§1. What to study in this course.

☆ Quantum Theory of <u>Interacting</u> Particles in Spacetime.

(# of particles not necessarily preserved)

$e^+ + e^- \to 2\gamma$, $q + \bar{q} + g$,

$g + g \to h +$ anything.

$p +$ N or O $\to$ many hadrons.

e^- and phonon $\Rightarrow$ change ~~the~~ gd. state, excitation.

$\vdots$

<u>Perturbation theory</u>.

☆ QFT's : QM systems w/ <u>∞ DOF</u>.

✓ photon : $H = \int d^3\vec{x}\ \frac{1}{2}(\vec{E}^2 + \vec{B}^2) = \int \frac{d^3\vec{p}}{(2\pi)^3}\ \frac{1}{2}(E_{\vec{p}} \cdot E_{-\vec{p}} + B_{\vec{p}} \cdot B_{-\vec{p}})$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}).$$

✓ fermions.

$$H = \sum_{i=1}^{N} \left[-\frac{1}{2m} \vec{\partial}_i^{\,2} + \varphi(\vec{x}_i) \right] + \sum_{i \neq j}^{N} \frac{c}{|\vec{x}_i - \vec{x}_j|}$$

$\Updownarrow$ equivalent.

$$\psi(\vec{x}) \equiv \sum_n \psi_n(\vec{x})\, a_n$$

$$H = \int d^3\vec{x}\ \psi^\dagger(\vec{x})\, H_0\, \psi(\vec{x}) + \int d^3\vec{x} \int d^3\vec{y}\ \psi^\dagger(\vec{y})\, \psi^\dagger(\vec{x}) \frac{c}{|\vec{x} - \vec{y}|} \psi(\vec{x})\, \psi(\vec{y})$$

for any "N"

$$\{\psi(\vec{x}), \psi^\dagger(\vec{y})\} = \delta^3(\vec{x} - \vec{y}).$$

$\Rightarrow$
- treat (field) & particle on an equal footing.
- DOF at each pt $\Rightarrow$ ∞ DOF [box $\Rightarrow$ discrete (IR); lattice $\Rightarrow$ finite (UV)]
- $\boxed{\text{Perturbation theory for QM systems w/ ∞DOF} \Rightarrow \text{divergence, renormali-zation}}$

§2 S-matrix in Interaction Picture.

§2.1 Schrödinger picture & Heisenberg picture.

There are 2 diff. ways in describing time-evolution.

☆ Schrödinger picture:

states evolve in time. in $\mathcal{H}$. (Hibert space).

$$|\psi\rangle \text{ at } t=t_1 \Longrightarrow \left(e^{-i\int_{t_1}^{t_2} H(t')dt'}|\psi\rangle\right) \in \mathcal{H} \equiv |\psi(t_2)\rangle$$

$\in \mathcal{H}$ at $t=t_2$.

(Schrödinger eq. $i\frac{\partial}{\partial t}\psi = H\psi$)

time evolution of an observable A (whose def. does not change over time.)

$$\boxed{\langle\psi(t)|\,A\,|\psi(t)\rangle}$$

☆ Heisenberg picture

specify states as they are in $\mathcal{H}$. at a specific moment.

~~say, take a specific moment~~ $t=t_*$

say, take $t=t_*$, ~~as~~ (take a snap shot)

$$|\psi\rangle = |\psi(t_*)\rangle$$

and let the observable operator evolve in time.

$$A(t) = e^{iH(t-t_*)}A\,e^{-iH(t-t_*)}$$

Hamilton eq. $\frac{dA(t)}{dt} = -i[A(t), H(t)]$

$$\boxed{\langle\psi|\,A(t)\,|\psi\rangle}$$

$\{\ \}_{\text{Poisson}} \to -i[\ \ ]_{\text{comm}}$

☆ ~~They are the s~~ They are equivalent

• switch from one to the other

$$\boxed{\begin{aligned}\langle\psi(t_*)|\,A(t)\,|\psi(t_*)\rangle &= \langle\psi(t_*)|\,e^{iH(t-t_*)}A\,e^{-iH(t-t_*)}|\psi(t_*)\rangle \\ &= \langle\psi(t)|\,A\,|\psi(t)\rangle\end{aligned}}$$

§2.2 Interaction Picture.

