Lorentz-invariant matrix element. If the initial state is a 1-particle state, decay rate of a particle X, $\Gamma(X \to a + b + c)$, is obtained by multiplying $1/(2m_X)$ to (1). If the initial state is a 2-particle state, cross section of two particles A and B, $\sigma(A+B \to a+b+c)$, is given by the integral (1) above multiplied by

$$\frac{1}{2E_A}\frac{1}{2E_B}\frac{1}{\Delta u_{AB}} = \frac{1}{4|E_A p_B^z - E_B p_A^z|} = \frac{1}{4\sqrt{(p_A\cdot p_B)^2 - m_A^2 m_B^2}} \simeq \frac{1}{2s_{AB}}. \tag{2}$$

   (a) Let us use the center-of-mass frame of the initial state. In the case of a scalar (i.e., spinless) particle decay, there is no special direction in the initial state, and hence the matrix element $\mathcal{M}$ has SO(3) symmetry. Thus, the phase space integral (1) can be reduced to an integral over 2 coordinates in such a situation; $3 \times 3 - 4 - 3 = 2$. Now, here is a problem. If all the three particles a, b and c in the final state are massless, and if we take the energy of two particles in the final state, $E_a = |\vec{p}_a|$ and $E_b = |\vec{p}_b|$, why is it that the integration region in the (E_a, E_b) plane is limited to the shaded region in Figure 1 (a)?



(a) (b1) and (b2)

Figure 1: 3-body decay phase space (a), and its Feynman diagram (b1). The soft and collinear divergence of this graph is cancelled by those in (b2).

(b) Where in this triangular region are the two particles b and c collinear? Where is the particle c soft? [This 3-body phase space can be used also for the process $e^- + e^+ \to \gamma^{(*)} \to q + \bar{q} + g$ Figure 1 (b1) (if the direction of the e^+e^- beam axis is ignored, and the spin of e^-e^+ averaged). This triangular space of the 3-body phase space integration looks quite similar to the space of Feynman parameters in the virtual vertex correction to the process $e^-e^+ \to q + \bar{q}$. Figure 1 (b2).]

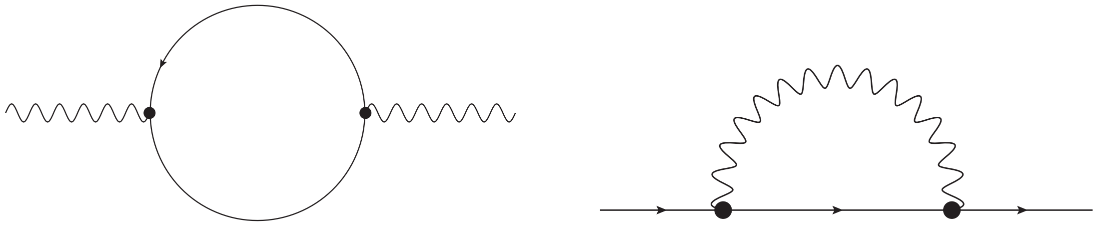
(c) In the case of a scalar particle X, the decay rate—$1/(2m_X)$ times eq. (1)—becomes[1]

$$d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{8m_X} dE_a dE_b |\mathcal{M}|^2. \tag{3}$$

Assuming that the matrix element $|\mathcal{M}|^2$ does not have any particular structure (such as poles) as a function of (E_a, E_b), and further assuming that $|\mathcal{M}|^2 \sim 1$, derive an estimate of the 3-body decay rate of this particle X. If we are to apply similarly crude argument for 2-body decay processes, which is to assume that $|\mathcal{M}|^2 \sim (m_X/2)^2$, what is the 2-body decay rate? [From this, one will see that 3-body decay rates of a given particle are generally $1/8\pi^2 \sim 10^{-2}$ times smaller than 2-body decay rates of the same particle, if there is no particular enhancement / suppression effects from the matrix elements.]

