§6.4 Wilson's interpretation of renormalization group.

(meaning of running coupling constants II)

technically ----

eg. quantum correction. $\Pi^{(1)}_{ren.(\mu)}(q^2)$

$$= \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left\{\ln\left(\frac{m_e^2 - x(1-x)q^2}{M_{reg}^2}\right) - \ln\left(\frac{m_e^2 + x(1-x)\mu^2}{M_{reg}^2}\right)\right\}$$

$\Uparrow$ counter term.

or... in fermion wavefcn renormalization. (subtract)

Originally ... from integrals like.

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2 + m^2 + x(1-x)q^2]^2} \Rightarrow i\frac{1}{16\pi^2}\int_0^{\Lambda^2} dK\,\frac{K}{[K+m^2-x(1-x)q^2]^2}$$

$$\approx \int^{\Lambda^2}\frac{dK}{K} \Longrightarrow \int^{\mu^2}\frac{dK}{K}$$

effectively replaced in this way

at large $K = (k^2)_E \gg m^2, |q^2|$ etc.

If we take $\mu^2 \simeq |q^2|$ (kinematics of interest)

Theory regularized at Λ^2, M_{reg}^2 with M, e. $\Longleftrightarrow$ Theory. loop momentum integration cut-off at μ^2. with $m(\mu)$ $e_r(\mu)$

In path-integral language.

$$Z = \int_{|k|\lesssim\Lambda, M_{reg}} \mathcal{D}A_\mu \mathcal{D}\Psi\mathcal{D}\bar\Psi\, e^{iS[A,\Psi,\bar\Psi;\,M,e]} \Longrightarrow Z = \int_{|k|\lesssim\mu}\mathcal{D}A_\mu\mathcal{D}\Psi\mathcal{D}\bar\Psi\, e^{\left(iS[\ \ ;\,m(\mu),e_r(\mu)] + \cdots\right)}$$

$$S \to S + \left(\int A_\mu J^\mu + \int \Psi^\dagger K + h.c.\right)$$

$Z[J,K]$

If interested only in $Z[J,K]$ for $J(q)$, $K(p)$ etc. integrate $A(k)$, $\Psi(k)$ etc. $|q|, |p| \ll |k|$ first!

$(+ - -)$ part. $\longrightarrow$ (add non-renormalizable operators)

low mom.
high mom

$\Rightarrow$

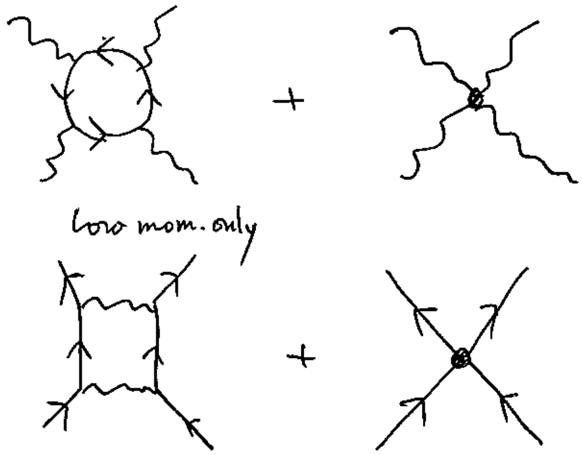
low mom. only
$(|k| \le \mu.)$

and $A_\mu . \psi . \; e_r(\mu)$ different normalization

$\Rightarrow$ already seen.

$\begin{pmatrix} \text{low} \\ + \\ \text{high} \end{pmatrix}$ loop momentum $\Rightarrow$

low mom. only



extra terms. $+\frac{1}{M^4} F..F..F..F..$, $+(\bar\psi\psi\bar\psi\psi)\frac{1}{M^2}$

- $\int dk \frac{k}{[k+m^2-q^2]^4}$ $\quad \int dk$ for $\mu^2 < k$
- such integrals ... dominated by IR region.

