§ 5.2 QED in Renormalized Perturbation Theory (Cont'd)

☆  $= -i\Sigma = -i\left[A(p^2, M^2)\not{p} + B(p^2, M^2)\right]$

when un-renormalized fields & coupling are used.

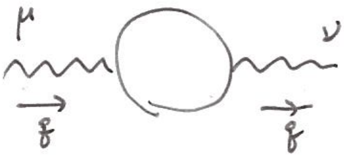
$\Psi_I = \sqrt{Z_2}\, \Psi_{I,r}$

$\Psi_{I,r}$: canonical normalization. in $\langle\Omega|T\{\Psi_{I,r}(x)\overline{\Psi}_r(y)\}|\Omega\rangle$ (*)

if $\quad Z_2 = \dfrac{(1-A)}{\left[(1-A)^2 + 2(A-1)\frac{\partial A}{\partial p^2} + 2(M+B)\frac{\partial B}{\partial p^2}\right]}\Bigg|_{p^2=m^2}$

Λ-dependence disappeared from (*)

☆

 $= i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)$

unrenormalized fields & coupling are used.

$$\boxed{\Pi^{(1)}(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(\frac{M_e^2 - x(1-x)q^2}{M_{reg}^2}\right)}$$

in Pauli-Villars regularization.

$$\langle\Omega|T\{A_\mu(x)A_\nu(y)\}|\Omega\rangle = \int\frac{d^4q}{(2\pi)^4}\,\frac{-i\,e^{-iq\cdot(x-y)}\,\eta_{\mu\nu}Z_3}{q^2+i\epsilon} + \cdots$$

$$\Longrightarrow \boxed{A_{I,\mu} = \sqrt{Z_3'}\, A_{I r,\mu}}$$

canonical if $\quad Z_3' = Z_3 = \dfrac{1}{\left[1-\Pi(q^2)\right]}\Big|_{q^2=0}$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Rightarrow -\frac{1}{4}F^r_{\mu\nu}F^{r\,\mu\nu} - \frac{1}{4}(Z_3'-1)\,F^r_{\mu\nu}F^{r\,\mu\nu}$$

$\Downarrow$ use this for $\mathcal{L}_0$ $\qquad\qquad$ $\Downarrow$ treat as a part of $\mathcal{L}_{int}$.

⇒ at 1-loop ($\mathcal{O}(e^2)$), $A^r_\mu - A^r_\nu$ self-energy.

$$= i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\left(\Pi(q^2) - \delta_{Z_3'}\right) \Rightarrow i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\left(\Pi^{(1)}(q^2) - \Pi^{(1)}(0)\right)$$

use $\Psi_{I.r}$

$\delta^{(1)}_{Z_3'}$

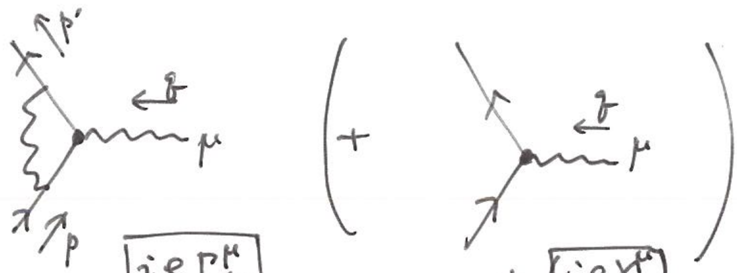
$$\boxed{\frac{e^2}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(\frac{M_e^2 - x(1-x)q^2}{M_e^2}\right)} = \boxed{\Pi^{(1)}_{ren}(q^2)}$$

$\overline{M}^2_{reg}$: disappear

☆ Finally

$$\underline{\mathcal{L}_{QED} \supset e(\bar{\Psi}\gamma^\mu A_\mu \Psi)}$$

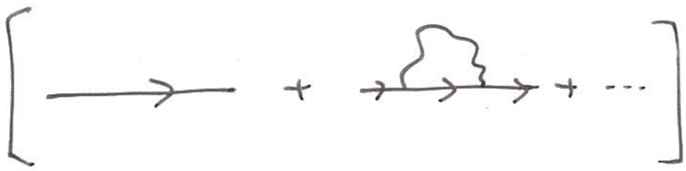
at 1-loop



$\boxed{ie\Gamma^\mu_{(1)}}$ $\quad + \boxed{ie\gamma^\mu}$

↳ divergent. (or large log) (if regulator ↗ ∞)

