RG equation :

relation among renormalized coupling constants
for renormalized conditions
at different energy scales.

$$\frac{\partial g(\mu)}{\partial \ln \mu} = \frac{\partial}{\partial \ln(\sqrt{-q^2})}\left[g^{ren}(q^2)\right]\Big|_{q^2=-\mu^2} + g(\mu) \times \frac{\partial}{\partial \ln \mu} \ln\left[\prod_i \left(\frac{Z_i}{Z_i^{(\mu)}}\right)^{-1/2}\right]$$

(irreducible amplitudes. (amputated) scale as $\prod_i \left(\frac{Z_i}{Z_i^{(\mu)}}\right)^{-1/2}$ (like coefficients))

$$= -\frac{\partial}{\partial \ln \Lambda}\left[g(q^2;\Lambda)\right] + g(\mu)\frac{\partial}{\partial \ln \Lambda} \ln\left(\prod_i Z_i^{-1/2}\right)$$

$$\begin{cases} \boxed{\Delta g^* \sim \ln\left(\frac{\Lambda^2}{q^2}\right) \Rightarrow g^{ren} \sim \ln\left(\frac{m^2}{q^2}\right).} \\ \boxed{Z^{(\mu)} \sim \ln\left(\frac{\Lambda^2}{m^2+\mu^2}\right), \quad Z \sim \ln\left(\frac{\Lambda^2}{m^2}\right)} \end{cases}$$

$$\begin{cases} \gamma \equiv -\frac{\partial}{\partial \ln \mu}\left(\ln\sqrt{\frac{Z}{Z^{(\mu)}}}\right) \\ \frac{\partial g(\mu)}{\partial \ln \mu} = \beta_g. \end{cases}$$

(β-fcn : determined from log divergence part.)

Dimensional Regularization

An easy way to $\left\{ \begin{array}{l} \text{calculate } \beta\text{-fn} \\ \text{renormalize. (regularize \& subtract)} \end{array} \right\}$

Loop momentum integration.

$$\frac{d^4k}{(2\pi)^4} \Rightarrow \frac{d^nk}{(2\pi)^n}(\mu)^{4-n} \Rightarrow i\,\frac{vol(S_{n-1})}{(2\pi)^n\cdot 2}\int dK\, K^{\frac{n}{2}-1}$$

after appropriate shift.

$vol(S_{n-1})$:

$$\int d^nx\; e^{-\sum_i (x_i)^2} = \left(\int_{-\infty}^{+\infty} dx\, e^{-x^2}\right)^n = \pi^{\frac{n}{2}}$$

$$\parallel$$

$$\int_0^{+\infty} dr\; r^{n-1}\, vol(S_{n-1})\cdot e^{-r^2} = \frac{vol(S_{n-1})}{2}\cdot\int_0^{+\infty} dR\; R^{\frac{n}{2}-1} e^{-R} = \frac{vol(S_{n-1})}{2}\Gamma\left(\frac{n}{2}\right)$$

$(R=r^2)$

$$\Rightarrow \boxed{\frac{vol(S_{n-1})}{2} = \frac{\pi^{n/2}}{\Gamma(n/2)}}$$

Idea: $\displaystyle\int_0^{\Lambda^2} dK\, \frac{K}{[K+m^2-x(1-x)q^2]^2} \simeq \ln\left(\frac{\Lambda^2+m^2-x(1-x)q^2}{m^2-x(1-x)q^2}\right)-1.$

Logarithmically divergent.

but

$$\lim_{\Lambda\to+\infty}\int^{\Lambda^2} dK\, \frac{K^{\frac{n}{2}-1}\quad \mu^{4-n}}{[K+m^2-x(1-x)q^2]^2} = \left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{2-\frac{n}{2}}\int_0^{+\infty} dy\, \frac{y^{\frac{n}{2}-1}}{(y+1)^2}$$

$$\parallel$$

$$\int_0^1 d\xi\, (1-\xi)^{\frac{n}{2}-1}\,\xi^{1-\frac{n}{2}}$$

convergent if $\frac{n}{2}>0$ and $2-\frac{n}{2}>0 \Leftrightarrow \boxed{4>n}$

$$= \left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{2-\frac{n}{2}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}$$

$$\int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \mu^{4-n} \frac{1}{[k^2 - m^2 + x(1-x)q^2]^2} = \int_0^1 dx \; i \frac{\pi^{n/2}}{(2\pi)^n \Gamma(n/2)} \Gamma\left(\tfrac{n}{2}\right) \Gamma\left(2-\tfrac{n}{2}\right) \left(\frac{\mu^2}{m^2 - x(1-x)q^2}\right)^{2-\frac{n}{2}}$$

$$= \int_0^1 dx \frac{i}{(4\pi)^2} \underbrace{\Gamma\left(2-\tfrac{n}{2}\right)\left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)^{2-\frac{n}{2}}}_{\text{small } (2-\frac{n}{2})}$$

