§5 Renormaliz~~ation~~ed Perturbation Theory

§5.1 Idea

✓ Observables dep. on high-energy DOF / theory ?

× need to know high-energy theory ?

○ an opportunity to get information on high-E. Theory?

✓ perturbation assuming $\underline{p^2 = M^2 \text{ mass shell}}$.

↳ propagator $\dfrac{i}{p^2 - M^2 + i\epsilon}$

↳ state $\boxed{|\vec{p}\rangle^{\text{free}} = a^\dagger_{\vec{p}} |0\rangle \sqrt{2E_{\vec{p}}}}$

non-sense ??

free theory $p^2 = \underline{\underline{m^2}}$ way off!

✓ residue.

$$\langle\Omega| T\{\Psi(x)\bar{\Psi}(y)\}|\Omega\rangle = \langle 0| T\{\Psi_I(x)\bar{\Psi}_I(y) \exp\left(-i\int dt' V_I(t')\right)\}|0\rangle_{\text{conn}}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{iZ_2\, e^{-ip\cdot(x-y)} \not{p}}{p^2 - m^2 + i\epsilon} + \cdots$$

$$\Psi_I(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(u(\vec{p})_s\, \underline{a_{\vec{p},s}}\, e^{-ip\cdot x} + v(\vec{p})_s\, \underline{b^\dagger_{\vec{p},s}}\, e^{ip\cdot x} \right)\Bigg|_{p^0 = E_{\vec{p}}}$$

$$|\vec{p}\rangle^{\text{in}} = \frac{1}{\sqrt{Z_2}\, \langle\Omega|0\rangle_{T_-}} e^{-iH(T_+ - T_-)} e^{iH_0(T_+ - T_-)} |\vec{p}\rangle^{\text{free}}.$$

$$\Rightarrow [\Psi_I(x)]_r \equiv \frac{1}{\sqrt{Z_2}} \Psi_I(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(u(\vec{p})_s [a_{\vec{p},s}]_r e^{-ip\cdot x} + v(\vec{p})_s [b_{\vec{p},s}]^\dagger_r e^{ip\cdot x} \right)\Bigg|_{p^0 = E_{\vec{p}}}$$

$$[a^\dagger_{\vec{p}}]_r \equiv \frac{1}{\sqrt{Z_2}} a^\dagger_{\vec{p}\,(\text{free})}$$

$\boxed{\text{(re-normalize)}}$

$\boxed{\text{QED action}}$

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - M\right]\Psi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}.$$

< for now. keep e & A_μ untouched > ← (allowed in QED)

$$\mathcal{L}_{free} = \bar{\Psi}\left[i\gamma^\mu\partial_\mu - M\right]\Psi$$

$$= Z_2\,\bar{\Psi}_r\left[i\gamma^\mu\partial_\mu - M\right]\Psi_r.$$

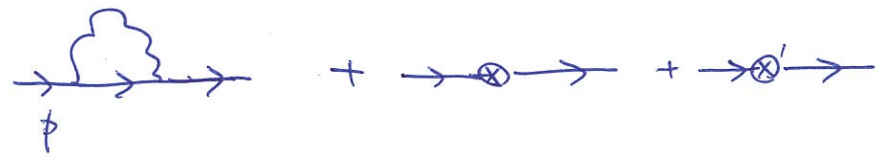
$$= \underset{(1+\delta_{Z2})}{\cancel{Z_2}}\,\bar{\Psi}_r\left[i\gamma^\mu\partial_\mu - (m+\delta M)\right]\Psi_r.$$

$$= \underbrace{\bar{\Psi}_r\left[i\gamma^\mu\partial_\mu - m\right]\Psi_r}_{\mathcal{L}_0} + \underbrace{(\delta_{Z2})\cdot\bar{\Psi}_r(i\gamma^\mu\partial_\mu)\Psi_r - (\delta_{Z2}m + \delta M + \delta_{Z2}\delta M)\bar{\Psi}_r\Psi_r.}_{\mathcal{L}_{int}}$$

$$\mathcal{L}_{int}: \underset{(1+\delta_{Z2})}{(\bar{\Psi}_r\,\gamma^\mu\Psi_r\,A_\mu e)} + \Uparrow$$

$\left(-i\Sigma^{1\text{-loop}} = iM^{1\text{-loop}}\right)$ $=$

$(\mathcal{O}(e^2))$



use e. $((\text{not}\ \underset{+}{\ }\ \delta_{Z2}\cdot e)$ $\quad i\,\delta_{Z2}\,\gamma^\mu p_\mu \quad -i(\delta_{Z2}M + \delta M$

Result from §4

$-i\Sigma(p^2, M^2)$ self-energy in un-renormalized fields.