(assume no explicit time-dep. in H for a while)

✓ $H = H_0 + V$

free. (bilinear) interaction part.

e.g. in QED. (ignore pure photon part)

$$H_\circ = -\bar{\Psi}\left\{ i\gamma^i(\partial_i - iA_i) + \gamma^0 A_0 - m \right\}\Psi \qquad i = 1.2.3.$$

$$\left[\quad {}_{\text{non}}(\tau^a = -1) \quad (\partial_\mu + i\tau^a A^a_\mu g = D_\mu) \right]$$

$$H_0 = -\bar{\Psi}\left(i\gamma^i(\partial_i) - m\right)\Psi$$

$$V = (\bar{\Psi}\gamma^\mu A_\mu \Psi)$$

✓ time-evolution due to H

⇓

time-evolution due to H_0 (easy to understand)

and remaining effect

easier part.

for $\mathcal{O}(\vec{x})$ define $e^{iH_0(t-t_*)}\mathcal{O}(\vec{x})e^{-iH_0(t-t_*)} \equiv \mathcal{O}_I(\vec{x},t)$

instead of full Heisenberg picture op. $\mathcal{O}(x,t)$

e.g. in scalar QED.

$$\mathcal{L} = [(\partial^\mu + iA^\mu)\phi^*][(\partial_\mu - iA_\mu)\phi] - m^2|\phi|^2 - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}$$

$$\phi(\vec{x}) = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + b^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$\phi^*(\vec{x}) = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(b_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$H_0 = \int\frac{d^3\vec{q}}{(2\pi)^3}\sqrt{(|\vec{q}|^2+m^2)}\left(a^\dagger_{\vec{p}}a_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}}\right)$$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q})$$

then $$\phi_I(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + b_{\vec{p}}^\dagger\, e^{ip\cdot x} \right) \Big|_{p^0 = E_{\vec{p}}}$$

§2.3 Time-ordered Correlation Function.

✓ $\langle \Omega | T\{ \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \} | \Omega \rangle$. ——(*)

$\mathcal{O}_i(x_i)$: full Heisenberg picture operators.

at spacetime pt $x_i = x_i^\mu$ $(\mu = 0.1.2.3)$.

$|\Omega\rangle$, $\langle\Omega|$: vacuum state : (more about this later)

$T\{ \cdots \}$ time-ordering.

if $x^0_{\sigma(1)} > x^0_{\sigma(2)} > \cdots$

$T\{\ \} = \mathcal{O}_{\sigma(1)}(x_{\sigma(1)})\, \mathcal{O}_{\sigma(2)}(x_{\sigma(2)}) \cdots$

✓ use this (*) as an example and study

how to deal with the full H evolution.

by ~~tw~~ separating into the easier H_0 evolution

and the remaining effect

— for simplicity assume. $x_1^0 > x_2^0 > \cdots > x_n^0$

then.

$$\mathcal{O}_i(x_i) = e^{iH(t_i-t_*)}\,\mathcal{O}_i(\vec{x}_i)\,e^{-iH(t_i-t_*)}$$

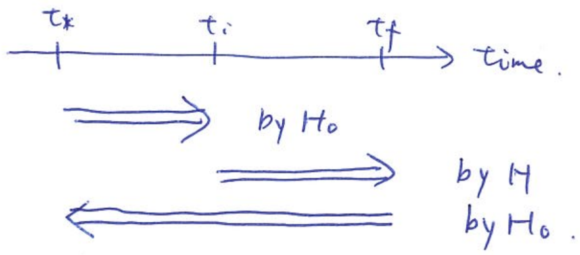
$$= e^{iH(t_i-t_*)}\,e^{-iH_0(t_i-t_*)}\,e^{iH_0(t_i-t_*)}\,\mathcal{O}_i(\vec{x}_i)\,e^{-iH_0(t_i-t_*)}\,e^{iH_0(t_i-t_*)}\,e^{-iH(t_i-t_*)}$$

$$= e^{iH(t_i-t_*)}\,e^{-iH_0(t_i-t_*)}\,\mathcal{O}_{i_I}(x_i)\,e^{iH_0(t_i-t_*)}\,e^{-iH(t_i-t_*)}.$$

$$\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2) = e^{iH(t_1-t_*)}\,e^{-iH_0(t_1-t_*)}\,\mathcal{O}_{1_I}(x_1)\,e^{iH_0(t_1-t_*)}\,e^{-iH(t_1-t_2)}\,e^{-iH_0(t_2-t_*)}\,\mathcal{O}_{2_I}(x_2)\,e^{iH_0(t_2-t_*)}\,e^{-iH(t_2-t_*)}$$

define.

