§ 4. Ultraviolet Divergence & Regularization.

(use self-energy as an example.)

§ 4.1 Evaluate 1-Loop Self-Energy

$$\Sigma^{1\text{-loop}} = (-ie^2)\int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu[(\not p+\not k)+m]\gamma_\mu}{[(p+k)^2-m^2+i\varepsilon][k^2+i\varepsilon]}$$

$$(\text{numerator}) = [-2(\not p+\not k)+4m]$$

$$(\gamma^\mu\gamma^\kappa\gamma_\mu = -2\gamma^\kappa, \quad \gamma^\mu\gamma_\mu = 4)$$

$$\left[\begin{array}{l} \dfrac{1}{AB} = \displaystyle\int_0^1 dx \frac{1}{[xA+(1-x)B]^2} = \int_0^1 dx_1\, dx_2 \frac{\delta(x_1+x_2-1)}{[x_1A+x_2B]^2} \\[2ex] \dfrac{1}{\prod_{i=1} A_i} = \displaystyle\int_0^1 dx_i \frac{\delta[(\sum_i x_i)-1]}{[\sum x_i A_i]^N}(N-1)! \end{array}\right]$$

$$\Sigma = (-ie^2)\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{[-2(\not p+\not k)+4m]}{[k^2+2xk\cdot p+xp^2-xm^2+i\varepsilon]^2}$$

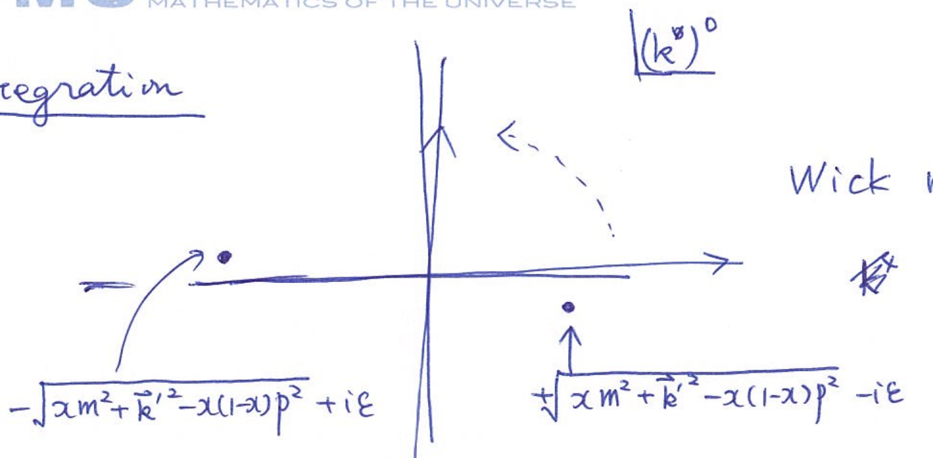
$$k^2+2xk\cdot p+xp^2-xm^2+i\varepsilon = (k+xp)^2+x(1-x)p^2-xm^2+i\varepsilon, \qquad k' \equiv k+xp$$

$$= (-ie^2)\int_0^1 dx \int \frac{d^4k'}{(2\pi)^4} \frac{[-2(\not p(1-x)+\not k')+4m]}{[(k')^2+x(1-x)p^2-xm^2+i\varepsilon]^2}$$

denominator : even under $k'_\mu \to -k'_\mu$.

$\Rightarrow$ drop $\not k'$ in numerator.

$(k')^0$ integration



$$\text{deformed contour}\int_{-i\infty}^{+i\infty} dk^0 \Rightarrow i\int_{-\infty}^{+\infty} dk^4$$

Euclidean signature.

$$\Sigma = \underbrace{(-ie^2)(i)}_{(+e^2)}\int_0^1 dx\int\frac{d^4k_E}{(2\pi)^4}\,\frac{[-2\not{p}(1-x)+4m]}{[k_E^2+xm^2-x(1-x)p^2]^2}$$

from the region $|k_E|^2 \gg m^2,\ p^2$,

$$\Sigma \sim e^2\int\frac{d|k_E|}{(16\pi^2)}\ k_E^3/k_E^4 = \frac{e^2}{(16\pi^2)}\times(\text{log divergence.})$$

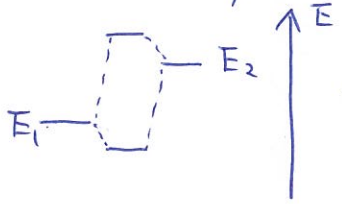
from large $|k_E|$ region.