$(-\gamma = -0.5772)$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{1}{z}\left(\Gamma(1) + \left.\frac{d\Gamma}{dz}\right|_{z=1} \times z + \cdots\right)$$

$$\left[\frac{1}{(2-\frac{n}{2})} + (-\gamma) + \cdots\right]\left[1 + \left(2-\tfrac{n}{2}\right)\ln\left(\frac{\mu^2 4\pi}{m^2 - x(1-x)q^2}\right) + \cdots\right]$$

$$= \frac{1}{(2-\frac{n}{2})} + (-\gamma) + \ln\left(\frac{\mu^2 4\pi}{m^2 - x(1-x)q^2}\right) + O\left(2-\tfrac{n}{2}\right).$$

- still divergent. when $n \to 4$.
- empirical rule: $\frac{1}{(2-\frac{n}{2})} \Longleftrightarrow \ln(\Lambda^2)$

  $\left(\nexists \; \frac{1}{(1-n/2)} \text{ pole} \Leftrightarrow \exists \text{ quadratic divergence.} \right)$

  β-function as coefficients of $\underline{\ln(\Lambda^2)}$

  $\Rightarrow \underline{\frac{1}{(2-n/2)}}$.

- renormalization at scale μ.

  simply subtract $\frac{1}{(2-\frac{n}{2})} + \left(-\gamma + \ln(4\pi)\right)$

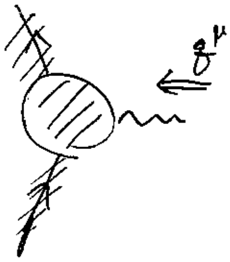
  renormalization scheme: ~~mini~~ $\overline{MS}$

  minimal subtraction or

§ 6.3 Meaning of Running Coupling Constants. I

☆ Observables (eg. $|\mathcal{M}|^2$ for a given kinematics)

should not depend on the choice of renormalization scale.



$=$

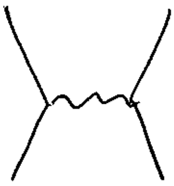
$+$ radiative correction.

$\approx (i e_r(|q|) \gamma^\mu)$ $\qquad \cdot = i e_r \gamma^\mu$

due to the difference between $\vec{q} = \vec{0}$ & $\vec{q} \neq \vec{0}$.

☆ good approximation at fixed order perturbation.

eg. QED scattering amplitude.

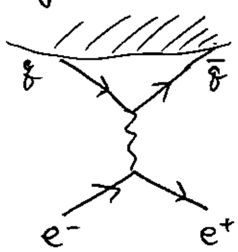


$+ \cdots$

if at tree level. $\left(i\mathcal{M} \sim i \frac{e_r^2(\mu)}{q^2} \eta_{\mu\nu} \times (\text{polarizatio}\right.$

corrections of order $\times \frac{\alpha_e}{\pi} \times \ln\left(\frac{-q^2}{\mu^2}\right)$ remain.

eg. total hadron σ



$$\sigma_{tot} = \frac{4\pi\alpha_e^2}{3s}(Q_q)^2 \times 3 \times \left[1 + \frac{\alpha_s(\mu^2)}{\pi} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 \left\{ c_2 + \pi b \ln\left(\frac{s}{\mu^2}\right)\right\} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3 \left[c_3 + \left\{\pi b \ln\left(\frac{s}{\mu^2}\right)\right\}^2 - \cdots \ln\left(\frac{s}{\mu^2}\right)\right] + \cdots \right]$$

$$\left(\frac{\partial}{\partial \ln \mu^2} \left(\frac{1}{\alpha_s(\mu^2)}\right) = b + \mathcal{O}(\alpha_s^*). \right)$$

$\boxed{\text{take } \mu^2 \simeq s \,!}$

☆ resum $\sum_k \left(\frac{\alpha_s}{\pi}\right)^k \left(\ln\left(\frac{\mu_1^2}{\mu_0^2}\right)\right)^k$

$$\alpha_s(\mu_1) \simeq \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0)\, b \ln\left(\frac{\mu_1^2}{\mu_0^2}\right)}$$

leading log resummation.

§6.4 Wilson's interpretation of renormalization group.

(meaning of running coupling constants II)

technically ....

eg. quantum correction. $\Pi^{(1)}_{\text{ren.}(\mu)}(q^2)$

$$= \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{reg}^2}\right) - \ln\left(\frac{m_e^2 + x(1-x)\mu^2}{M_{reg}^2}\right)\right\}$$

$\Uparrow$ counter term.

or... in fermion wavefcn renormalization. (subtract)

Originally ... from integrals like.

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2 + m^2 + x(1-x)q^2]^2} \Rightarrow i\frac{1}{16\pi^2}\int_0^{\Lambda^2} dK\, \frac{K}{[K + m^2 - x(1-x)q^2]^2}$$

$$\approx \int^{\Lambda^2}\frac{dK}{K} \Longrightarrow \int^{\mu^2}\frac{dK}{K}$$

$\boxed{\text{effectively replaced in this way}}$

at large $K = (k^2)_E \gg m^2, |q^2|$ etc.

