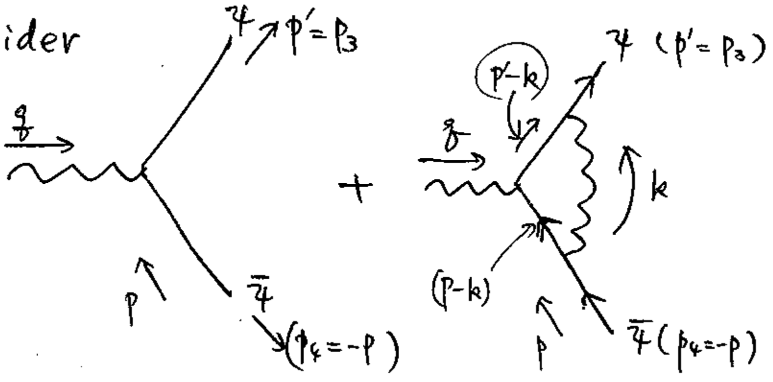
# § 9. Soft and Collinear Divergence

## §9.1 Divergence in Virtual Correction

Consider



in QED.

$$(ie\Gamma^\mu) = ie\gamma^\mu + ie\left(\frac{e^2}{16\pi^2}\right)\underset{\substack{\blacktriangle\\0\ \ 1}}{\int} dx\,dy\left(2\gamma^\mu\left\{\ln\left(\frac{(1-x-y)\Lambda^2+(x+y)^2m_e^2-xy\,q^2}{(x+y)^2m_e^2-xy\,q^2}\right)+\frac{\left[m_e^2\{1-4(1-x-y)+(1-x-y)^2\}+\cancel{xy}(1-x)(1-y)\,q^2\right]}{(x+y)^2m_e^2-xy\,q^2}\right\}+[\gamma^\mu.\gamma^\nu]\text{ part.}\right)$$

<u>dx dy</u> (1-loop part)

- <u>fixed y small x</u>

$$\simeq\left(ie\frac{e^2}{16\pi^2}2\gamma^\mu\right)\int dy\int_0 dx\,\frac{+(1-y)q^2+m_e^2\{\quad\}}{(2y\,m_e^2-y\,q^2)\,x+y^2m_e^2}$$

$$\left(\int_0^{x_*}dx\ \Downarrow\right)$$

$$\simeq ie\left(\frac{e^2}{16\pi^2}2\gamma^\mu\right)\int dy\,\frac{+(1-y)q^2+m_e^2\{\,\}}{2y\,m_e^2-y\,q^2}\ln\left(\frac{(2y\,m_e^2-y\,q^2)x_*+y^2m_e^2}{y^2m_e^2}\right)$$

If. $q^2 \gg m_e^2$ (eg. $f^+ + f^- \longrightarrow e^+ + e^-$ s-channel $q^2 = S$.) (center of mass energy)

$$\Rightarrow\simeq(ie\gamma^\mu)\times\frac{-e^2}{8\pi^2}\int\frac{dy\,(1-y)}{y}\ln\left(\frac{-q^2}{m_e^2}\right).$$

if $m_e \simeq 0$. ($\ll S$)

<u>log divergence.</u>

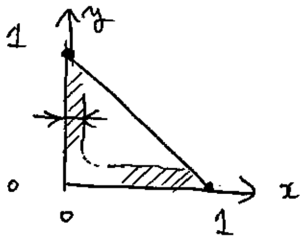
- <u>fixed x small y</u>

the same.

· both $x, y$ small

$$\int \cancel{dx\,dy} \frac{(ie)e^2}{16\pi^2} 2\gamma^\mu \int dx\,dy \frac{q^2}{(x+y)m_e^2 - xy\,q^2} \Leftarrow \int_0 \frac{d\lambda\,\lambda}{\lambda^2}$$

$\boxed{\text{summary}}$



Feynman parameter

## Origin of this divergence

$$\Gamma_{(1)}^{\mu} \sim \int dx\,dy\,dz \int \frac{d^4k}{(2\pi)^4} \frac{2\delta(x+y+z-1)\ \left[p, k, p' \dots \text{(no } x, y \text{ yet)}\right]}{\left\{x[(p-k)^2 - m_e^2] + y[(p'-k)^2 - m_e^2] + z[k^2]\right\}^3}$$

divergence from finite "k" (not from large k region).