$\boxed{\text{Ultraviolet divergence}}$

§ 4.2. Time-ordered Perturbation Theory (digression)

Quantum mechanics w/ two state system

$$H = \begin{pmatrix} E_2 & v \\ v^* & E_2 \end{pmatrix}$$



$$E_\pm = \frac{(E_1+E_2)}{2} \pm \sqrt{\left(\frac{E_2-E_1}{2}\right)^2 + |v|^2}$$

$$E_- = E_1 + v^* \frac{1}{(E_1-E_2)} v - v^* \frac{1}{(E_1-E_2)} v \frac{1}{(E_1-E_2)} v^* \frac{1}{(E_1-E_2)} v + \cdots$$

perturbation series.

E_- shifts. from E_1. because of the interaction.

states w/ E_n closer to E_1 contribute more .....?

position space Feynman rule $\Longleftrightarrow$ momentum space Feynman rule

$$i\mathcal{M} (2\pi)^4 \delta^4(q^{in} - p^{out}) = \int d^4 y_i \int \frac{d^4 k_a}{(2\pi)^4} \frac{i e^{-ik_a \cdot (\sum_i \epsilon_{ia} y_i)}}{(k_a^2 - m_a^2 + i\epsilon)} [P_a]_{\alpha' s} V$$

$$(= i[V_{y_i}]_{\alpha' s})$$

$$\longrightarrow \int \frac{d^4 k_a}{(2\pi)^4} \frac{i[P_a]_{\alpha' s}}{[k_a^2 - m_a^2 + i\epsilon]} \prod_i \left[(2\pi)^4 \delta^4\left(\sum_a \epsilon_{ia} k_a\right) - i[V_i]_\alpha\right]$$

◦ carry out all $d^3 y_i$ integral

$$= \int dy_1^0 \sim dy_N^0 \int \frac{d^4 k_a}{(2\pi)^4} \frac{[P]\, i e^{-ik_a^0(\epsilon_i y_i^0)}}{[k_a^2 - m_a^2 + i\epsilon]} \prod_i (2\pi)^3 \delta^3\left(\sum_a \epsilon_{ia} \vec{k}_a\right) - i[V_i]$$

$\Downarrow$

sum of $N!$ contributions. according to the ordering of y_i^0 $\quad i = 1 \sim N$

from $y_1^0 < y_2^0 < \cdots < y_N^0$.

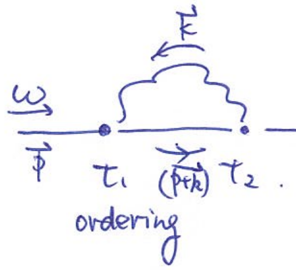
◦ carry out all dk_a^0 integral

$$\int \frac{dk_a^0}{2\pi} \frac{i\, e^{-ik_a^0(t_p - t_q)}}{[(k_a^0)^2 - E_{\vec{k}_a}^2 + i\epsilon]} = \begin{cases} (t_p > t_q) & \dfrac{e^{-iE_{\vec{k}_a}(t_p - t_q)}}{(2E_{\vec{k}_a})} \\ t_p > t_q\ (t_q > t_p) & \dfrac{e^{-iE_{\vec{k}_a}(t_q - t_p)}}{(2E_{\vec{k}_a})} \end{cases}$$

$$= \cancel{\int_0^{+\infty} d(t_2 - t_1) \int_0^{+\infty} d(t_3 - t_2)}$$

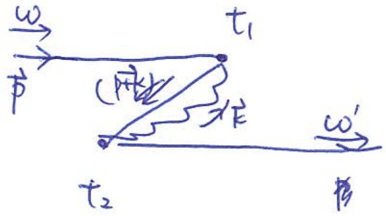
$$= \int_{t_1 \le t_2 \le \cdots \le t_N} dt_i (-iV_i) \int \left(\frac{d^3 \vec{k}}{(2\pi)^3}\right)^{\# loop} (2\pi)^3 \delta^3\left(\sum \vec{q}^{\,in} - \sum \vec{p}^{\,out}\right) \left[\prod_a \frac{1}{(2E_{\vec{k}_a})}\right] \times \left[\left(e^{-iE(\Delta t's)}\right)'s\right]$$

