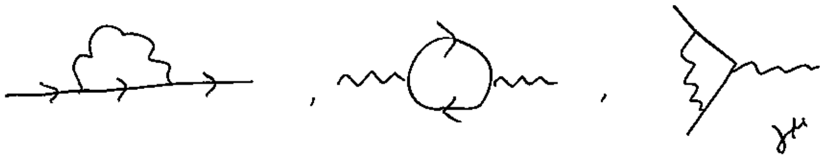
§5.3 Superficial Degree of Divergence

In QED (at 1-loop level)

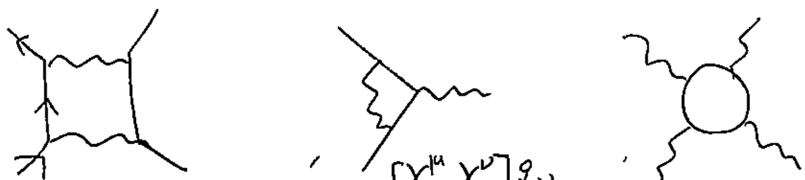
4 renormalization conditions.

- e^- (and e^+) (pole) mass : m
- electric charge (at $q^\mu = 0$): e_r.
- Ψ_e field normalization : 1
- A_μ field : : 1.

to rewrite $(M, e; \Lambda)$.



divergence appear. (taken care of).



$[\gamma^\mu, \gamma^\nu] q_\nu$

+ ...

$\boxed{\text{any other amplitudes with divergence?}}$

In QED, we've done all possible "renormalization".

How do we know?

fermion propagator

$$\frac{i[\not p + M]}{[p^2 - M^2 + i\varepsilon]} \Rightarrow -1$$

vector boson propagator (Feynman gauge)

$$\frac{-i\eta_{\mu\nu}}{(p^2 + i\varepsilon)} \Rightarrow -2.$$

scalar propagator

$$\frac{i}{[p^2 - M^2 + i\varepsilon]} \Rightarrow -2.$$

(scalar)^2 – vector vertex

$$\propto (p_1 - p_2)^\mu \Rightarrow +1.$$

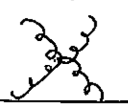
(fermion)^2 – vector vertex

$$\gamma^\mu \Rightarrow 0.$$

(vector)^3 vertex (non-Abelian)

 $\propto p^\mu \Rightarrow +1$

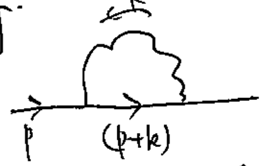
(vector)^4 vertex

 $\propto (\eta^{\mu\nu}\eta^{\rho\sigma} + \dots) \Rightarrow 0.$

1-loop momentum $\dfrac{d^4 p}{(2\pi)^4} \Rightarrow +4$

$\boxed{\text{UV divergence: mass, external momenta don't matter.}}$

eg.

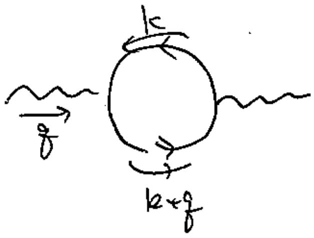


$$\Rightarrow D = \underset{}{(-2)}^{\gamma} + (-1)^{\psi} + 4 = +1$$

truncate $\xrightarrow{\text{odd part.}}$ 0

$\int d^4k \; \gamma^\mu (\not p + \not k) \gamma_\mu \Rightarrow$ use $\not p$ not $\not k$

| logarithmic div.



$$\Rightarrow D = (-1)^{\psi} \times 2 + 4 = \underline{\underline{+2}} \quad \text{for } (\;)_{\mu\nu}.$$

but gauge symmetry :

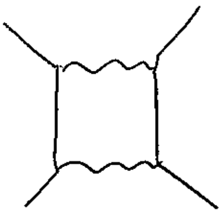
$$(\quad)_{\mu\nu} = i\underline{(q^2 \eta_{\mu\nu} - q_\mu q_\nu)}\,\Pi(q^2)$$

use q^2. or $q_\mu \cdot q_\nu$ not k^2 $\Rightarrow$ 0

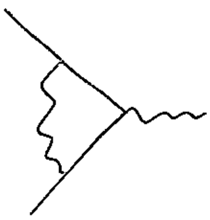
| logarithmic. div

$\Rightarrow$ $D = 0$. logarithmic.