$\parallel$

$-i\left[A(p^2,M^2)\not p + B(p^2,M^2)\right]$

propagator

$$\frac{i}{\not p - M - \Sigma(p,M^2)} = \frac{i}{\not p - M - A\not p - B}$$

$$= \frac{i\left[(1-A)\not p + (M+B)\right]}{(1-A)^2 p^2 - (M+B)^2}$$

$$\begin{cases} A(p^2,M^2) \equiv A^{(1)}(p^2,M^2) + O(e^4) \\ B(p^2,M^2) \equiv B^{(1)}(p^2,M^2) + O(e^4) \end{cases}$$

$$-i\Sigma^{(1)}_{(p,M)} = \frac{-ie^2}{16\pi^2}\int_0^1 dx \left[-2\not p(1-x) + 4M\right] \ln\left(\frac{(1-x)\Lambda^2 + xM^2 - x(1-x)p^2}{xM^2 - x(1-x)p^2}\right)$$

pole

solution of $\cancel{(1-A)^2p^2}$ $(1-A(p^2,M^2))^2 p^2 - (M+B(p^2,M^2))^2 = 0$.

$$p^2 = m^2 \Rightarrow M^2 + \left[2M^2 A^{(1)}(M^2,M^2) + 2M B^{(1)}(M^2,M^2)\right] + O(e\ldots$$

$$M^2 = m^2 - \left[2m^2 A^{(1)}(m^2,m^2) + 2m B^{(1)}(m^2,m^2)\right] + O(\ldots$$

→ homework.

$$\boxed{(M-m)} \equiv \boxed{(\delta M) = -m A^{(1)}(m^2,m^2) - B^{(1)}(m^2,m^2) + O(e^4)}$$

residue.

$$Z_2 = \frac{(1-A(p^2,M^2))\big|_{p^2=m^2}}{\frac{\partial}{\partial p^2}\left[(1-A)^2p^2 - (M+B)^2\right]} = \frac{(1-A)}{\left[(1-A)^2 + 2(A-1)m^2\frac{\partial A}{\partial p^2} - 2(M+B)\frac{\partial B}{\partial p^2}\right.}$$

at $p^2=m^2$

(diff. from $\frac{1}{(1-A)}$)

$$\boxed{\delta_{Z2}^{(1)} = \left[A^{(1)} + 2m^2\frac{\partial A^{(1)}}{\partial p^2} + 2m\frac{\partial B^{(1)}}{\partial p^2}\right]_{\text{evaluated at}\,(p^2=m^2,\ M^2\sim m^2)} + \mathcal{O}(e^4)}$$

 in renormalized field

$$\Rightarrow\ -i\Sigma(p^2, \underline{m}^2)$$

$\boxed{\text{Therefore}}$

$$\boxed{-i\Sigma^{(1)} = i\mathcal{M}^{(1)}(p^2,m^2)} = -i\left(A^{(1)}(p^2,m^2)\not{p} + B^{(1)}(p^2,m^2)\right)$$

$$+ i\left[A^{(1)}(m^2,m^2) + 2m^2\left(\frac{\partial A^{(1)}}{\partial p^2}\right)(m^2,m^2) + 2m\left(\frac{\partial B^{(1)}}{\partial p^2}\right)(m^2,m^2)\right]\not{p}$$

$$- i\left[\cancel{mA^{(1)}(m^2,m^2)} + 2m^3\left(\frac{\partial A^{(1)}}{\partial p^2}\right)(m^2,m^2) + 2m^2\left(\frac{\partial B^{(1)}}{\partial p^2}\right)(m^2,m^2)\right.$$
$$\left.\cancel{-mA^{(1)}(m^2,m^2)} - B^{(1)}(m^2,m^2)\right]$$

$$\langle\Omega|T\{\Psi_r(x)\bar{\Psi}(y)_r\}|\Omega\rangle = \frac{i}{\not{p} - m - \Sigma^{(1)}(p^2,m^2)} + \cdots + \mathcal{O}(e^4)$$

$$= \frac{i}{\left[\begin{array}{l}1 - A^{(1)}(p^2,m^2) + A^{(1)}(m^2,m^2) \\ \quad + 2m^2\frac{\partial A}{\partial p^2}\Big| + 2m\frac{\partial B}{\partial p^2}\Big|\end{array}\right]\not{p} - \left[\begin{array}{l}m + B^{(1)}(p^2,m^2) - B^{(1)}(m^2,m^2) \\ \quad + 2m^3\frac{\partial A^{(1)}}{\partial p^2}\Big| + 2m^2\frac{\partial B^{(1)}}{\partial p^2}\Big|\end{array}\right]}$$

still different from $\frac{i}{\not{p}-m}$ quantum corr.

