§10.2 Optical Theorem

$$\mathcal{S}_{\beta\alpha} \equiv {}^{out}\langle \beta | \alpha \rangle^{in} \equiv \mathbb{1}_{\beta\alpha} + (2\pi)^4 \delta^4(p_{in} - p_{out})\, i\mathcal{M}_{\beta\alpha}$$

$\mathcal{S}$-matrix

scattering amplitude or matrix element.

$\mathcal{S}_{\beta\alpha}$: unitary matrix.

(conservation of probability in quantum mechanics)

$$\Rightarrow \boxed{\mathcal{S}^\dagger \mathcal{S} = \mathbb{1}}$$

$$\Rightarrow (2\pi)^4 \delta^4(p_{in} - p_{out}) \times \left[\begin{aligned} &\mathcal{M}^\dagger_{\beta\gamma} \mathcal{M}_{\gamma\alpha} (2\pi)^4 \delta^4(p_{in} - p_{out}) \\ &+ i\{\mathcal{M}_{\beta\alpha} - (\mathcal{M}^\dagger)_{\beta\alpha}\} \end{aligned} \right] = 0.$$

$$\boxed{\frac{1}{i}[\mathcal{M} - \mathcal{M}^\dagger]_{\beta\alpha} = \prod_{i\in\gamma} \int \frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \mathcal{M}^\dagger_{\beta\gamma} \mathcal{M}_{\gamma\alpha} (2\pi)^4 \delta^4(p_{in} - p_{out})}$$

if $\beta = \alpha$.

$$\boxed{\frac{1}{i}[\mathcal{M} - \mathcal{M}^\dagger]_{\alpha\alpha} = \prod_{i\in\gamma} \int \frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{p_i}} |\mathcal{M}_{\gamma\alpha}|^2 (2\pi)^4 \delta^4(p_{in} - p_{out}).}$$

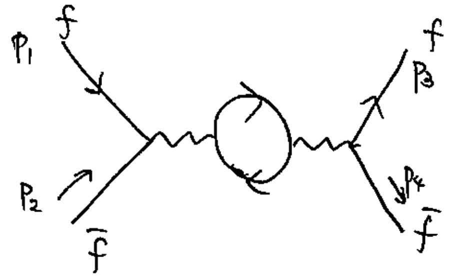
α: 1 particle state

$$\Gamma = \frac{1}{2m} \times (\text{RHS})$$

α: 2 particle state

$$\sigma = \frac{1}{(2E_A)(2E_B)(v_{AB})} \times (\text{RHS})$$

2-body → 2-body forward scattering amplitude.



$$i\mathcal{M} = ie^2\left[\bar{u}_{r'}(\vec{p}_3)\gamma^\mu v_{s'}(\vec{p}_4)\right]\frac{1}{s}\left\{1+\Pi^{(1)}_{ren}(s+i\varepsilon)+\cdots\right\}\left[\bar{v}_s(\vec{p}_2)\gamma_\mu u_r(\vec{p}_1)\right]$$

set $\beta=\alpha$ (i.e. $\vec{p}_3=\vec{p}_1$, $\vec{p}_4=\vec{p}_2$, $r'=r$, $s'=s$)

take spin average $\frac{1}{2}\sum_r \ \frac{1}{2}\sum_s$

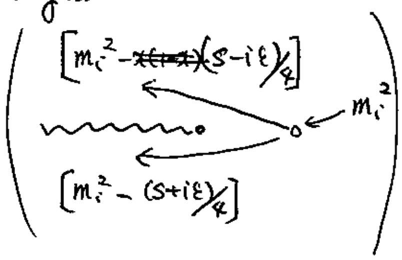
$$\Rightarrow \ i\bar{\mathcal{M}} = ie^2\,\frac{1+\Pi^{(1)}_{ren}(s+i\varepsilon)+\cdots}{s}\ \underbrace{\frac{1}{4}\mathrm{Tr}\left[\gamma^\mu(\not{p}_2-m)\gamma_\mu(\not{p}_1+m)\right]}_{\substack{\shortparallel \\ \left[-(s+2m_f^2)\right]}}$$

$(\bar{\mathcal{M}}-\bar{\mathcal{M}}^\dagger)$ comes from $\underline{\Pi^{(1)}_{ren}(s+i\varepsilon)}$.

