§ 5.4 Renormalized Perturbation Theory of "Non-Renormalizable" Theories.

Historically

Renormalizable theories :

Renormalizable QFT's. which need only a finite # of renormalized coefficients.

$$\boxed{D = 4 - \left(\tfrac{3}{2}E_\psi + E_\phi + E_A\right) + \sum_i (\Delta_i - 4)\, V_i}$$

Δ_i : (naive) operator dim.

✓ If $(\Delta_i - 4) > 0$ in one of interaction terms..

$D > 0$ unlimitedly.

✓ ~~set~~ infinite # of renormalization conditions.

what's wrong?

divergent ⇒ subtract by counter terms.

$$(g_i)_r = \text{fn.}(\{g_k\text{'s}\} ; \Lambda).$$

$\uparrow$ many $\qquad \uparrow$ many.

doable? well-defined?

✓ In the real world..

ν mass may be due to $\mathcal{L}_{int} = \frac{1}{M}\, \text{``}\overbrace{\Psi\Psi}^{\text{lepton}}\overbrace{\phi\phi}^{\text{Higgs field}}\text{''}$

$\Delta = 5$ dimension-5 operator.

☆ matrix element $\mathcal{M}$.

S-matrix : $[\text{mass}]^{(\cancel{4}-E)}$ (out-state) bra, (in-state) ket : $[\text{mass}]^{-1}$.

$$S = (2\pi)^4 \delta^4(p_{in} - p_{out})\, i\mathcal{M} \qquad \langle \vec{p} \,|\, \vec{q} \rangle = (2\pi)^3 \delta^3(\vec{p}-\vec{q})(2E_{\vec{p}})$$

$$\Rightarrow \quad \mathcal{M} : [\text{mass}]^{(4-E)}$$

spinor fields: spinor polarization : $[\text{mass}]^{+1/2}$.

$\mathcal{M}$ w/o polarization: $\boxed{[\text{mass}]^{4-\left(\frac{3}{2}E_\psi + E_\phi + E_A\right)}}$

☆ coefficients of mass dimensions.

$$\begin{cases} \mathcal{L}_{int} \supset m^{(4-\Delta_j)} \mathcal{O}_j & \text{renormalizable operators} \quad (4-\Delta_j \geq 0) \\ \mathcal{L}_{int} \supset \dfrac{1}{M^{\Delta_i - 4}} \mathcal{O}_i & \text{non-renormalizable operators.} \quad (\Delta_i - 4 > 0) \end{cases}$$

$$\mathcal{M} \propto \underbrace{\prod_i \left(\frac{1}{M^{(\Delta_i-4)V_i}} \right) \prod_j \left(m^{(4-\Delta_j)\cdot V_j} \right)}_{[\text{mass}]^{-\sum_i (\Delta_i - 4)V_i - \sum_j (\Delta_j - 4) V_j \,\equiv (-\nu)}}$$

$\Rightarrow$ after subtraction $\left(\begin{array}{l}\text{renormalization conditions} \\ \text{set } \cancel{\text{fo}} \text{ at momenta} \ll M\end{array}\right)$

$$\mathcal{M} \sim \prod_i \left(\frac{1}{M^{(\Delta_i - 4)V_i}} \right) (\text{Energy})^{4 - \left(\frac{3}{2}E_\psi + E_\phi + E_A\right) + \sum_i (\Delta_i - 4)V_i}$$

$$\sim (\text{Energy})^{4 - \left(\frac{3}{2}E_\psi + E_\phi + E_A\right)} \cdot \left(\frac{E}{M}\right)^{\nu}.$$

renormalized perturbation theory for a given amplitude (matrix element) **at a given level of precision $\underline{\nu}$**

$\Rightarrow$ only a finite # of non-renormalizable vertices contribute.

§6. Renormalization Group.

§6.1. Variations of Renormalization Conditions.

For QED, we chose (in §5)

- ✓ $\langle \psi \bar{\psi} \rangle$ pole mass : $\boxed{m}$
- ✓ $\psi - \bar{\psi} - A_\mu$ coupling (electric charge) at $q_\mu = 0$: $\boxed{e_r}$
- ✓ $\langle \psi \bar{\psi} \rangle$ canonical normalization at $p^2 = m^2$.
- ✓ $\langle A_\mu A_\nu \rangle$ " " at $p^2 = 0$ (pole: on-shell)

but not unique ....