<u>mass</u>



pole: defined as the observed mass m. ($m \neq M$)

<u>electric charge</u>

charge of $e^- + e^-(e^+e^-) + \cdots$ cloud : defined as the observed charge e_r ($e_r \neq e$)

$$ie\Gamma^\mu_{(1)} = ie\left(\gamma^\mu V_1(q^2) + [\gamma^\mu, \gamma^\nu] q_\nu \times (\ ) + \cdots\right).$$

<u>$\Psi_r - \bar{\Psi}_r - A_{r\mu}$ coupling</u>

$$(ie\gamma^\mu + ieV_1\gamma^\mu + \cdots) Z_2\sqrt{Z_3} \simeq i\gamma^\mu\left(e + e\left(V_1^{(1)} + \delta^{(1)}_{Z_2} + \tfrac{1}{2}\delta^{(1)}_{Z_3}\right) + \cdots\right)$$

$$\Rightarrow \boxed{e + e\left(V_1^{(1)}\Big|_{q^2=0} + \underbrace{\delta^{(1)}_{Z_2} + \tfrac{1}{2}\delta^{(1)}_{Z_3}}_{q^2=0}\right) + O(e^5) = e_r}$$

$$\boxed{M^2 + \left(2M^2 A^{(1)}\Big|_{p^2=M^2} + 2M\, B^{(1)}\Big|_{p^2=M^2}\right) + O(e^4) = m^2}$$

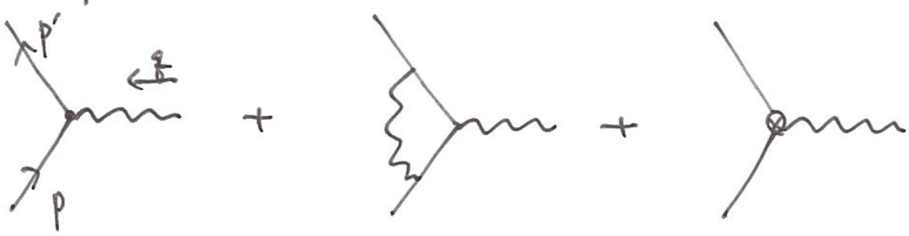
Renormalized perturbation Theory :

use $(\Psi_r,\ A_{r\mu},\ m,\ e_r)$ for perturbation.

$$\mathcal{L}_{QED} = \bar{\Psi}_r\left[i\gamma^\mu(\partial_\mu - ie_r A_{r\mu}) - m\right]\Psi_r - \frac{1}{4}F^{(r)}_{\mu\nu}F^{(r)\mu\nu} \quad = \mathcal{L}_0$$

$$\left.\begin{aligned} &+ \delta_{Z2}\,\bar{\Psi}_r(i\gamma^\mu\partial_\mu)\Psi_r \qquad - \frac{1}{4}(\delta_{Z3})F^r_{\mu\nu}F^{r\mu\nu} \\ &- (Z_2 M - m)(\bar{\Psi}_r\Psi_r) \\ &+ (Z_2\sqrt{Z_3}\,e - e_r)(\bar{\Psi}_r\gamma^\mu A_{r\mu}\Psi_r) \end{aligned}\right\} \mathcal{L}_{int}.$$

$\boxed{\text{at 1-loop}}$



$$ie_r\gamma^\mu \quad + \quad ie_r\Gamma^\mu_{(1)} \quad + \quad i\left\{e - e_r + e_r\left(\delta^{(1)}_{Z2} + \frac{1}{2}\delta^{(1)}_{Z3}\right)\right\}\gamma^\mu$$

$$= ie_r\gamma^\mu + ie_r\left(\Gamma^\mu_{(1)} - V_1(q^2{=}0)\gamma^\mu\right) + i\gamma^\mu\left[e - e_r + e_r\left\{V^{(1)}_1(q^2{=}0) + \delta^{(1)}_{Z2} + \delta^{(1)}_{Z3}/2\right\}\right]$$