$\boxed{\text{small } x, y}$ $\Rightarrow$ $k^2 \simeq 0$. $\underline{k^\mu \rightsquigarrow \lambda k^\mu_*}$

$$[(p-k)^2 - m_e^2] = \underset{\downarrow\ 0}{(p^2 - m_e^2)} - \underset{\lambda}{2p\cdot k} + \underset{\lambda^2 \to \text{ignore}}{k^2}.$$

$\underline{x \rightsquigarrow \lambda x_*,\ y \rightsquigarrow \lambda y_*}$

$$\left[\begin{array}{l} \{x[\quad] + y[\quad] + z[k^2]\} \sim \lambda^2 \{\quad\}_* \\ dx\,dy\,d^4k \sim \lambda^6\,d \end{array}\right.$$

$\Rightarrow$ log div.

from soft $\gamma$ mom. region.

$\boxed{\text{soft divergence}}$

$\boxed{\text{small } x \text{ (finite } y)}$

$$\Rightarrow \left\{\begin{array}{l} y[(p'-k)^2 - m_e^2] + (1-y)k^2 = 0 \\ y(k-p')^\mu + (1-y)k^\mu = 0 \end{array}\right. \left(\downarrow \frac{\partial}{\partial k_\mu}\right) \longrightarrow \underline{\underline{k^\mu = y\,p'^\mu}} \ (y \text{ finite}).$$

if $m_e \simeq 0$

$$p' \sim (E, E, \vec{0}) \quad (\cancel{p^3})$$

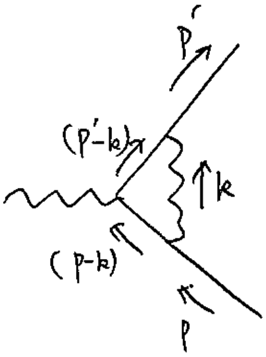
$$k^\mu \sim (yE + \lambda^2 E,\ yE - \lambda^2 E,\ \lambda \vec{k}_{T*}). \Rightarrow \begin{array}{l} (p'\cdot k) \sim 2\lambda^2 E^2 \\ (k^2) \sim 4y\lambda^2 E^2 - \lambda^2 |\vec{k}_{T*}|^2 \end{array}$$

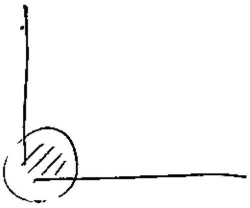
set $x \sim \lambda^2$.

$$\int dy \frac{\underbrace{dx\,d^4k}_{\underbrace{\lambda^2 \cdot 1 \cdot \lambda^2 \cdot \lambda \cdot \lambda}_{\lambda^6}}}{\{\quad\}^3 \rightsquigarrow \lambda^6 \{\quad\}^3_*}$$

$\Rightarrow$ log divergence. from a region $k \mathbin{/\!/} p'$

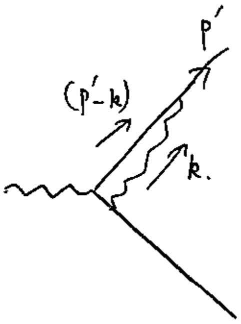
$\boxed{\text{collinear divergence}}$





soft divergence. ( x.y. small ).