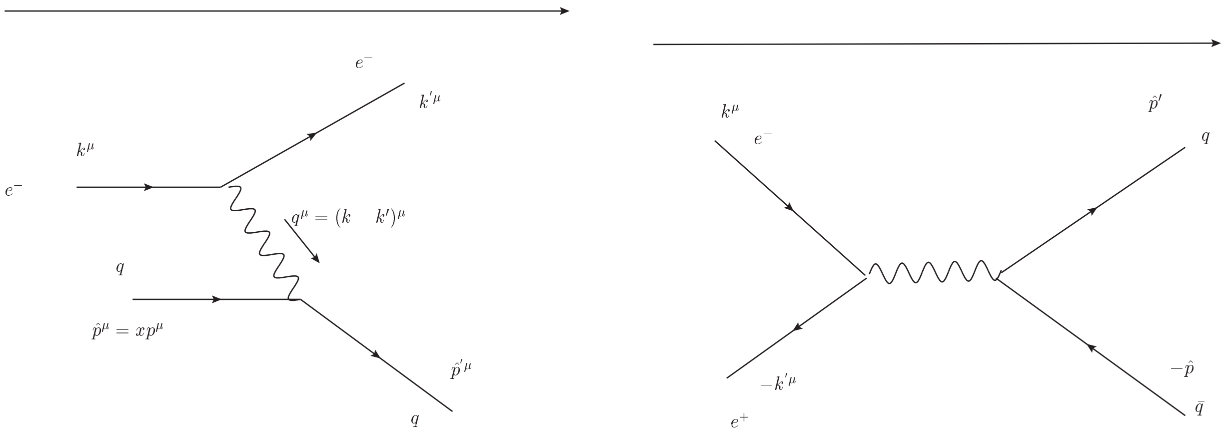
(d) (not a problem) In reality, the matrix element is not usually structureless. Especially in the case most of the decay process goes through $X \to a + a'$, $a' \to b + c$, then such decay events appear on a line of $E_a =$ const.. The dot-distribution of a 3-body decay of X on the (E_a, E_b) plane—called Dalitz plot—was introduced for the purpose of finding such structures in matrix elements from experimental data.

[1] If you have a plenty of time, you can also try to derive this from (1). Ref.: e.g., "Kinematics" review article in the "Review of Particle Physics" from Particle Data Group.



(a) (b)

Figure 2: Are the 2 graphs IR divergent?



(a) t channel (b) s channel

Figure 3: parton–lepton QED interaction at tree level: t-channel diagram in (a) and s-channel diagram in (b).

3. **IR sensitivity (pinch surface) and power counting** [B]

   In QED with a massless fermion, are the graphs Figure 2 (a) and (b) IR finite or divergent? If they are (or either one of them is) IR divergent, from which region of the space of Feynman parameter (and loop momenta) does the divergence arise? Here, assume that fermions and vector boson are massless.

4. **DIS 1 (tree): $2 \to 2$ t-channel Parton Scattering** [B]

   (a) Compute the spin-averaged (for initial states) spin-summed (for final states) Lorentz-invariant matrix element of $e^- + q \to e^- + q$ scattering (Figure 3 (a)). The result should be
   $$\overline{\sum |\mathcal{M}|^2} = 2(e^4 Q_q^2) \left(\frac{\hat{s}^2}{t^2} + \frac{\hat{u}^2}{t^2} \right), \tag{4}$$
   where $\hat{s}$, t and $\hat{u}$ are the Mandelstam variables of the $e^- + q \to e^- + q$ scattering. [Note that the matrix element of s-channel scattering (Fig. 3 (b)) $e^- + e^+ \to q + \bar{q}$ [homework

II-5] and the one we obtain above are identical after replacing $\hat{s} \leftrightarrow t$. This phenomenon is called crossing symmetry.]

(b) Rewrite the (spin-averaged/summed) matrix element by using the kinematical variables of DIS (deep inelastic scattering) such as Q^2, y, x and s, rather than referring to momenta of the quark parton $\hat{p}^\mu$ and $(\hat{p}+q)^\mu$.