$\hookrightarrow O(e^5)$: ignore

✓ when $q^2=0$. $\Rightarrow ie_r\gamma^\mu + (\text{non-}\gamma^\mu)$

✓ $$\Gamma^\mu_{(1)} = \left(\frac{e^2}{16\pi^2}\right)\int_\Delta dx\,dy \left(2\gamma^\mu\left[\ln\left(\frac{(1-x-y)\Lambda^2 + (x+y)^2M^2 - xyq^2}{(x+y)^2M^2 - xyq^2}\right) + \frac{\cancel{M^2(1-x-y)}\ M^2\{1-4(1-x-y)+(1-x-y)^2\} + (1-x)(1-y)q^2}{(x+y)^2M^2 - xyq^2}\right] \cancel{+(\text{non-}\gamma^\mu)} + [\gamma^\mu,\gamma^\nu]q_\nu\frac{(1-x-y)(x+y)M}{(x+y)^2M^2 - xyq^2}\right)$$

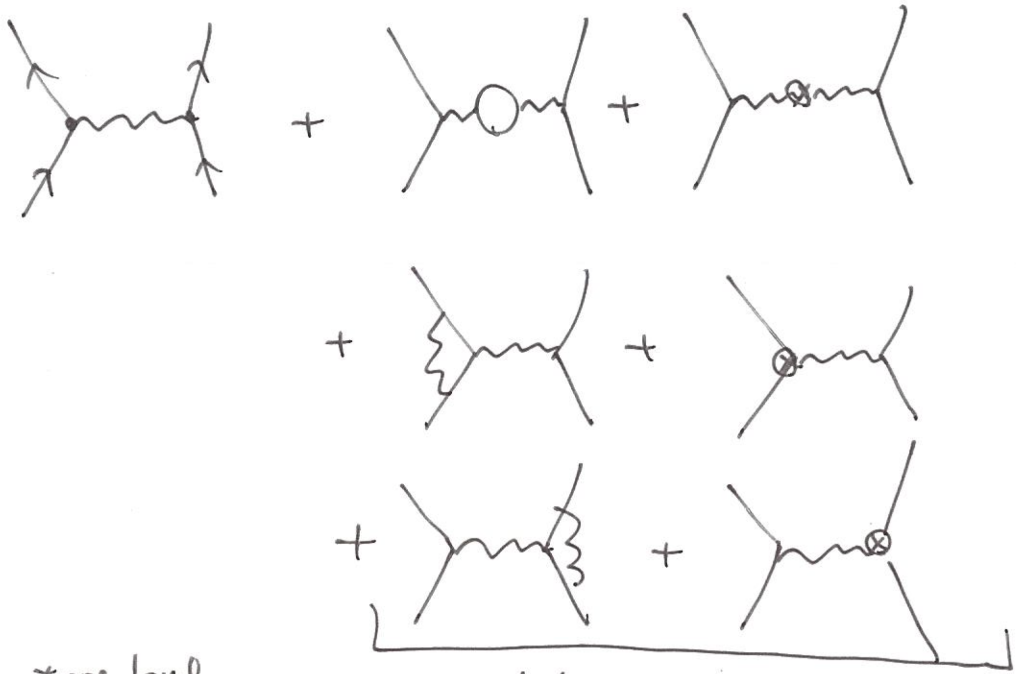
$\Rightarrow \cancel{\Gamma^\mu_{(1)}} -$

$$\Rightarrow \left(\Gamma^\mu_{(1)}(q^2) - V_{(1)}(0)\gamma^\mu\right) =$$

$\left\{\begin{array}{l}\left[\text{quantum correction}\right] \\ \left[\Lambda^2,\ \bar{M}^2_{reg.}\ \text{disappeared.}\right]\end{array}\right.$