$\Rightarrow$ unimportant if $m^2, |q|^2 \ll \mu^2$.

Non-renormalizable operators

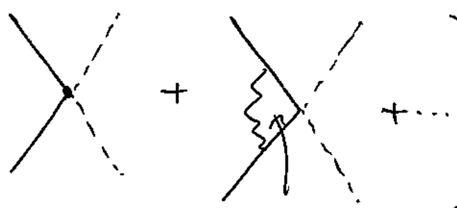
like dim.5 ν mass $\Delta\mathcal{L} = (\bar\Psi\Psi\phi\phi)\frac{1}{M}$

β-decay 4-fermi op. $\Delta\mathcal{L} = \frac{1}{M^2}(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi)$

[C ]

high cut-off theory. low / high } loop momentum.

$\Rightarrow$

[C′ ]

low mom. only $|k| < \mu$.

C′: μ-dependent.

§ 7. Low-energy Effective Theory

Wilson's interpretation on "renormalization at scale μ"

When all the external lines (probes) have momenta $\ll \mu$.

integrate out all the D.O.F. w/. $k \gtrsim \mu$ first!
(= take care of)

$\Longrightarrow$ change coupling constant values. (renormalize coupling constants)

additional operators.

(effectively do the job of high mom. modes)

Take one step further.

✓ for a massive particle ϕ with $\mu \ll M_\phi$.

some $\mathcal{D}\phi(k)$ left. but no on-shell modes for $k^\mu \ll \mu$.

$\longrightarrow$ does not come out as an external state.

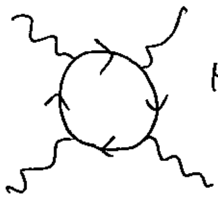
$\Longrightarrow$ integrate out ϕ completely! get it done!

eg. 1. QED w/ $e^+ e^-$, $\mu^+ \mu^-$

$m_e \simeq 0.511$ MeV, $m_\mu \simeq 105$ MeV.

what if we take $\boxed{m_e \ll \mu \ll m_\mu}$??

(i)



$\mu^\pm$ loop.

carry out this integral completely.

$$\Rightarrow \Delta \mathcal{L} = \frac{2\alpha_{QED}^2}{45 m_\mu^4}\left\{\frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{7}{16}\left(F_{\mu\nu}F_{\kappa\lambda}\frac{\epsilon^{\mu\nu\kappa\lambda}}{2}\right)^2\right\}$$

(ii) photon propagator. $\dfrac{e^2}{q^2(1-\Pi_{ren})}$

$$\boxed{\left(\frac{1-\Pi_{ren}(q^2)}{e^2}\right) = \frac{1}{e^2_{*(e\mu)}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 - x(1-x)q^2}{m_i^2 + x(1-x)E^2}\right).}$$

$$\left(\text{cf.} \quad = \frac{1}{e^2_{\overline{MS}}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 - x(1-x)q^2}{E^2}\right)\right).$$

scheme dependence.

$$☆ \quad \frac{1}{e^2_{*(e\mu)}(E)} = \frac{1}{e^2_{*(e\mu)}(E_0)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\sum_i \ln\left(\frac{m_i^2 + x(1-x)E^2}{m_i^2 + x(1-x)E_0^2}\right)$$

$$\boxed{\frac{\partial(1/e^2_{*(e\mu)})}{\partial \ln(E^2)} = -\frac{1}{2\pi^2}\sum_i \underbrace{\int_0^1 dx\, x(1-x)\left(\frac{x(1-x)E^2}{m_i^2 + x(1-x)E^2}\right)}} \rightarrow \begin{cases} 1/6 & (m_i^2 \ll E^2) \\ \frac{1}{30}\left(\frac{E^2}{m_i^2}\right) & (E^2 \ll m \\ \ll 1. \end{cases}$$

☆ when $E \ll m_\mu$.