5. **DIS 2 (tree): Tree-level OPE and its Evaluation** [B]

The twist-2 (parton) contribution to the DIS (deep inelastic scattering) structure functions is characterized best in the language of OPE (operator product expansion). The DIS cross section is given in terms of the following matrix element

$$iT^{\mu\nu} \equiv (iQ_q)^2 \int d^4y e^{iq\cdot(x-y)} \langle h(\vec{p})| T\left\{ \left(\overline{\Psi}_q \gamma^\mu \Psi_q\right)(x) \left(\overline{\Psi}_q \gamma^\nu \Psi_q\right)(y) \right\} |h(\vec{p})\rangle. \tag{5}$$

(a) Before evaluating the matrix element of operators in the hadron states $\langle h(\vec{p})|$ and $|h(\vec{p})\rangle$, let us first work on the operators inside. There are two separate series of twist-2 operators contributing at tree level to the OPE of the two QED currents. One is

$$\begin{aligned} &(iQ_q)^2 \Delta_{(q)} \left[\int d^4y e^{iq\cdot(x-y)} T\left\{ \left(\overline{\Psi}_q \gamma^\mu \Psi_q\right)(x) \left(\overline{\Psi}_q \gamma^\nu \Psi_q\right)(y) \right\} \right] \\ \sim\ & (-iQ_q^2) \left[\overline{\Psi}_q \frac{\gamma^\mu \left(\not{q} + \frac{i}{2} \overleftrightarrow{\not{D}} \right) \gamma^\nu}{\left(q + \frac{i}{2} \overleftrightarrow{D} \right)^2} \Psi_q \right] \left(\frac{x+y}{2} \right), \end{aligned} \tag{6}$$

and the other is

$$\begin{aligned} &(iQ_q)^2 \Delta_{(\bar{q})} \left[\int d^4y e^{iq\cdot(x-y)} T\left\{ \left(\overline{\Psi}_q \gamma^\mu \Psi_q\right)(x) \left(\overline{\Psi}_q \gamma^\nu \Psi_q\right)(y) \right\} \right] \\ \sim\ & (+iQ_q^2) \left[\overline{\Psi}_q \frac{\gamma^\nu \left(\not{q} - \frac{i}{2} \overleftrightarrow{\not{D}} \right) \gamma^\mu}{\left(q - \frac{i}{2} \overleftrightarrow{D} \right)^2} \Psi_q \right] \left(\frac{x+y}{2} \right). \end{aligned} \tag{7}$$

Here, $\overleftrightarrow{D} \equiv \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$. See the lecture note for derivation of the expressions above. The denominators in the two expressions above can be approximated by

$$\frac{1}{\left(q \pm \frac{i}{2} \overleftrightarrow{D} \right)^2} \to \frac{1}{\left(q^2 \pm i q^\mu \overleftrightarrow{D}_\mu \right)}. \tag{8}$$

Expanding these denominators into a power series of $\mp(1/q^2) q \cdot i\overleftrightarrow{D}$, and using the relation

$$\gamma^\rho \gamma^\lambda \gamma^\sigma = \left[\eta^{\rho\lambda} \gamma^\sigma + \eta^{\sigma\lambda} \gamma^\rho - \eta^{\rho\sigma} \gamma^\lambda \right] - i\epsilon^{\rho\lambda\sigma\kappa} \gamma_5 \gamma_\kappa, \tag{9}$$

rewrite the operators $\Delta_{(q)}[\cdots]$ and $\Delta_{(\bar{q})}[\cdots]$ as a sum of local operators.

(b) Let us now insert those local operators in the hadron bra and ket. Let us denote the matrix elements of the twist-2 operators as follows:[2]

$$
\begin{aligned}
\langle h(\vec{p})| \left[\overline{\Psi}_q \gamma^{\lambda_1} \left(\frac{i}{2}\overleftrightarrow{D}\right)^{\lambda_2} \cdots \left(\frac{i}{2}\overleftrightarrow{D}\right)^{\lambda_j} \Psi_q \right] |h(\vec{p})\rangle &= p^{\lambda_1} p^{\lambda_2} \cdots p^{\lambda_j} A_j, \\
\langle h(\vec{p})| \left[\overline{\Psi}_q \gamma_5 \gamma^{\lambda_1} \left(\frac{i}{2}\overleftrightarrow{D}\right)^{\lambda_2} \cdots \left(\frac{i}{2}\overleftrightarrow{D}\right)^{\lambda_j} \Psi_q \right] |h(\vec{p})\rangle &= 0; \qquad (10)
\end{aligned}
$$