$$= \left(\frac{e^2}{16\pi^2}\right)\int_\Delta dx\,dy\left(2\gamma^\mu\left[\ln\left(\frac{(x+y)^2M^2}{(x+y)^2M^2 - xyq^2}\right) + \frac{m^2\{1-4(\ )+(\ )^2\} + \cdots q^2}{(x+y)^2m^2 - xyq^2} - \frac{\{1-4(\ )+(\ )^2\}m^2}{(x+y)^2m^2}\right] - [\gamma^\mu,\gamma^\nu]q_\nu \times \frac{(1-x-y)(x+y)M}{(x+y)^2m^2 - xyq^2}\right)$$

$\boxed{\text{at 1-Loop}}$



tree level $\mathcal{O}(e_r^2)$

1-loop level. $\mathcal{O}(e_r^4)$.

$$(e_r\gamma^\mu)\frac{\eta_{\mu\nu}}{q^2}(e_r\gamma^\nu) \quad + (e_r\gamma^\mu)\frac{\eta_{\mu\nu}}{q^2}\left[\Pi(q^2)-\Pi(0)\right](e_r\gamma^\nu) \quad + \mathcal{O}(e_r^6)$$

$$+ e_r\left(\Gamma^\mu_{(1)} - \gamma^\mu V_1^{(1)}(0)\right)\frac{\eta_{\mu\nu}}{q^2}(e_r\gamma^\nu)$$

$$+ (e_r\gamma^\mu)\frac{\eta_{\mu\nu}}{q^2}\, e_r\left(\Gamma^\nu_{(1)} - \gamma^\nu V_1^{(1)}(0)\right)$$

☆ external states : properly normalized : no need for $\left(\prod_i Z_2^{-1/2}\right)$ in LSZ now.

☆ rewritten in terms of (m, e_r) from (M, e).

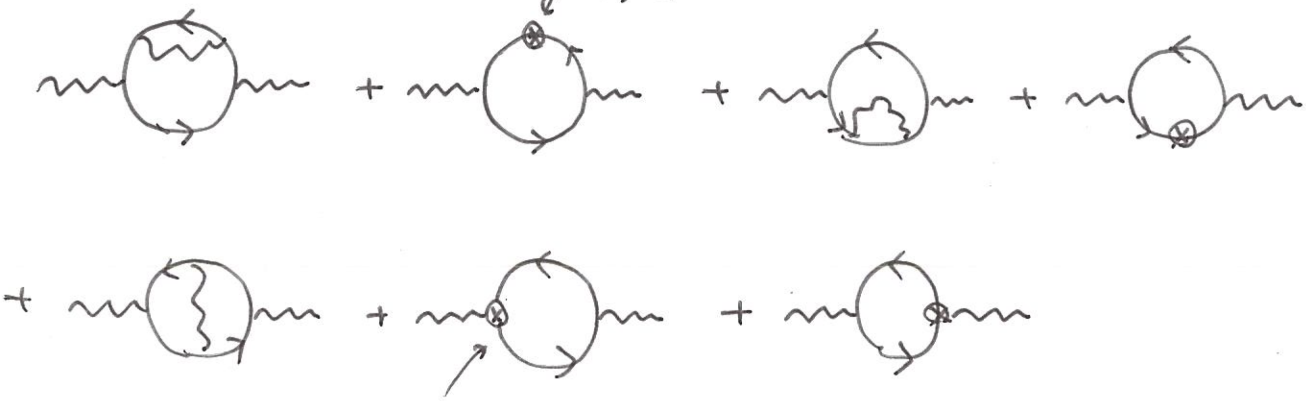
$\Rightarrow$ regularization parameters Λ, $\bar{M}^2_{reg}$. disappeared.

☆ perturbative (quantum) correction calculated in this way.

eg. anomalous magnetic moment.