$$\frac{1}{e^2_{*(e)}(E)} \equiv \frac{1}{e^2_{*(e\mu)}(E)} - \frac{1}{2\pi^2}\int dx\, x(1-x)\ln\left(\frac{m_\mu^2}{m_\mu^2 + x(1-x)E^2}\right)$$

$$\Rightarrow \left(\frac{1-\Pi_{ren}(q^2)}{e^2}\right)\cdot q^2 = q^2\left\{\frac{1}{e^2_{*(e)}(E)} - \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\frac{m_e^2 - x(1-x)q^2}{m_e^2 + x(1-x)E^2}\right)\right\} + \frac{1}{2\pi^2}\left(\frac{1}{30}\right)\frac{(q^2)^2}{m_\mu^2} + \cdots$$

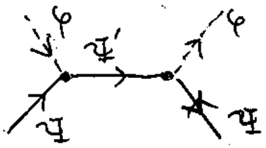
$\Downarrow$ quantum correction in QED w/ $e^+ e^-$ only.

$\Downarrow$ effective operator

$$\Delta\mathcal{L} \propto \left(\frac{e^2}{2\pi^2}\right)\frac{1}{30}\frac{1}{m_\mu^2}\left(F_{\mu\nu}\partial^2 F\right.$$

eg. 2 "see-saw mechanism" $\Leftarrow$ [Weyl fermion (not Dirac fermion) should be used in reality]

$$\mathcal{L} = \bar{\Psi}(i\gamma\cdot D)\Psi + \bar{\Psi}'(i\gamma\cdot D)\Psi' - M\bar{\Psi}'\Psi'$$
$$+ (D^\mu\varphi)^*(D_\mu\varphi) \qquad + \lambda\bar{\Psi}'\Psi\varphi + \lambda^*\bar{\Psi}\Psi'\varphi^*$$



$\boxed{\text{at energy scale } E \ll M.}$

(incl. momentum transfer)

propagator $\dfrac{i[\not{q}+M]}{q^2-M^2+i\epsilon} \simeq -i\dfrac{M}{M^2} = \dfrac{-i}{M}.$

$\cancel{\mathcal{L}_{eff}} = i\mathcal{M} \sim (i\lambda\cdot i\lambda^*)[\varphi^*\bar{\Psi}]\left(\dfrac{-i}{M}\right)[\varphi\Psi]$

$\Uparrow$ reproduce.

$$\boxed{\mathcal{L}_{eff} = \frac{|\lambda|^2}{M}(\varphi^*\bar{\Psi}\Psi\varphi)}$$

in theory w/o Ψ'

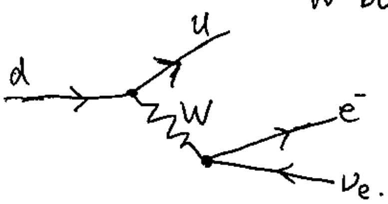
$\langle\varphi\varphi^*\rangle \sim v^2$

$\Rightarrow \Psi_{mass} \sim \dfrac{|\lambda|^2 v^2}{M}$

tiny if $v \ll M$.

eg. 3. 4-fermi operator. (β-decay)

at $E \ll m_W$ W-boson propagator $\dfrac{-i\eta_{\mu\nu}}{q^2-m_W^2} \Rightarrow \dfrac{i\eta_{\mu\nu}}{m_W^2}$



$$\boxed{\mathcal{L}_{eff.} = \left[\bar{u}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)d\right]\left[\bar{e}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)\nu\right]\left(\frac{-g^2}{2m_W^2}\right)}$$

$\uparrow$ in QCD $\times$ QED

$$i\mathcal{M} = (-ig)\left[\bar{e}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\nu\right]\frac{-i\eta_{\mu\nu}}{q^2-m_W^2}(-ig)\left[\bar{u}\gamma^\nu\left(\frac{1-\gamma_5}{2}\right)d\right]\frac{1}{2}$$
$$\hookrightarrow \simeq -i\frac{g^2}{2m_W^2}[\bar{e}\cdots\nu]^\mu[\bar{u}\cdots d]_\mu.$$

Low-Energy Effective Theory

heavy particles can be integrated out for low energy description.