the latter holds true when the hadron state is not polarized (when the hadron spin is not pointing to a particular direction). Use this to show that the twist-2 contributions are given by the following:

$$
\begin{aligned}
\Delta_{(q)+(\bar{q})} T^{\mu\nu} = &\left(\left[-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right] + \frac{2x}{p \cdot q} \left[p^\mu - \frac{(p \cdot q)}{q^2} q^\mu \right] \left[p^\nu - \frac{(p \cdot q)}{q^2} q^\nu \right] \right) \\
&(Q_q)^2 \left[\sum_j \left[1 + (-1)^j \right] \left(\frac{1}{x}\right)^j \left(\frac{A_j}{2}\right) \right]. \qquad (11)
\end{aligned}
$$

(c) (not a problem) From this result, we can derive $T_2 = 2xT_1$, and

$$
\begin{aligned}
T_1 &= \frac{(Q_q)^2}{4\pi} \sum_j \left[1 + (-)^j \right] \frac{1}{x^j} \left(\frac{A_j}{2}\right) = -(Q_q)^2 \int \frac{dj}{2i} \frac{1 + e^{-\pi i j}}{\sin(\pi j)} \left(\frac{1}{x}\right)^j \frac{A_j^{(+)}}{8\pi}, \\
2\mathrm{Im} T_1 &= (Q_q)^2 \int \frac{dj}{2\pi i} \left(\frac{1}{x}\right)^j \frac{A_j^{(+)}}{4}. \qquad (12)
\end{aligned}
$$

6. **Cutkosky rule: exercise** [B]

Cutkosky rule tells you how to compute imaginary part of a forward scattering amplitude; the prescription is that, for $2\mathrm{Im}\mathcal{M}_{\alpha\alpha}$,

- write down the foward amplitude $i\mathcal{M}_{\alpha\alpha}$ first, using Feynman rule,
- introduce a line cutting the Feynman diagram into two parts somewhere in the middle (see Figure 4 for example)
- multiply (-1),
- replace all the cut propagators

$$
\frac{i}{(p_a^2 - m_a^2 + i\epsilon)} \quad \text{by} \quad (2\pi)\delta(p_a^2 - m_a^2), \qquad (13)
$$

[2] To be more precise, the left hand sides should be replaced by matrix elements of those operators with $\lambda_1 \cdots \lambda_j$ made symmetric and traceless. Similarly, on the right hand side, terms proprotional to the hadron mass and $\eta^{\lambda_{\sigma(p)}\lambda_{\sigma(q)}}$ should be subtracted in order to make it traceless. We ignore this, for now, however. It is O.K. for the purpose below just to use the right hand side for the matrix element on the left hand side.