$\boxed{\text{at 2-loop}}$

$\delta_{Z_2}^{(1)},\ \delta_m^{(1)}$



$\cancel{\delta_{Z_1}^{(1)}}$ $\quad e - e_r + e_r\left(V_1^{(1)}(0) + \delta_{Z_2}^{(1)} + \frac{1}{2}\delta_{Z_3}^{(1)}\right)$

$+$ 

$$\boxed{\cancel{\delta_{Z_3}^{(2)}}} \Uparrow \boxed{-i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\delta_{Z_3}^{(2)}}$$

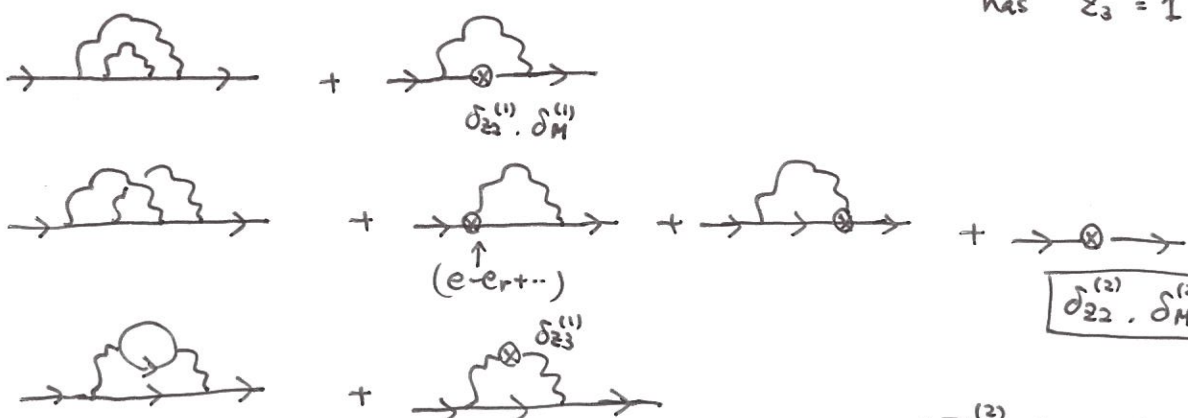
$$= i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi_{\text{ren}}^{(2)}(q^2).$$

$\delta_{Z_3}^{(2)}$: determined so that $\Pi_{\text{ren}}^{(2)}(q^2) = 0$

$$\Updownarrow$$

$$\frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon]\left(1-(\Pi_{\text{ren}}^{(1)}+\Pi_{\text{ren}}^{(2)})(q\right.}$$

has "Z_3" $= 1$.



$\delta_{Z_2}^{(1)},\ \delta_M^{(1)}$

$(e - e_r + \cdots)$

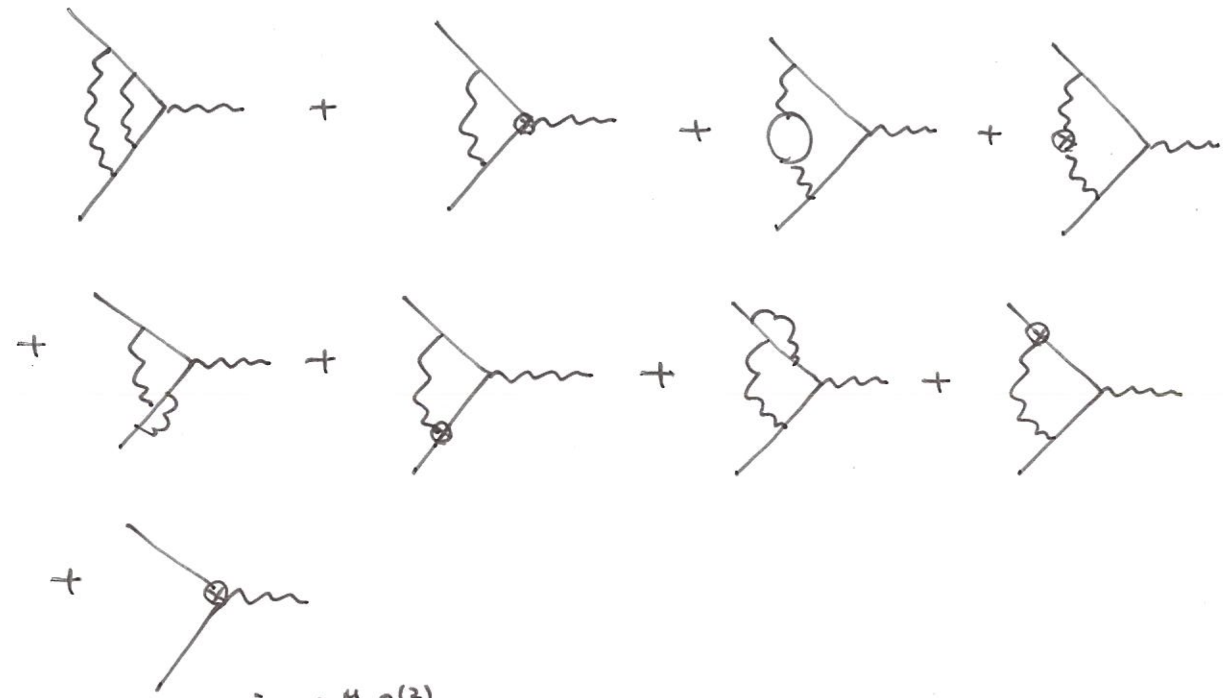
$\boxed{\delta_{Z_2}^{(2)},\ \delta_M^{(2)}}$