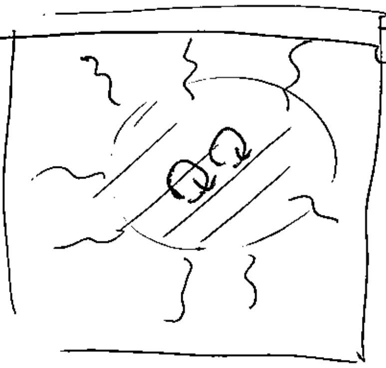
- couplings : renormalized.
- effective operators generated.. no other footprints.

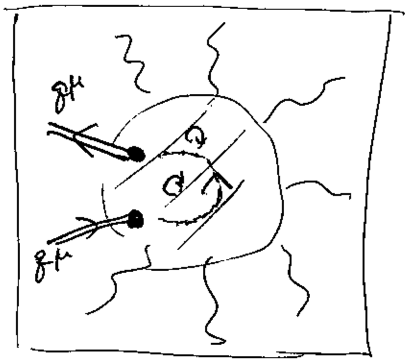
Standard Model is yet another effective theory of some more fund. theory.

§8. Operator Product Expansion.

Low-energy Eff. Theory

by carrying out high-~~loop~~ mom. loop first.





[all ext. mom. are soft.]

[all ext. mom. are soft except 2]

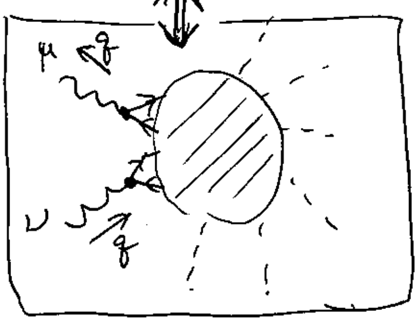
Two operators combined ⇒ no net momentum. flow outside.

Any effective description.

Consider.

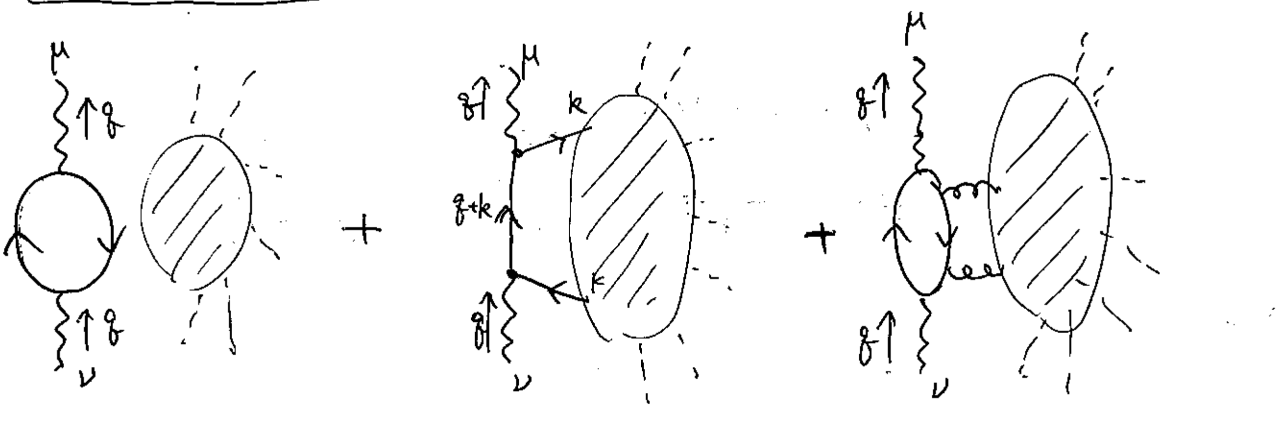
$$-e^2\iint \langle\Omega|T\{\cdots\ J^\mu(x)\, J^\nu(y)\ \cdots\}|\Omega\rangle\ e^{iq'\cdot x}\ e^{-iq\cdot y}\ d^4x\, d^4y.$$