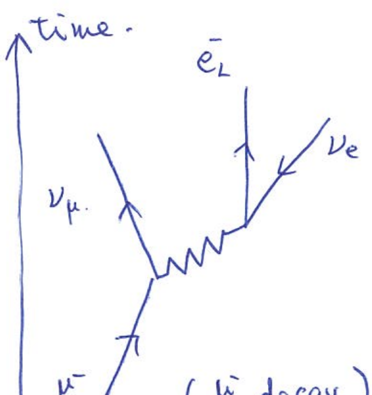
<u>eg. 1</u> Electroweak theory. $SU_L(2) \times U(1)_Y \to U(1)_{QED}$.
symmetry breaking by a scalar field condensation.

<u>parameters</u>

$$\begin{cases} g,\ g' & (\text{coupling constants of } SU(2)_L \times U(1)_Y) \\ \langle \phi \rangle = v/\sqrt{2}. \end{cases}$$

<u>observables</u> $\alpha_{QED} = \left(e^2/4\pi\right)$, m_Z, m_W, $G_\mu(\mu \to \nu_\mu + e + \bar{\nu}_e)$

at tree level.



(μ^- decay)

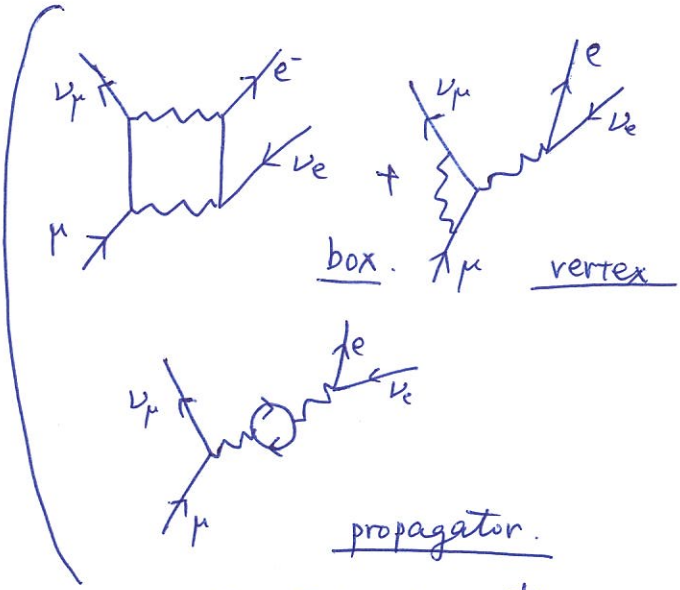
$$m_W = g\frac{v}{2}, \quad m_Z = \sqrt{g^2 + g'^2}\,\frac{v}{2}, \quad e = \frac{g g'}{\sqrt{g^2 + g'^2}}, \quad \frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8 m_W^2}$$

$$\Rightarrow \boxed{\frac{G_\mu}{\sqrt{2}} = \frac{\pi\, \alpha_{QED}}{2 m_W^2 \left(1 - \frac{m_W^2}{m_Z^2}\right)}}$$

At 1-loop level.

$$\frac{\pi\alpha_{QED}}{2m_W^2\left(1-\frac{m_Z^2}{m_W^2}\right)} \neq \frac{G_\mu}{\sqrt{2}}$$

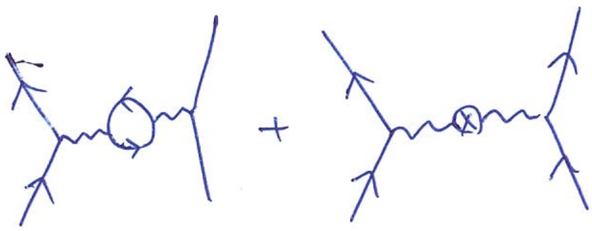
$$\equiv \frac{\pi\alpha_{QED}}{2m_W^2\left(1-\frac{m_Z^2}{m_W^2}\right)(1-\Delta r)}$$



quantum correction to $(\mu^- \to \nu_\mu e^- \bar{\nu}_e)$

use 3 observables to fix the renormalized values of the theory parameters. g, g', v.