$k^\mu \approx 0$.  $\left\{ \begin{array}{l} k^\mu \sim \lambda . \\ x.y \sim \lambda \end{array} \right.$



collinear divergence

$k^\mu \simeq y\, p'^\mu$

$\Rightarrow (p'-k)^\mu \simeq (1-y)\, p'^\mu$  $\left\{ \begin{array}{l} \vec{k}_T \sim \lambda \\ p'\cdot k \sim \lambda^2 \\ x \sim \lambda^2 \end{array} \right.$  ( small x )

$(-k^\mu) \simeq x(-p^\mu)$

$\Rightarrow [-(p-k)]^\mu \simeq (1-x)[-p]^\mu .$  (small y).

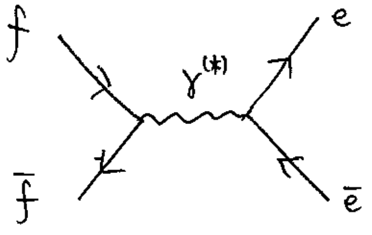
- adding soft photon in the intermediate state.
- splitting a nearly massless fermion into the fermion and a collinear photon w/. energy ratio $(1-y) : (y)$ or $(1-x) : (x)$

The energy cost can be (arbitrarily) small.
(virtuality)

$\Rightarrow$ contribute a lot in perturbation!

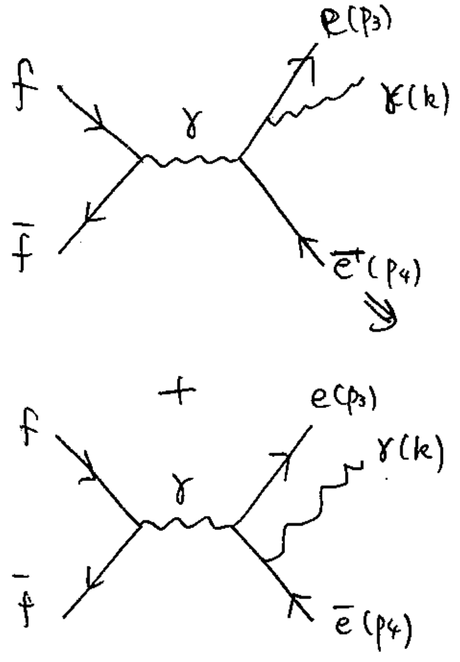
# § 9.2 Divergence in Real Emission

$f+\bar{f} \to e^+ + e^- + \underline{\gamma}$ 3-body final state. in QED s-channel.



$$\Rrightarrow \quad i\mathcal{M} = \left[ie\,\bar{v}(p_2)\gamma_\mu u(p_1)\right]\frac{-i}{s}\left[\bar{u}(p_3)\gamma^\mu v(p_4)\ ie\right]$$

$$\sigma(f+\bar{f}\to e^- + e^+) \propto \int\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{1}{2E_{p_3}}\int\frac{d^3\vec{p}_4}{(2\pi)^3}\frac{1}{2E_{p_4}}(2\pi)^4\delta^4(p^\mu_{in}-p^\mu_{out}) \times |\mathcal{M}|^2$$



$$\left[ie\,\bar{u}(p_3)\left(i\gamma^\lambda e\right)\frac{i[(\not{p}_3+\cancel{\not{k}})+m_e]}{(p_3+k)^2-m_e^2+i\varepsilon}\gamma^\mu v(p_4).\right]$$

ignore $\not{k}$, $m_e$ and keep. $\not{p}_3$ in [ ].

$$\begin{cases} \gamma^\lambda \not{p}_3 = \{\gamma^\lambda, \not{p}_3\} - \not{p}_3\gamma^\lambda = 2p_3^\lambda - \not{p}_3\gamma^\lambda. \\ \bar{u}(p_3)\cdot[\not{p}_3 - m_e] = 0 \quad \text{(Dirac eq)} \\ \quad (\Rightarrow \bar{u}(p_3)\not{p}_3 \simeq 0\ ) \end{cases}$$

$$\downarrow$$

(eikonal approximation) $\Leftarrow$

$$\left[(ie)\cdot\ \bar{u}(p_3)\gamma^\mu v(p_4)\right]\times\frac{2ie\,p_3^\lambda}{2p_3\cdot k+i\varepsilon}$$