$\Updownarrow$



large momentum q^μ. $(q \simeq q')$

necessarily flow from $J^\nu(y)$ to $J^\mu(x)$.



$$\boxed{\int [ie J^\mu(x)]\,[ie J^\nu(y)]\, e^{iq'\cdot x}\, e^{-iq\cdot y}\, d^4y \quad \text{in } T\{\cdots\}.}$$

$$= i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi^{(1)}_{\text{ren}}(q^2)\cdot e^{i(q'-q)\cdot x}\ \mathbb{1}.$$

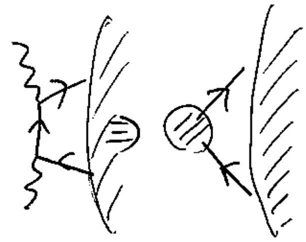
$$+\int d^4y\ e^{iq'\cdot x}\left[ie\,\bar\psi(x)\gamma^\mu\right]\underbrace{\int\frac{d^4k}{(2\pi)^4}\frac{i[(\cancel{q}+\cancel{k})+m]\,e^{-i(q+k)\cdot(x-y)}}{(q+k)^2-m^2+i\varepsilon}}_{\langle 0|T\{\psi_I(x)\bar\psi_I(y)\}|0\rangle.\ \text{propagator.}}(ie)\left[\gamma^\nu\psi(y)\right]e^{-iq\cdot y}$$

approximation. (or expansion).

$$\frac{i[(\cancel{q}+\cancel{k})+m]}{(q+k)^2-m^2+i\varepsilon}\ \to\ \frac{i\cancel{q}}{q^2}$$

$$\int d^4y\,(-ie^2)\frac{q_\lambda}{q^2}\int\frac{d^4k}{(2\pi)^4}\,e^{i(q'-q-k)\cdot x}\,e^{i(q+k-q)\cdot y}\left[\bar\psi(x)\gamma^\mu\gamma^\lambda\gamma^\nu\psi(y)\right]$$

$$\left(\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)} = \delta^4(x-y)\right)$$



$$\simeq\ -ie^2\frac{q_\lambda}{q^2}e^{i(q'-q)\cdot x}\left[\bar\psi(x)\gamma^\mu\gamma^\lambda\gamma^\nu\psi(y)\right]$$

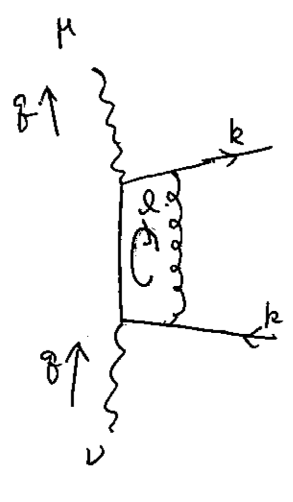
$$= \sum_I C_I(q^\kappa)\cdot O_I(x).$$

k^λ's : momentum in $\psi(y)$ or $\bar\psi(x)$.

$\Rightarrow$ ∂ on ψ or ∂ on $\bar\psi$.

$\Rightarrow$ derivative expansion. $\dfrac{q\cdot\overset{\rightarrow\text{ or }\leftarrow}{\partial}}{q^2}\,O.$

loop correction.



$|q| \ll \mu$: finite integral (ℓ UV)

$\mu \ll q$: log ~~divergence~~ correction. (ℓ)

propagator $\left\{\dfrac{i(\cancel{q}+\cancel{\ell}+\cancel{k})+m}{(q+\ell+k)^2-m^2} \simeq \dfrac{i\cancel{q}}{q^2}\right.$

$$\boxed{C_I(q^2;\alpha_s;\mu)\ \left[O_I(x)\right]_\mu.}$$

{ tree + 1-loop $\mu<\ell$ }

only $k<\mu$.