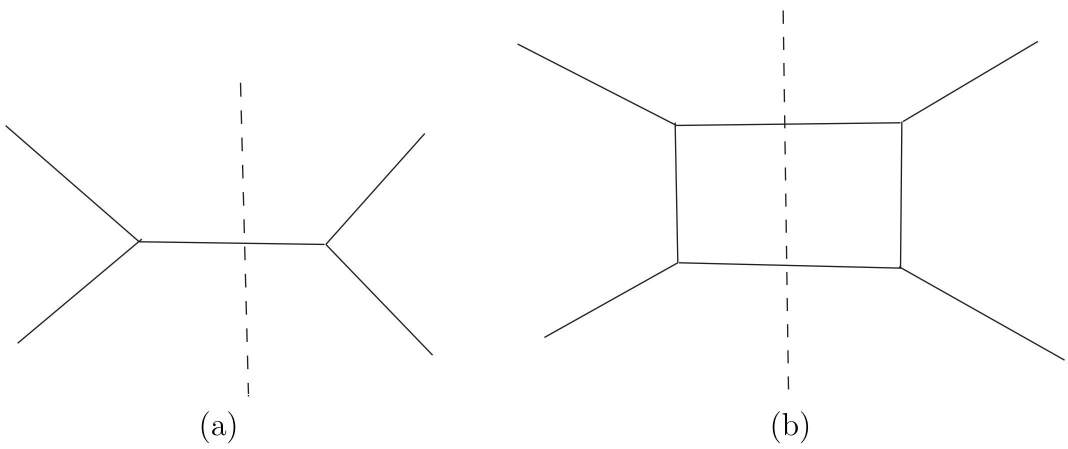


Figure 4: Forward amplitudes in a φ^3 theory, with cut lines.

- sum up for all possible ways of introducing a cut line.

A proof is found, e.g., in Peskin–Schröder textbook section 7.3.

Let us confirm that this prescription is right by using explicit examples. For simplicity, we use scalar φ^3 theory.

(a) Let us first use a forward amplitude given in Figure 4 (a). Compute, first, $2\mathrm{Im}\mathcal{M}$ by directly extracting the imaginary part of $\mathcal{M}$; use

$$\frac{1}{i}\left(\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon}\right)=-(2\pi)\delta(x). \tag{14}$$

As an alternative method, the Cutkosky rule above can be used to calculate $2\mathrm{Im}\mathcal{M}$. Check that the both results are the same.

(b) Let us now move on to another amplitude Figure 4 (b). Verify that what we obtain after applying the Cutkosky rule to the cut diagram Figure 4 (b) is $\int d\Pi|\mathcal{M}|^2$; we can then see from the optical theorem that it is the same as $2\mathrm{Im}\mathcal{M}$.

7. **DIS 3: the relation between Im M and Im T** [B]

Because of the optical theorem, the DIS total cross section is given by

$$\sigma_{\mathrm{DIS}} \simeq \frac{1}{4k\cdot p}2\mathrm{Im}\mathcal{M}(e^-+h\to e^-+h), \tag{15}$$

where k^μ and p^μ are momenta of e^- and the target hadron h (such as proton), respectively.

(a) Use the prescription in the previous problem for $2\text{Im}\mathcal{M}(e^- h \to e^- h)$ and $2\text{Im}T^{\mu\nu}(\gamma^* h \to \gamma^* h)$ to show[3] that

$$\begin{aligned}\sigma_{\text{DIS}} &\simeq \frac{1}{4k\cdot p}\int \frac{d^4k'}{(2\pi)^4}\frac{e^4}{(q^2)^2} 2\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right](2\pi)\delta^4((k')^2)2\text{Im}T^{\mu\nu}, \\ &= \frac{1}{4k\cdot p}\int \frac{d^3\vec{k}'}{(2\pi)^3}\frac{1}{2E_{\vec{k}'}}\frac{e^4}{(q^2)^2} 2\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right] 2\text{Im}T^{\mu\nu}. \qquad (16)\end{aligned}$$

Here, we assume that the incoming e^- beam is not polarized, so that we can take a spin average.

(b) If neither the e^- nor hadron h is polarized, there is no special azimuthal angle around the e^-–h collision axis. Thus, the integration over the azimuthal angle can be carried out first. Show that the integration measure becomes

$$\int \frac{d^3\vec{k}'}{(2\pi)^3}\frac{1}{2E_{\vec{k}'}} \longrightarrow \frac{1}{4(2\pi)^2}\int dQ^2 dy. \qquad (17)$$

[hint: This integration measure is Lorentz invariant. Thus, it is OK to use any Lorentz frame. The rest frame of the target hadron will be useful.]

8. **DIS 4 (tree): Light Ray Opeartor for pdf with Cutkosky rule** [B]

The DIS structure functions are given (at tree level) by simple expressions involving quark and anti-quark PDF. In the lecture, this relation was derived by showing that the Mellin transform of both the structure functions and the $q+\bar{q}$ PDF become the same thing: the matrix elements of twist-2 spin j operators. This relation between the structure function and PDF can be seen more directly, however, by using the Cutkosky rule.