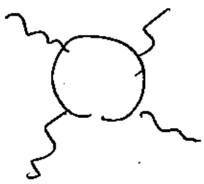
eg.



$$\Rightarrow D = (-1)^{\psi} \times 2 + (-2)^{\gamma} \times 2 + 4 = -2$$

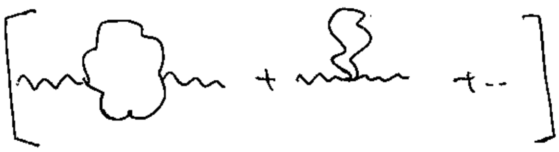


$\Rightarrow$ $D = 0$ .. but. $[\gamma^\mu, \gamma^\nu]\, \underline{q_\nu}$. $\Rightarrow$ -1.



$D = 0$. but eventually $\Rightarrow -4$

| no divergence



$\Rightarrow$ logarithmic

$$D = 4\cdot(\#L) - I_\psi - 2I_\gamma$$

$$2\times(\#V) = 2I_\psi + E_\psi$$

$$(\#V) = 2I_\gamma + E_\gamma$$

$$D = 4\left[(\#C) + (\#I_{tot}) - (\#V)\right] - I_\psi - 2I_\gamma$$

$$= 4 + (3I_\psi + 2I_\gamma) - \left[\overset{4(\#V)}{\cancel{3(\#V) + (\#V)}}\right]$$

$$\cancel{= 4 + (3I_\psi + 2I_\gamma) - \left[\left(3I_\psi + \tfrac{3}{2}E_\psi\right) + (2I_\gamma + E_\gamma)\right]}$$

$$= 4 + \left(3(\#V) - \tfrac{3}{2}E_\psi\right) + \left((\#V) - E_\gamma\right) - 4(\#V) = \boxed{4 - \tfrac{3}{2}E_\psi - E_\gamma}$$

for any higher loop amplitudes.

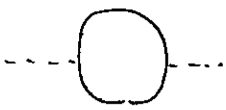
$D \le 4$, $E_\psi, E_\gamma > 0$: only finite amplitudes w/ $D > 0$

All the $D \ge 0$ amplitudes exist in $\mathcal{L}_{QED}$ (w counterterms).

Yukawa theory

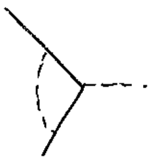
eg. $\mathcal{L} = \bar\Psi[i\gamma^\mu\partial_\mu - m]\Psi + \frac{1}{2}(\partial_\mu\phi)^2 - g\bar\Psi\Psi\phi - \frac{M^2}{2}\phi^2$ $\quad$ ϕ: real scalar.

$$\boxed{D = 4 - \tfrac{3}{2}E_\psi - E_\phi}$$



$D=2$.

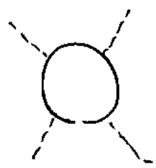
mass, wavefn ren. of ϕ.



$D=0$

g + g_r.

but



$D=0$.

need $\frac{\lambda}{4!}\phi^4$ term in $\mathcal{L}$.

otherwise, regulator scale remains.