$$\left[A^{(1)}(p^2,m^2) - A^{(1)}(m^2,m^2)\right] = \frac{e^2}{16\pi^2}\int_0^1 dx\,[-2\cdot(1-x)]\left\{\ln\left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)p^2}{xm^2 - x(1-x)p^2}\right) - \ln\left(\frac{(1-x)\Lambda^2 + xm^2 - x(1-x)m^2}{xm^2 - x(1-x)\cancel{p^2}m^2}\right)\right\}$$

$$\Downarrow$$

$$\left\{\ln\left(\frac{\cancel{x}\,[xm^2]}{\cancel{x}\,[m^2-(1-x)p^2]}\right)\right\}$$

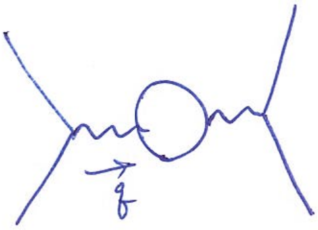
$\boxed{\text{sweep under the carpet (rug)}} \Leftarrow \boxed{\text{Large } \ln\left(\Lambda^2/m^2\right) \text{ disappear}} + \boxed{\text{power suppressed } \frac{1}{\Lambda^2} \text{ corr.}}$

$\boxed{\text{insensitive to high E. Theory}}$

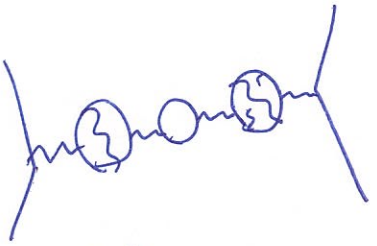
$\boxed{\text{when we use re-normalized fields \& observed (physical) parameters}}$

§ 5.2 QED in Renormal~~ization~~ized Perturbation Theory

beyond electron self-energy.



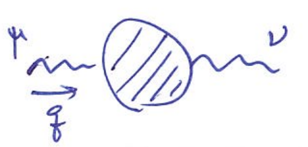
and



...

$$\frac{-i\eta_{\mu\nu}}{q^2} \Longrightarrow \left[\frac{-i\left[\eta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right]}{q^2\ (1-\Pi(q^2))} + \frac{-i\ q_\mu q_\nu}{q^2 q^2} \leftarrow (\times\xi=1)\right]$$

Feynman gauge propagator



$$i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\,\Pi(q^2).$$

electron mass

$$\boxed{\Pi^{(1)}(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left(\frac{M^2-x(1-x)q^2}{M_{reg.}^2}\right).}$$

Pauli-Villars

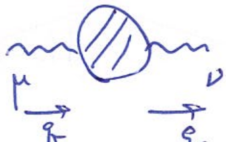
↔ homework. regulator mass.

- no mass shift
- residue. $\frac{-i\eta_{\mu\nu}Z_3}{[q^2+i\varepsilon]}$; $Z_3 = \frac{1}{1-\Pi(q^2)\big|_{q^2=0}} \simeq 1+\Pi^{(1)}(q^2{=}0)$
  $= 1+\delta_{Z_3}^{(1)}$

$$A_{\mu,B}(x)\frac{1}{\sqrt{Z_3}} \equiv A_{\mu,r}(x) \quad \text{renormalized field}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Rightarrow -\frac{1}{4}Z_3(F^r_{\mu\nu}F^{r\mu\nu}) = \underbrace{-\frac{1}{4}(F_{\mu\nu}F^{\mu\nu})_r}_{\mathcal{L}_0}\underbrace{-\frac{\delta_{Z_3}}{4}(F_{\mu\nu}F^{\mu\nu})_r}_{\mathcal{L}_{int}}.$$

$\Rightarrow$



$$= i(q^2\eta_{\mu\nu}-q_\mu q_\nu)\left[\Pi(q^2)-\Pi(0)\right]$$

$$-i\delta_{Z_3}(q^2\eta^{\mu\nu}-q^\mu q^\nu)$$

$$\boxed{\left[\Pi(q^2)-\Pi(0)\right] = \frac{e^2}{2\pi}\int_0^1 dx\ x(1-x)\ln\left(\frac{m_e^2-x(1-x)q^2}{m_e^2}\right)}$$