eg. 2 scattering at large momentum transfer.



$$\frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}\Big[\underset{\substack{\uparrow\\ \text{tree}}}{1} + \underset{\substack{\uparrow\\ \text{1-loop}}}{\Pi^{(1)}_{ren}(q^2)} + \underset{\substack{\uparrow\\ \text{2-loop.}}}{\cdots}\Big]$$

$$\Pi^{(1)}_{ren}(q^2) = \Pi^{(1)}(q^2) - \Pi^{(1)}(q^2=0) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{reg}^2}\right) - \ln\left(\frac{m_e^2}{M_{reg}^2}\right)\right\}$$

fixed order

$$[1+\Pi^{(1)}_{ren}(q_0^2)] \not\approx \frac{1}{[1-\Pi^{(1)}_{ren}(q_0^2)]} \quad \text{if} \quad \frac{\alpha_{QED}}{\pi}\ln\left(\frac{(-q)^2}{m_e^2}\right) \not\ll 1$$

$\Rightarrow$ ~~why not using~~

$$\cancel{\langle A^\mu_r(q) A^\nu_r(-q)\rangle = \frac{-i\eta^{\mu\nu}}{[q^2+i\epsilon][1-\Pi^{(1)}_{ren}(q_0^2)]} \sim \frac{-i\eta^{\mu\nu}}{[q^2+i\epsilon][1-\Pi^{(1)}_{ren}(q^2)]}}$$

when $q^2 \sim q_0^2$. ?

remember

$$\langle \{A_*^\mu(q)\, A_*^\nu(-q)\}\rangle \sim \frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon]\,[1-\Pi^{(1)}(q^2)]}$$

↳ large log quant. corr.

$[1-\Pi^{(1)}(0)]\times$ ⇓

$$\langle \{A_r^\mu(q)\, A_r^\nu(-q)\}\rangle \sim \frac{-i\eta^{\mu\nu}[1-\Pi^{(1)}(q^2=0)]}{[q^2+i\varepsilon]\,[1-\Pi^{(1)}(q^2)]} \sim \frac{-i\eta^{\mu\nu}}{[q^2+i\varepsilon]\,[1-\underbrace{[\Pi^{(1)}(q^2)-\Pi^{(1)}(0)]}_{\Pi^{(1)}_{ren}(q^2)}]}$$

why not $\times\,[1-\Pi^{(1)}(q^2=-\vec{q}_0^{\,2})]$ instead?

$$\begin{cases} A_*^\mu = A_r^\nu\sqrt{Z_3} = A_r^\nu \dfrac{1}{\sqrt{1-\Pi(0)}} \\ A^\nu = A_{r(q_0)}^\nu\sqrt{Z_3^{(q_0)}} = A_{r(q_0)}^\nu \dfrac{1}{\sqrt{1-\Pi(-\vec{q}_0^{\,2})}} \end{cases}$$

$$\left(A_{r(q_0)}^\nu = \sqrt{\frac{Z_3}{Z_3^{(q_0)}}}\, A_r^\nu \right)$$

$$\langle \{A_{r(q_0)}^\mu(q)\, A_{r(q_0)}^\nu(-q)\}\rangle \sim \frac{-i\eta^{\mu\nu}[1-\Pi^{(1)}(-|\vec{q}_0|^2)]}{[q^2+i\varepsilon]\,[1-\Pi^{(1)}(q^2)]} \sim \frac{-i\eta_{\mu\nu}}{[q^2]\left[1-\Pi^{(1)}_{ren}(q^2) - \Pi^{(1)}_{ren}(-|\vec{q}_0|^2)\right]}$$

- loop expansion in perturbation with $A^\mu_{r(q_0)}$

$$: 1 + [\Pi^{(1)}_{ren}(q^2) - \Pi^{(1)}_{ren}(q_0^2)] + [\quad]^2 + \cdots$$

$\Rightarrow$ good effieient expansion if $|q^2| \sim |-|\vec{q}_0|^2|$

- finitely different $\begin{bmatrix}\text{counterterm} \\ \text{field normalization}\end{bmatrix}$

only cost to pay

good enough renormalization condition.

$$\langle\Omega|T\{\phi_{r(q_0)}(p)\,\phi_{r(q_0)}(-p)\}|\Omega\rangle \sim \frac{i\,(Z/Z^{(q_0)})}{p^2-m^2+i\varepsilon}.$$

$Z/Z^{(q_0)}$ not necessarily $1 \Rightarrow \underbrace{\times \prod_i \left(Z_i/Z_i^{(q_0)}\right)^{1/2}}_{\text{do not forget to}}$ for S-matrix.

to amputated amplitudes

§ 6.2 Renormalization at Energy Scale "E".