(λ_r may just happen to be 0 though)

To recap

- particle species a: propagator p^{-k_a}
- vertex (interaction) type i: (field "a") $\times\, n_{ai}$ and $d_i \times (\partial_\mu)$

$$\boxed{D = 4L - \sum_a (k_a \cdot I_a) + \sum_i (d_i \cdot V_i)}$$

- for particle species "a": $\boxed{(2I_a + E_a) = \sum_i n_{ai} V_i}$
- $\boxed{L - (C-1) = \left(\sum_a I_a\right) - \left(\sum_i V_i\right)}$

$$\Rightarrow D = 4 + \sum_a (4-k_a) I_a + \sum_i (d_i - 4) V_i$$

$$= 4 + \sum_i \Big(\underbrace{\sum_{a} \left(\tfrac{4-k_a}{2}\right) n_{ai} + d_i}_{= \Delta_i} - 4 \Big) V_i - \sum_a \left(\tfrac{4-k_a}{2}\right) E_a .$$

- $\Delta_i \equiv \sum_a \left(\frac{4-k_a}{2}\right) n_{ai} + d_i$ (eg) (naive) operator dim.

? $\boxed{(4-\Delta_i): \text{ mass-dimension of the coefficient.}}$ of the vertex "i"

~~scalar~~ $\left(\frac{4-k_a}{2}\right)$: (naive) mass-dim. of field "a"

$$\begin{cases} \text{scalar} : \dfrac{i}{p^2 - m^2 + i\varepsilon} & \Rightarrow 1 \\ \text{vector} : \dfrac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon} & \Rightarrow 1 . \quad \Psi : \dfrac{i(\not p + m)}{p^2 - m^2} \Rightarrow 3/2 . \end{cases}$$

$\boxed{\text{QED}, \quad (\text{Yukawa} + \phi^4) \text{ theory}}$.

Both have only interactions where $\Delta_i = 4$ $(4-\Delta_i = 0)$

$\Rightarrow$ limited variety of amplitudes ($\{E_a\}$) where $D \geq 0$ (divergent).

$\boxed{\text{If all the interactions of a theory satisfy } \Delta_i \leq 4. \ (4-\Delta_i \geq 0).}$

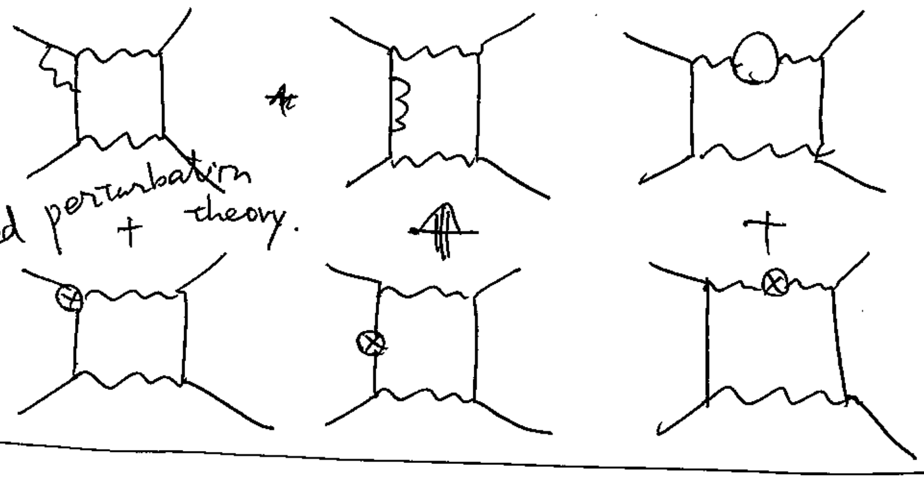
Renormalizable QFT

If all possibly divergent amplitudes have corresponding terms in $\mathcal{L}$ ~~so that~~ ($\Leftrightarrow$ $\exists$ counter terms),
such amplitudes can be written in terms of observed (renormalized) coupling coefficients (and kinematical variables) w/o referring to the regulator scale.

subtlety 1

subdiagram may diverge even when $D<0$.

eg.



but in renormalized perturbation theory

subtlety 2

$\exists$ counter term alone. is not enough.

$$\text{eg.}\quad \Lambda^2 \ln\left(\frac{\Lambda^2}{p^2}\right) - \underset{\text{c.t.}}{\Lambda^2 \ln\left(\frac{\Lambda^2}{m^2}\right)} \Rightarrow \Lambda^2 \ln\left(\frac{m^2}{p^2}\right)$$

finite at the kinematics for the renormalization condition.