$\delta_{Z_3}^{(1)}$

$$= -i\Sigma_{\text{ren}}^{(2)}(p, m)$$
$$= -i\left(A_{\text{ren}}^{(2)}\not{p} + B_{\text{ren}}^{(2)}\right)$$

$\delta_{Z_2}^{(2)},\ \delta_M^{(2)}$ determined so that

- $\boxed{\left.\left(1 - A_{\text{ren}}^{(1)} - A_{\text{ren}}^{(2)}\right)^2 m^2 = \left(m + B_{\text{ren}}^{(1)} + B_{\text{ren}}^{(2)}\right)^2\right|_{p^2=m^2} \quad \text{mod } \mathcal{O}(e^6)}$
- $\boxed{\left.\left[\left(1 - A_{\text{ren}}^{(1)} - A_{\text{ren}}^{(2)}\right)^2 + 2\left(A_{\text{ren}}^{(1)} + A_{\text{ren}}^{(2)} - 1\right)\frac{\partial A_{\text{ren}}^{(1)+(2)}}{\partial p^2} - 2\left(m + B_{\text{ren}}^{(1)+(2)}\right)\frac{\partial B_{\text{ren}}^{(1)+(2)}}{\partial p^2}\right]\right|_{p^2=m^2} = \left.\left(1 - A_{\text{ren}}^{(1)+(2)}\right)\right|_{p^2=m}}$



$ie_r\gamma^\mu \underline{\underline{\delta_{Z_1}^{(2)}}}$ determine so that $F_1(q^2)\big|_{q^2=0} = 1$

$$ie_r\Gamma^\mu_{\text{ren}} = ie_r\left(F_1(q^2)\gamma^\mu - \frac{F_2(q^2)}{4m}[\gamma^\mu,\gamma^\nu]q_\nu\right.$$

$$+ \text{mod}\,[\times(\not p - m)]$$

$$+ \text{mod}\,[(\not p' - m)\times]$$

$\left\{\begin{array}{l} F_1 : \cancel{\text{Pauli}}\ \text{Dirac} \\ F_2 \quad\ \ \text{Pauli} \end{array}\right.$

$\boxed{\text{Renormalization conditions. (on-shell)}}$

- ✓ Ψ_r. canonically normalized. ("Z_2" = 1)
- ✓ fermion pole mass = m.
- ✓ $A_{r,\mu}$ canonically normalized ("Z_3" = 1)
- ✓ $F_1(q^2=0) = 1$

$\Rightarrow$ determine. $\underbrace{\delta_{Z_2}^{(n)},\ \delta_{Z_3}^{(n)},\ (M-m),\ (e-e_r)\quad (\text{or } \delta_{Z_1}^{(n)})}_{\Uparrow}$

order by order.

$\boxed{\text{counter terms.}}$