$(\gamma\gamma)$

$$\boxed{\Sigma^{(1)}(q^2; M_{reg}^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\ x(1-x)\ \ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{reg}^2}\right).}$$

$(\psi\bar\psi)$

$$\boxed{\Sigma^{(1)}(p; \Lambda^2) = \frac{e^2}{16\pi^2}\int_0^1 dx\left[-2\not p(1-x) + 4m_e\right]\ln\left(\frac{(1-x)\Lambda^2 + x m_e^2 - x(1-x)p^2}{x m_e^2 - x(1-x)p^2}\right)}$$

$(\psi\bar\psi A)$

$$ie\Gamma^\mu_{(1)} \cong (ie)\frac{e^2}{16\pi^2}\int dx dy \left(2\gamma^\mu\left[\ln\left(\frac{(1-x-y)\Lambda^2 + (x+y)^2 M_e^2 - xyq^2}{(x+y)^2 m_e^2 - xyq^2}\right) + \frac{\left[m^2\{1 - 4(1-x-y) + (1-x-y)^2\} + (1-x)(1-y)q^2\right]}{\{(x+y)^2 m_e^2 - xyq^2\}}\right] - [\gamma^\mu, \gamma^\nu]\, q_\nu \frac{(1-x-y)(x+y)m_e}{\{(x+y)^2 m_e^2 - xyq^2\}}\right)$$

$$\Sigma^{(1)}_{ren}(q^2) \equiv \Sigma^{(1)}(q^2; M_{reg}^2) - \Sigma^{(1)}(\underline{q^2=0}; M_{reg}^2) \sim \ln\left(\frac{m^2}{m^2 - x(1-x)q^2}\right) \quad [q^2=0]$$

$$\Sigma^{(1)}(p;\Lambda) - \Sigma^{(1)}(\underline{p^2 = m_e^2}; \Lambda) \sim \ln\left(\frac{x m_e^2 - x(1-x) m_e^2}{x m_e^2 - x(1-x)p^2}\right) \quad [p^2 \to m_e^2]$$

$$\Gamma^\mu_{(1)}(q;\Lambda) - \Gamma^\mu_{(1)}(\underline{q^2=0};\Lambda) \sim 2\gamma^\mu \ln\left(\frac{(x+y)^2 m_e^2 - xy\times 0}{(x+y)^2 m_e^2 - xyq^2}\right). \quad q^2=0$$

$\Longrightarrow$ always leave $\frac{\alpha_e}{\pi}\ln\left(\frac{E^2}{q^2}\right)$ quantum corrections.

(E: energy scale of renormalization conditions.)

if we choose $E^2 \sim 0$ or m_e^2 ...
perturbative expansion is not efficient when $m_e^2 \ll |q^2|$.

$\Longrightarrow$ $\boxed{\text{renormalize at energy scale } E}$ (often use μ instead.)

different renormalization condition

$\Longrightarrow$ different values for the renormalized coupling constants.

eg. QED $(\psi\bar{\psi}A)$ coupling.

$$ie_r\gamma^\mu \quad (+\, ie_r\Gamma^\mu_{(1),\mathrm{ren}}) \Longrightarrow \left(ie_r\gamma^\mu + ie_r\Gamma^\mu_{(1),\mathrm{ren}}\right)\times\frac{1}{\sqrt{1-A_{\mathrm{ren}}(p^2=-\mu^2)}^{\,2}\sqrt{1-\Pi_{\mathrm{ren}}}}$$