If we take $\mu^2 \approx |q^2|$ (kinematics of interest)

$\boxed{\text{Theory } \boxed{\text{regularized at } \Lambda^2, M_{reg}^2 \text{ with } M, e.} \Longleftrightarrow \text{Theory } \boxed{\text{loop momentum integration cut-off at } \mu^2. \text{ with } m(\mu)\ \ e_r(\mu)}}$

In path-integral language.

$$\boxed{Z = \int_{|k|\lesssim \Lambda, M_{reg}} \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{iS[A,\psi,\bar\psi;\, M, e]}} \Longrightarrow \boxed{Z = \int_{|k|\lesssim\mu}\mathcal{D}A_\mu\mathcal{D}\psi\mathcal{D}\bar\psi\left(e^{iS[\quad;\, m(\mu), e_r(\mu)]} + \cdots\right)}$$

$$S \to S + \left(\int A_\mu J^\mu + \int \psi^\dagger K + h.c.\right)$$

$Z[J, K]$

$\boxed{\text{If interested only in } Z[J,K] \text{ for } J(q), K(p) \text{ etc. } \underline{\text{integrate}}\ A(k), \psi(k) \text{ etc. } |q|, |p| \ll |k| \text{ first!}}$

$\underline{(+\,--\,)\text{ part.}}$ $\longrightarrow$ (add non-renormalizable operators)

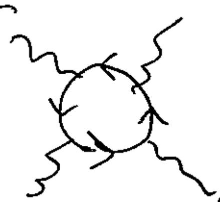
low mom. + high mom

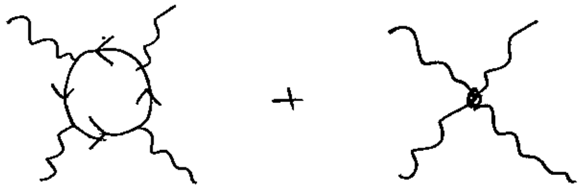
$\Longrightarrow$

low mom. only ($|k| \leq \mu$.)

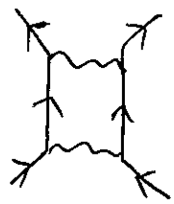
and A_μ. Ψ. $e_r(\mu)$ different normalization

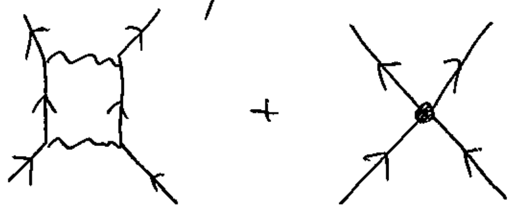
$\Rightarrow$ already seen.





$\left(\begin{matrix}\text{low}\\+\\\text{high}\end{matrix}\right)$ loop momentum $\Rightarrow$ low mom. only





extra terms. $+\frac{1}{M^4} F..F..F..F..$, $+(\bar\Psi\Psi\bar\Psi\Psi)\frac{1}{M^2}$.

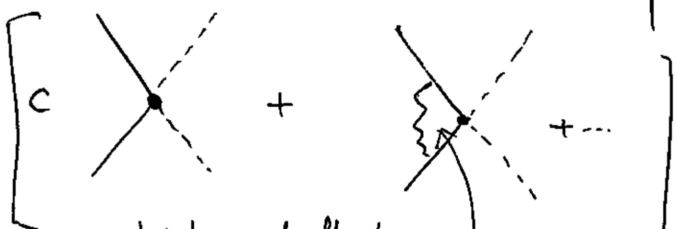
- $\int dK \frac{K}{[K+m^2-q^2]^4}$ $\quad$ $\left[\int dK \text{ for } \mu^2 < K\right]$.
- such integrals ... dominated by IR region.

$\Rightarrow$ unimportant if $m^2, |q|^2 \ll \mu^2$.

Non-renormalizable operators

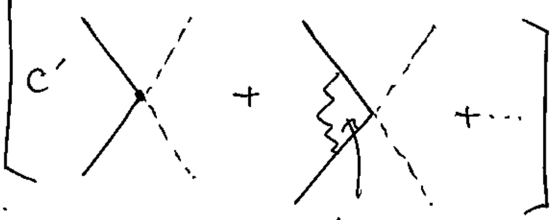
like dim.5 ν mass $\Delta\mathcal{L} = (\bar\Psi\Psi\phi\phi)\frac{1}{M}$.

β-decay 4-fermi op. $\Delta\mathcal{L} = \frac{1}{M^2}(\bar\Psi\gamma^\mu\Psi)(\bar\Psi\gamma_\mu\Psi)$



high cut-off theory. low / high } loop momentum.

$\Rightarrow$



low mom. only $|k| < \mu$.

C': μ-dependent.