Bogoliubov – Parasiuk, Hepp. . Zimmermann.

renormalizable

§ 5.4 Renormalized Perturbation Theory of "Non-Renormalizable" Theories.

Historically

Renomalizable theories :

Renormalizable QFT's. which need only a finite # of renormalized coefficients.

$$\boxed{D = 4 - \left(\frac{3}{2}E_\psi + E_\phi + E_A\right) + \sum_i (\Delta_i - 4) V_i}$$

Δ_i : (naive) operator dim.

✓ If $(\Delta_i - 4) > 0$ in one of interaction terms..

$D > 0$ unlimitedly.

✓ ~~set~~ infinite # of renormalization conditions.

what's wrong?

divergent ⇒ subtract by counter terms.

$(g_i)_r = \mathrm{fn}_i(\{g_k\text{'s}\} ; \Lambda).$

↑ many ↑ many.

doable? well-defined?

✓ In the real world..

ν mass may be due to $\mathcal{L}_{int} = \frac{1}{M}\text{"}\Psi\Psi\phi\phi\text{"}$

lepton Higgs field.

$\Delta = 5$ dimension-5 operator.

★ matrix element $\mathcal{M}$.

S-matrix : $[\text{mass}]^{(\cancel{4}-E)}$ (out-state) bra, (in-state) ket : $[\text{mass}]^{-1}$.

$$S = (2\pi)^4 \delta^4(p_{in} - p_{out})\, i\mathcal{M} \qquad \langle \vec{p} \,|\, \vec{q} \rangle = (2\pi)^3 \delta^3(\vec{p} - \vec{q})\,(2E_{\vec{p}})$$

$$\Rightarrow \quad \mathcal{M} \; : \; [\text{mass}]^{(4-E)}$$

spinor fields: spinor polarization : $[\text{mass}]^{+1/2}$.

$\mathcal{M}$ w/o polarization: $\boxed{[\text{mass}]^{4-(\frac{3}{2}E_\psi + E_\phi + E_A)}}$

★ coefficients of mass dimensions.

$\mathcal{L}_{int} \supset m^{(4-\Delta_j)} \mathcal{O}_j$ renormalizable operators $(4-\Delta_j \geq 0)$

$\mathcal{L}_{int} \supset \dfrac{1}{M^{\Delta_i - 4}} \mathcal{O}_i$ non-renormalizable operators. $(\Delta_i - 4 > 0)$

$$\mathcal{M} \propto \underbrace{\prod_i \left(\frac{1}{M^{(\Delta_i - 4)V_i}} \right) \prod_j \left(m^{(4-\Delta_j)\cdot V_j} \right)}_{[\text{mass}]^{-\sum_i (\Delta_i - 4)V_i - \sum_j (\Delta_j - 4)V_j \,\equiv\, (-\nu)}}$$

$\Rightarrow$ after subtraction (renormalization conditions set ~~to~~ at momenta $\ll M$)

$$\mathcal{M} \sim \prod_i \left(\frac{1}{M^{(\Delta_i - 4)V_i}} \right) (\text{Energy})^{4 - (\frac{3}{2}E_\psi + E_\phi + E_A) + \sum_i (\Delta_i - 4)V_i}$$

$$\sim (\text{Energy})^{4 - (\frac{3}{2}E_\psi + E_\phi + E_A)} \cdot \left(\frac{E}{M} \right)^{\nu}.$$

renormalized perturbation theory for a given amplitude (matrix element) at a given level of precision $\underline{\nu}$

$\Rightarrow$ only a finite # of non-renormalizable vertices contribute.