vanish at $q^\mu=0$

coupling at $q^\mu=0$

field : residue $=1$ on-shell

coupling at $[q^2=-\mu^2]$

$(q^2=-\mu^2)$

field residue $=1$ at $(-p^2=\mu^2 \,/\, -q^2=\mu^2.)$

(*) $\Longrightarrow$
$$\boxed{\begin{aligned} ie_r(\mu) \simeq ie_r &\left\{1+\frac{e_r^2}{16\pi^2}\int dx\,dy\; 2\ln\left(\frac{(x+y)^2m_e^2}{(x+y)^2m_e^2+xy\mu^2}\right)\right\}\\ &\times\left\{1+\frac{2}{2}\frac{e_r^2}{16\pi^2}\int_0^1 dx\; -2(1-x)\ln\left(\frac{x^2m_e^2}{xm_e^2+x(1-x)\mu^2}\right)\right\}\\ &\times\left\{1+\frac{1}{2}\frac{e_r^2}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left(\frac{m_e^2+x(1-x)\mu^2}{m_e^2}\right)\right\}\end{aligned}}$$

for $\mu^2 \ll m_e^2$: $\dfrac{\partial e_r(\mu)}{\partial\ln\mu} \sim \dfrac{e_r^3}{\pi^2}\times\left(\dfrac{m_e^2}{\mu^2}\right)$ power suppressed.

$\Rightarrow$ ignore.

$m_e^2 \ll \mu^2$: $\dfrac{\partial e_r(\mu)}{\partial\ln(\mu^2)} \simeq e_r\left(-\dfrac{e^2}{16\pi^2}+\dfrac{e^2}{16\pi^2}+\dfrac{1}{2}\dfrac{e^2}{2\pi^2}\times\dfrac{1}{6}\right)$ $\quad \int_0^1 dx\,x(1-x)$

$$=\frac{e^3}{24\pi^2}$$

renormalization group equation at 1-loop. $\Longrightarrow$ $\boxed{\dfrac{\partial(4\pi/e^2)}{\partial\ln\mu} = -\dfrac{2}{3\pi}}$

$(m_e \ll \mu \ll m_\mu)$

$$\boxed{\left(\frac{4\pi}{e^2}\right)(\mu) = \frac{1}{\alpha(\mu)} \simeq (\mathrm{const}) - \frac{2}{3\pi}\ln\left(\frac{\mu}{m_e}\right)}$$

(*) correct only for log-part.

running coupling constant.

RG equation :

$$\boxed{\begin{array}{l}\text{relation among renormalized coupling constants} \\ \quad \text{for renormalized conditions} \\ \qquad \text{at different energy scales.}\end{array}}$$

$$\frac{\partial g(\mu)}{\partial \ln \mu} = \frac{\partial}{\partial \ln(\sqrt{-q^2})}\left[g^{ren}(q^2)\right]\Bigg|_{q^2=-\mu^2} + g(\mu) \times \frac{\partial}{\partial \ln \mu} \ln\left[\prod_i \left(\frac{Z_i}{Z_i^{(\mu)}}\right)^{-1/2}\right]$$

$$\left(\begin{array}{l}\text{irreducible amplitudes} \\ \text{(amputated)} \qquad \text{scale as} \quad \prod_i \left(\frac{Z_i}{Z_i^{(\mu)}}\right)^{-1/2} \\ \quad \text{(like coefficients)}\end{array}\right)$$

$$= -\frac{\partial}{\partial \ln \Lambda}\left[g(q^2;\Lambda)\right] + g(\mu)\frac{\partial}{\partial \ln \Lambda}\ln\left(\prod_i Z_i^{-1/2}\right)$$

$$\left\{\begin{array}{l}\boxed{\Delta g^{r} \sim \ln\left(\frac{\Lambda^2}{q^2}\right) \Rightarrow g^{ren} \sim \ln\left(\frac{m^2}{q^2}\right).} \\ \boxed{Z^{(\mu)} \sim \ln\left(\frac{\Lambda^2}{m^2+\mu^2}\right), \quad Z \sim \ln\left(\frac{\Lambda^2}{m^2}\right)}\end{array}\right.$$

$$\left\{\begin{array}{l}\gamma \equiv -\frac{\partial}{\partial \ln \mu}\left(\ln\sqrt{\frac{Z}{Z^{(\mu)}}}\right) \\ \frac{\partial g(\mu)}{\partial \ln \mu} \equiv \beta_g.\end{array}\right.$$

(β- fcn : determined from log divergence part.)