similarly

$$\left[(ie)\ \bar{u}(p_3)\gamma^\mu v(p_4)\right]\times\frac{-2ie\,p_4^\lambda}{2p_4\cdot k+i\varepsilon}$$

$$\Rrightarrow \boxed{\left|\mathcal{M}^{\kappa}_{e\bar{e}\gamma}\,\varepsilon^{(\lambda)}_{\kappa}\right|^2 \simeq \varepsilon_\kappa(k)\,\varepsilon^*_\lambda(k)\left(\frac{p_3^\kappa}{p_3\cdot k}-\frac{p_4^\kappa}{p_4\cdot k}\right)\Big(\qquad\Big)^\lambda(e^2) \times\left|\mathcal{M}_{e\bar{e}}\right|^2}$$

$k^2=0$ ( $\gamma$ : on-shell ).

$\gamma$ spin sum (polarization) $\Rrightarrow$ $\left[\varepsilon_\kappa(k)\,\varepsilon^*_\lambda(k)\Rrightarrow -\eta_{\kappa\lambda}.\right]$.

$$\Rightarrow |\mathcal{M}_{e\bar{e}}|^2\times e^2\left\{\frac{2(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)} - \underbrace{\frac{m_e^2}{(p_3\cdot k)^2}-\frac{m_e^2}{(p_4\cdot k)^2}}\right\}$$

↪ power suppressed. → ignore.

$$\sigma(f+\bar{f}\to e+\bar{e}+\gamma) \quad \cancel{\sigma(f+\bar{f}\to e+\bar{e})} \times$$

$$\propto \int\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{1}{2E_3}\int\frac{d^3\vec{p}_4}{(2\pi)^3}\frac{1}{2E_4}\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}(2\pi)^4\delta^4(p_{in}^\mu-p_{out}^\mu)\left|\mathcal{M}_{ee\gamma}\right|^2$$

$$= \underline{\sigma(\to e+\bar{e})\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}\times(2e^2)\frac{(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)}}$$

center of mass frame. of $f+\bar{f}$ collision.

$$\Rightarrow \quad p_3^\mu \sim E(1,\vec{v})$$

$$p_4^\mu \sim E(1,\vec{v}') \qquad \vec{v}'\simeq-\vec{v} \qquad \frac{(p_3\cdot p_4)}{(p_3\cdot k)(p_4\cdot k)} = \frac{(1-\vec{v}\cdot\vec{v}')}{k^2(1-\vec{n}\cdot\vec{v})(1-\vec{n}\cdot\vec{v}')}$$

$$k^\mu \sim k(1,\vec{n}).$$

$$\boxed{\sigma(\to e\bar{e}\gamma)\simeq\sigma(\to e\bar{e})\frac{e^2}{(2\pi)^3}\int\frac{dk}{k}\int d^2\Omega_{\vec{n}}\,\frac{(1-\vec{v}\cdot\vec{v}')}{(1-\vec{n}\cdot\vec{v})(1-\vec{n}\cdot\vec{v}')}}$$

from a region $\vec{n}$ almost parallel to $\vec{v}$ (angle $\theta$.

$$\frac{e^2}{(2\pi)^3}\int\frac{dk}{k}(2\pi)\int^1 d\cos\theta\,\frac{1}{1-\cos\theta\,|v|} \qquad |v|=\frac{|\vec{p}_3|}{E}\simeq1-\frac{m_e^2}{2E^2}$$

$$\hookrightarrow 1-\left(1-\frac{m_e^2}{2E^2}\right)\cos\theta.$$

$$\simeq\frac{e^2}{(2\pi)^2}\int\frac{dk}{k}\left(-\ln\left(\frac{m_e^2}{2E^2}\right)\right) = \frac{e^2}{(2\pi)^2}\int\frac{dk}{k}\ln\left(\frac{s}{2m_e^2}\right)$$

$$*\;\boxed{\begin{array}{l}\sigma(\to e\bar{e}\gamma) \simeq \sigma(\to e\bar{e})\times\cancel{\int\frac{dk}{k}}\left[\frac{e^2}{4\pi^2}\int\frac{dk}{k}\ln\left(\frac{s}{2m_e^2}\right)\right]\\ \quad |k_\gamma|\ll\sqrt{s}.\end{array}}$$