$$U(t_f, t_i; t_*) \equiv e^{iH_0(t_f-t_*)}\,e^{-iH(t_f-t_i)}\,e^{-iH_0(t_i-t_*)}$$



net effect $\approx (H - H_0 = V)$ during $(t_i \rightsquigarrow t_f)$

when $[H, H_0] \neq 0$.

ignore

$$\frac{\partial}{\partial t_f} U(t_f, t_i; t_*) = -i\,e^{iH_0(t_f-t_*)}\,(H-H_0)\,e^{-iH(t_f-t_i)}\,e^{-iH_0(t_i-t_*)}$$

$$= -i\,V_I(t_f)\,U(t_f, t_i; t_*)$$

$$\Rightarrow \boxed{U(t_f, t_i; t_*) = T\exp\left(-i\int_{t_i}^{t_f} dt'\,V_I(t')\right)}$$

$$\boxed{(*) = \langle \Omega | e^{iH(t_1 - t_*)} e^{-iH_0(t_1 - t_*)} T\left\{ \mathcal{O}_{1I}(x_1)\, \mathcal{O}_{2I}(x_2) \cdots \mathcal{O}_{nI}(x_n) \exp\left(-i\int_{t_n}^{t_1} dt' V(t')\right)\right\} e^{iH_0(t_n - t_*)} e^{-iH(t_n - t_*)} |\Omega\rangle}$$

$(t_1 > \cdots > t_n)$

— consider

$$e^{-iH(t_* - T_-)} e^{+iH_0(t_* - T_-)} |0\rangle$$

$\Big(|0\rangle$: vac. state of the free theory $\Big)$

$$= e^{-iH(t_* - T_-)} |0_{(T_-)}\rangle$$

$H_0 |0\rangle = E_0 |0\rangle$.

$$= e^{-iH(t_* - T_-)} \sum_n |n_{(T_-)}\rangle \langle n_{(T_-)} | 0_{(T_-)}\rangle$$

: eigen state decomposition under H. $\sum_n |n\rangle\langle n|$

Take a limit $T_- \to -\infty \times (1 - i\varepsilon)$

$\Rightarrow$ excited state loose a factor by $e^{-(\Delta E)|T_-|(-\varepsilon)} \to 0$.

so.

$$\boxed{\lim_{T_- \to -\infty(1-i\varepsilon)} e^{-iH(t_* - T_-)} e^{iH_0(t_* - T_-)} |0_{(t_*)}\rangle = |\Omega_{(t_*)}\rangle \langle \Omega_{(T_-)} | 0_{(T_-)}\rangle}$$

$$\boxed{\langle \Omega | T\{ \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\} |\Omega\rangle = \lim_{\substack{T_- \to -\infty(1-i\varepsilon) \\ T_+ \to +\infty(1-i\varepsilon)}} \frac{\langle 0 | T\left\{ \mathcal{O}_{1I}(x_1) \cdots \mathcal{O}_{nI}(x_n) \exp\left(-i\int_{T_-}^{T_+} dt' V_I(t')\right)\right\} |0\rangle}{\langle 0 | T\left\{ \exp\left(-i\int_{T_-}^{T_+} dt' V_I(t')\right)\right\} |0\rangle}}$$

$$\boxed{\lim_{T_+ \to +\infty(1-i\varepsilon)} \langle 0_{(t_*)} | e^{iH_0(T_+ - t_*)} e^{+iH(T_+ - t_*)} = \langle \Omega_{(t_*)} | \times \left(\langle 0_{(T_+)} | \Omega_{(T_+)} \rangle \right)}$$

§ 2.4 S-matrix

✓ In a free theory.