$\left(e^{-iE(\Delta t)}\right)$'s



$$e^{-i\omega t_1} \quad e^{-iE^f_{p+k}(t_2-t_1)} \quad e^{-iE^\gamma_k(t_2-t_1)} \quad e^{i\omega' t_2}$$

$$= e^{-i(E^f_{\vec{p}+\vec{k}}+E^\gamma_{\vec{k}}-\omega)(t_2-t_1)} \; e^{i(\omega'-\omega)t_2}.$$



$$e^{-i\omega t_1} \quad e^{-iE^f_{p+k}(t_1-t_2)} \quad e^{-iE^\gamma_k(t_1-t_2)} \quad e^{i\omega' t_2}$$

$$= e^{-i(E^f_{p+k}+E^\gamma_k+\omega')(t_1-t_2)} \; e^{i(\omega'-\omega)t_1}$$

$$(E^f_{p+k}+E^\gamma_k+\omega+\omega')-\omega$$

[off-shell-ness from t' to t'']

(general) $t_1 < t_2 < \cdots < t_N$.

$$e^{-i(E_{1\to 2}-\omega^{in})(t_2-t_1)} \; e^{-i(E_{2\to 3}-\omega^{in})(t_3-t_2)} \cdots$$

$$e^{-i(E_{N-1\to N}-\omega^{in})(t_N-t_{N-1})} \; e^{i(\omega^{out}-\omega^{in})t_N}$$

$$\int_0^{+\infty} d(t_2-t_1)\int_0^{+\infty} d(t_3-t_2)\cdots\int_0^{+\infty} d(t_N-t_{N-1})\int_{-\infty}^{+\infty} dt_N \;-(+i)^N$$

($(\omega^{in}-E)$ w. positive ($i\epsilon$) for convergence.)

- carry out all t_i integral $\Rightarrow$

$$\frac{1}{(E_{1\to 2}-\omega^{in})}\frac{1}{(E_{2\to 3}-\omega^{in})}\cdots\frac{1}{(E_{N-1\to N}-\omega^{in})}\times(2\pi)\delta(\omega^{out}-\omega^{in})(-i)$$

$$\boxed{iM = i\sum_{\text{time ordering}}\left[\int\frac{d^3k_a}{(2\pi)^3}\right]^{\#\text{loop}}\prod_i V_i \prod_a \left[\frac{[P_a]}{(2E_{k_a p_a})}\right]\underbrace{\frac{1}{(E_{12}-\omega^{in})(E_{23}-\omega^{in})\cdots(E_{N-1\to N}-\omega^{in})}}}$$

all possible intermediate states

just like in QM perturbation theory

(off shell-ness / virtuality) of intermediate states

UV divergence = many DOF w/ large $|k_a|$ win over the cost of large virtuality

§4.3 Regularization

UV divergence from UV (high energy) DOF.

— don't know much actually.

condensed matter :

a_i, $a_i^\dagger$ electron at site i.

$$H = E \sum_i a_i^\dagger a_i - t \sum (a_{i+1}^\dagger a_i + a_i^\dagger a_{i+1}) = \begin{pmatrix} E & -t & & \\ -t & \ddots & \ddots & \\ & \ddots & \ddots & -t \\ & & -t & E \end{pmatrix}$$

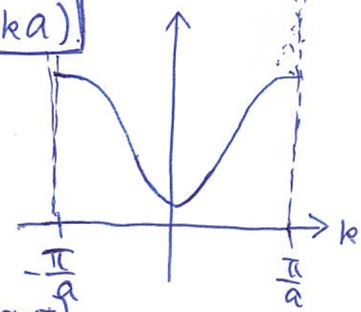
dispersion relation

$$\sum_n a_n^\dagger e^{i\delta n} |0\rangle \Rightarrow E_k^{(k)} = E - 2t\cos(\delta)$$

$$\boxed{= E - 2t\cos(ka)}$$

$$e^{i\delta n} \Leftrightarrow e^{i\overbrace{\left(\frac{\delta}{a}\right)}^{=k}(an)}$$

a: lattice spacing.