$\vec{n}_\gamma$ almost // $\vec{p}_3$

<u>collinear divergence</u> $\left\{\begin{array}{l}\gamma \text{ emission // to } e^- \\ \quad " \quad\;\; \text{// to } e^+\end{array}\right\}$ if $(m_e\ll s)\Leftarrow\left\{\begin{array}{l}\frac{1}{p_3\cdot k}\sim\frac{1}{0}\\ \frac{1}{p_4\cdot k}\sim\frac{1}{0}\end{array}\right.$ $(k^2=0)$

<u>soft divergence</u>

<u>propagator is nearly on-shell.</u>

# § 10 Cancellation of IR divergence.

## Observation

- soft divergence : massless ~~state~~ particle ($\gamma$) arbitrarily low energy

$\Rightarrow$ can we see it?

- collinear divergence : kinematically possible. for massless particles.

$$p^\mu \to \lambda_1 p^\mu + (1-\lambda_1) p^\mu \to \lambda_1 p^\mu + \lambda_2 p^\mu + (1-\lambda_1-\lambda_2) p^\mu$$

$$\to \cdots\cdots$$

$$\to \sum_i (\lambda_i p^\mu) . \quad \text{so that } \left(\sum_i \lambda_i = 1\right).$$

can we distinguish them?

$|\vec{k}_\gamma| \ll \sqrt{s}$

$\vec{k}_\gamma // \vec{p}_e$ or $\vec{p}_{\bar{e}}$ part of $\sigma(\to e\bar{e}\gamma)$ should be treated as a part of $\sigma(\to e\bar{e})$.

## Look at the collinear part

$$\bullet\ \sigma(\to e\bar{e}) \simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left|1 - \frac{e^2}{8\pi^2}\int dy \frac{1-y}{y} \ln\left(\frac{-s}{m_e^2}\right)\right|^2$$

$$\simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left[1 - \frac{e^2}{4\pi^2}\int dy \frac{1}{y} \ln\left(\frac{s}{m_e^2}\right).\right] \quad \left(\overset{\text{approx.}}{1-y \simeq 1.}\right).$$

$$\bullet\ \sigma(\to e\bar{e}\gamma) \simeq \sigma(\to e\bar{e})_{\text{tree}} \times \left[ + \frac{e^2}{4\pi^2}\int \frac{dk}{k} \ln\left(\frac{s}{2m_e^2}\right) \right].$$

$$\ln\left(\frac{1}{2m_e^2/s}\right) \Rightarrow \ln\left(\frac{1-\cos\theta_*}{2m_e^2/s}\right) + \ln\left(\frac{1}{1-\cos\theta_*}\right).$$

$$\boxed{\sigma(\to e\bar{e}\gamma)_{|k|>k_*,\ \theta>\theta_*} \simeq \sigma(\to e\bar{e}) \times \underbrace{\left[\frac{e^2}{4\pi^2}\int_{k_*} \frac{dk}{k} \ln\left(\frac{1}{1-\cos\theta_*}\right)\right]}}$$ $\Leftarrow$ positive

$$\boxed{\begin{array}{l}\sigma(\to e\bar{e}) \\ +\,\sigma(\to e\bar{e}\gamma)_{|k|>k_*,\ \theta_*>\theta}\end{array} \simeq \sigma(\to e\bar{e}) \times \left[1 - \frac{e^2}{4\pi^2}\int_{2k_*/\sqrt{s}} \frac{dy}{y} \ln\left(\frac{2}{1-\cos\theta_*}\right)\right]}$$

negative correction