For simplicity, let us focus on a linear combination of the two DIS structure functions, $2\text{Im}T^{\mu\nu}\eta_{\mu\nu} = 2\pi/x(F_2 - 6xF_1)$. At tree level, this is supposed to be $-(4\pi)(Q_q)^2[f_q + f_{\bar{q}}]$, where f_q and $f_{\bar{q}}$ are the quark and anti-quark PDF's.

Use the expressions (6) and (7) and apply the Cutkosky rule in order to rewrite $2\text{Im}\Delta_{(q)}T^{\mu\nu}$ and $2\text{Im}\Delta_{(\bar{q})}T^{\mu\nu}$ in terms of quark and anti-quark PDF, respectively. Note that one can use $\not{D}\Psi = 0$, $D^2\Psi = 0$, and the fact that

$$(2\pi)\delta(A) = \int_{-\infty}^{+\infty} d\lambda e^{i\lambda A}. \qquad (18)$$

9. **DIS at tree level in parton model (DIS 5)** [B]

[3] This argument, which relies on the Cutkosky rule, provides an alternative justification for the relation between σ_{DIS} and $\text{Im}T^{\mu\nu}$. This argument is easier and more intuitive than the one adopted in the lecture, while this one relies on perturbative (quark and gluon) picture.

(a) Show that the tree-level results $F_1(x;Q^2) = (Q_q^2/2)[f_q + f_{\bar{q}}]$ and $F_2(x;Q^2) = 2xF_1(x;Q^2)$ are obtained in the parton model, which is to replace the hadron bra and ket, $\langle h(\vec{p})|\cdots|h(\vec{p})\rangle$, by

$$\int_0^1 \frac{d\xi}{\xi}\ [\ f_q(x)\langle q(\xi\vec{p})|\ \cdots\ |q(\xi\vec{p})\rangle + f_{\bar{q}}(x)\langle \bar{q}(\xi\vec{p})|\ \cdots\ |\bar{q}(\xi\vec{p})\rangle\ ]\,. \tag{19}$$

(b) Show that the deep inelastic scattering process regarded as q–e^- scattering,

$$\int d\sigma_{\rm DIS} = \int d\xi\ [\ f_q(\xi)\sigma(q(\xi\vec{p}) + e^-(k) \to q + e^-) + f_{\bar{q}}(\xi)\sigma(\bar{q}(\xi\vec{p}) + e^- \to \bar{q} + e^-)\ ]\,, \tag{20}$$

reproduces the correct expression for the DIS differential cross section $d\sigma^2/dxdQ^2$. Use the result of the homework problem “DIS-1” (XI-4).

10. **DIS 6 (NLO): Remaining $\delta(1 - x/z)$ Part of the Splitting Function** [C]

In the lecture, by evaluating the 4 real gluon emission diagrams, a partial contribution to the quark-to-quark splitting function

$$\Delta P_{qq}(\chi) = \frac{\alpha_s}{2\pi}C_2(R)\frac{1+\chi^2}{(1-\chi)} \tag{21}$$

was derived. By including the vertex corrections and external line (quark line) wavefunction renormalization, and defining the factorization scale dependent PDF, $f_q(x;\mu_F)$, properly, derive the remaining contribution to the splitting function:[4]

$$P_{qq}(\chi) = \frac{\alpha_s}{(2\pi)}C_2(R)\left[\frac{1+\chi^2}{(1-\chi)_+} + \frac{3}{2}\delta(1-\chi)\right]. \tag{22}$$

During the course of deriving this expression, make sure that the soft divergence cancels. See any one of standard textbooks covering QCD (e.g. Peskin–Schrëder, pink textbook) for the definition of $1/(1-\chi)_+$ in the expression above.

[4] It is easier to arrive at the expression above by using a condition $\int_0^1 d\chi\ P_{qq}(\chi) = 0$, but it is more illuminating to derive by identifying the collinear divergence hidden in the vertex correction and wavefunction renormalization.