$E(k)$: not quadratic. $\left[\left(\frac{k^2}{2m}\right) + \text{const}\right]$

for $k \sim \left(1/a\right)$

where continium approx. not good.

$\exists$ (cut-off scale).

- cut-off in integration (state spectrum)
- change in dispersion relation

⋮

e^-, γ, phonon system. such cut off scale.

modify QED Lagrangian at high energy.

e.g.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{4\Lambda^2} F_{\mu\nu} D^\kappa D_\kappa F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi .$$

- higher covariant derivative.
- Pauli - Villars
- string theory ...

$$\Sigma^{1\text{-loop}} = (-ie^2)\int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu[(\not p+\not k)+m]\gamma_\mu}{[(p+k)^2-m^2+i\varepsilon]\,\underbrace{\left[k^2-\frac{k^4}{\Lambda^2}+i\varepsilon\right]}_{=\,[-k^2(k^2-\Lambda^2)]/\Lambda^2}}$$

$$= (-ie^2)\int_0^1 dx\int_0^{1-x} dy \int \frac{[-2(\not p+\not k)+4m](-\Lambda^2)\,2}{[k^2+2xp\cdot k - xm^2 - y\Lambda^2 + xp^2 + i\varepsilon]^3}\frac{d^4k}{(2\pi)^4}$$

$0 \le x+y \le 1$

$(x+y+z=1)$

$k+xp = k'$

$$= (-ie^2)\int_0^1 dx\int_0^{1-x} dy \int \frac{(-2\Lambda^2)[-2(\not p(1-x)+\not k')+4m]}{[k'^2 - xm^2 - y\Lambda^2 + x(1-x)p^2 + i\varepsilon]^3}\frac{d^4k'}{(2\pi)^4}$$

drop $(k'_\mu\gamma^\mu)$ safely. $(k'_\mu \to -k'_\mu$ odd$)$.

Wick rotation $\int_{-i\infty}^{+i\infty} dk'^0 = i\int_{-\infty}^{+\infty} dk^4$

$$= (e^2)\int_0^1 dx\int_0^{1-x} dy \int \frac{d^4k'_E}{(2\pi)^4} \frac{(+2\Lambda^2)[-2\not p(1-x)+4m]}{[k_E'^2 + xm^2 + y\Lambda^2 - x(1-x)p^2]^3}$$

$\mathrm{vol}(S^3) = 2\pi^2$

$K = |k_E|^2$

$$= \frac{e^2}{(16\pi^2)}\int_0^1 dx\int_0^{1-x} dy\int_0^\infty dK \frac{K\,(2\Lambda^2)[-2\not p(1-x)+4m]}{[K+xm^2+y\Lambda^2-x(1-x)p^2]^3}$$

$K = |k_E|^2$ integration:

$$\int dK \frac{K\,\Lambda^2}{(K+\Lambda^2)^3} \sim \Lambda^2 < K.\ \int\frac{dK}{K^2} \Rightarrow \text{converge.}$$

$$= \frac{e^2}{(16\pi^2)}\int_0^1 dx\int_0^{1-x} dy\, \frac{2\Lambda^2\,[-2\not p(1-x)+4m]}{[xm^2+y\Lambda^2-x(1-x)p^2]} \times \frac{1}{2}$$

y- integration

$$\boxed{\Sigma^{1\text{-loop}} = \frac{e^2}{16\pi^2}\int_0^1 dx \left[-2\not{p}(1-x)+4m\right] \ln\left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2}\right)}$$

$$\Sigma^{1\text{-loop}} \equiv A\not{p} + B$$

$$A = \frac{e^2}{16\pi^2}\int_0^1 dx \; -2(1-x) \ln\left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2}\right).$$

$$B = \frac{e^2}{16\pi^2} 4m \int_0^1 dx \quad \ln\left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2}\right)$$

finite if Λ is finite

under control ... regularization.