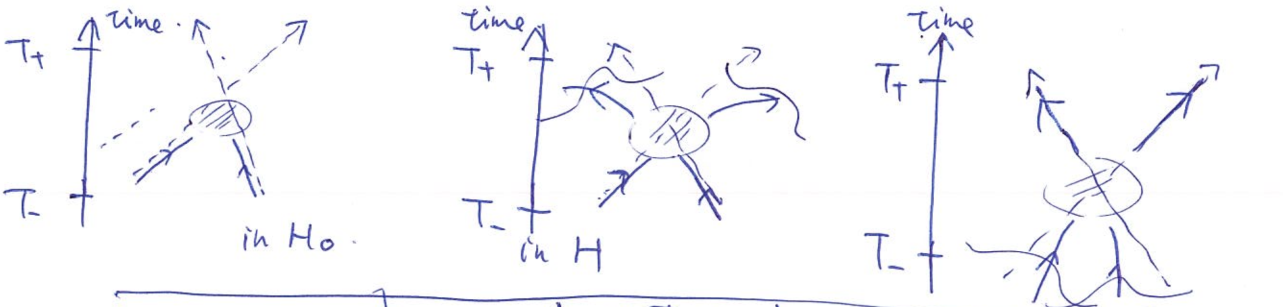
Hilbert space $\mathcal{H}$ spanned by

$$|0\rangle, \quad \frac{a^\dagger_{n_1,\vec{p}_1}|0\rangle}{\sqrt{2E_{\vec{p}}}} \quad \frac{a^\dagger_{n_1,\vec{p}_1}a^\dagger_{n_2,\vec{p}_2}|0\rangle}{\prod_i\sqrt{2E_{\vec{p}_i}}}, \;\cdots\cdots$$

n_i : label diff. particle spieces.

$\Longrightarrow$ want to find a similar description of $\mathcal{H}$ in Heisenberg picture for an interacting theory by using excited states on $|\Omega\rangle$.

✓ intuition



in-state

out-states

$$|n_1,\vec{p}_1;n_2,\vec{p}_2;\cdots\rangle^{in} \propto e^{iH(t_*-T_-)}\, e^{-iH_0(T_--t_*)}|n_1,\vec{p}_1;\cdots\rangle_{free}$$

$$|n_1,\vec{p}_1;n_2,\vec{p}_2;\cdots\rangle^{out} \propto e^{-iH(t_*-T_+)}\, e^{-iH_0(T_+-t_*)}|n_1,\vec{p}_1;\cdots\rangle_{free}$$

in-states, out-states .. normalized so that

$$\boxed{{}^{out}\langle n,\vec{p}\,|\,m,\vec{q}\rangle^{in} = \delta_{nm}\,(2E_{\vec{p}})\,\delta^3(\vec{p}-\vec{q})}$$

S-matrix

$$\boxed{S_{\beta\alpha} \equiv {}^{out}\langle\beta|\alpha\rangle^{in}}$$

: difference between the in-states & out-states consider

$$\int \langle \Omega | T\{ \phi_n(x) \phi_m(y) \} | \Omega \rangle e^{ip\cdot(x-y)} d^4(x-y)$$

$$= \frac{i Z_n \delta_{nm}}{p^2 - m_n^2 + i\varepsilon} + \cdots$$

(from Lorentz invariance.)

pole + branch cut

~~contain~~

$$\propto \langle 0 | T\{ \phi_{n_I}(x) \phi_{m_I}(y) \exp\left(-i\int dt' V_I(t')\right) \} | 0 \rangle$$

contains $\frac{i}{p^2 - m^2 + i\varepsilon}$ + extra contributions.

$\Rightarrow Z \neq 1$.

one can show that

$$\left(e^{-iH(T_*-T_+)} e^{-iH_0(T_+-T_*)} | n, \vec{p} \rangle_{free} \right)^\dagger \left(e^{-iH(T_*-T_-)} e^{-iH_0(T_--T_*)} | m, \vec{q} \rangle_{free} \right)$$

$$= Z_n \delta_{nm} (2E_{\vec{p}}) \delta^3(\vec{p} - \vec{q}).$$

Therefore.

$$\boxed{S_{\beta\alpha} = {}_{free}\langle \beta | T\exp\left(-i\int_{T_-}^{T_+} dt' V_I(t')\right) | \alpha \rangle_{free} \prod_i Z_{m_i}^{-1/2} \prod_j Z_{n_j}^{-1/2}}$$

$\beta = \{n_j, \vec{q}_j\}$, $\alpha = \{m_i, \vec{p}_i\}$

$\|$ (LSZ reduction formula)

Residue of

$$\boxed{\begin{gathered}\int d^4y_j\, e^{iq_j\cdot y_j} \langle 0 | T\{ \prod_j \phi_{m_{j_I}}(y_j) \prod_i \phi_{m_{i_I}}(x_i) \exp\left(-i\int_{T_-}^{T_+} dt' V_I(t')\right) \} | 0 \rangle e^{-ip_i\cdot x_i} d^4x_i \\ Z_{n_j}^{-1/2} \qquad Z_{m_i}^{-1/2} \\ \text{at} \left[\frac{i}{(q_j^2 - m_{n_j}^2 + i\varepsilon)} \quad \frac{i}{(p_i^2 - m_{m_i}^2 + i\varepsilon)} \right]\end{gathered}}$$