$$\Pi^{(1)}_{ren}(s+i\varepsilon) = \frac{e^2}{2\pi^2}\int_0^1 dx\ x(1-x)\ln\left[\frac{m_i^2-x(1-x)\{s+i\varepsilon\}}{m_i^2+x(1-x)\mu^2}\right]$$

from Dirac fermion "i".

log...



$$\frac{1}{i}\left[\Pi^{(1)}_{ren}(s+i\varepsilon)-\Pi^{(1)}_{ren}(s-i\varepsilon)\right]$$

$$= \frac{e^2}{2\pi^2}\sum_i\int_{x_{min}}^{x_{max}} dx\ x(1-x)\times\frac{[-2\pi i]}{i}$$

$$x_{max/min} = \frac{1}{2}\left(1\pm\sqrt{1-\frac{4m_i^2}{s}}\right) \qquad \cong -\frac{e^2}{\pi}\sum_{4m_i^2<s}\frac{1}{6}$$

if $4m_i^2<s$.

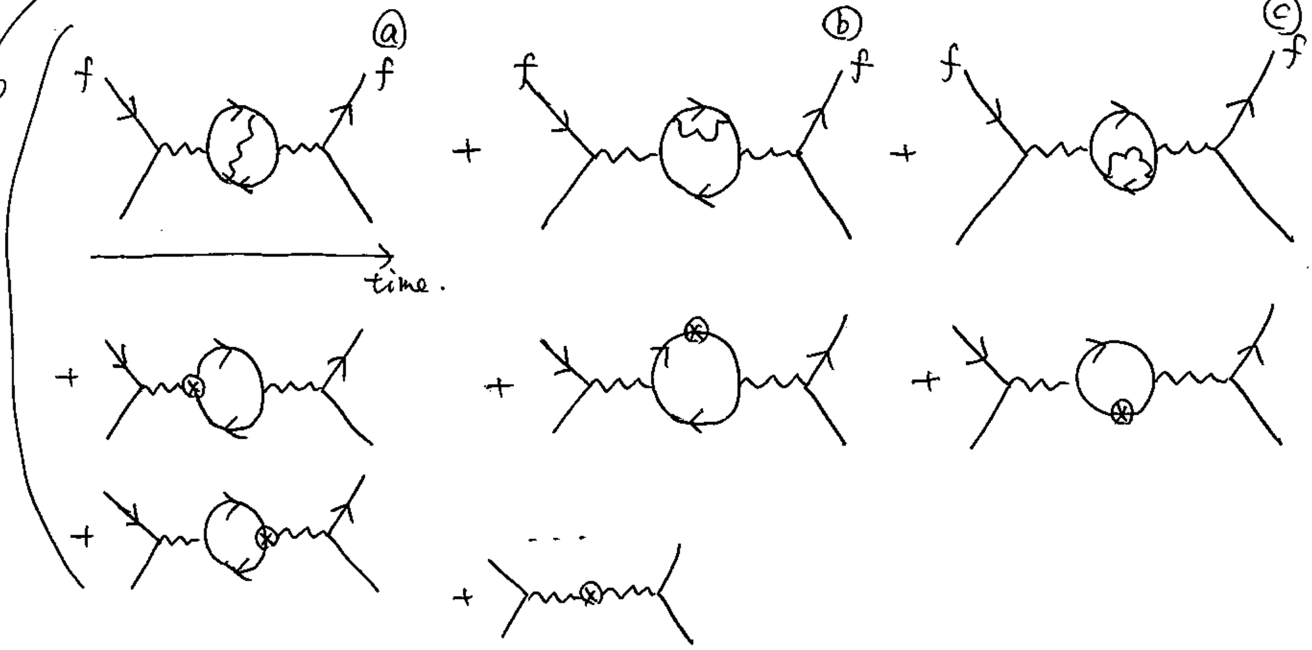
$$\Rightarrow \ \frac{1}{i}\left[\bar{\mathcal{M}}-\bar{\mathcal{M}}^\dagger\right] \simeq (-e^2)\times\left(-\frac{e^2}{6\pi}\right)\times\#[\text{light Dirac fermions}]$$

$$\sigma_{tot}(f+\bar{f}\to\gamma^{(*)}\to\text{any}) \cong \frac{1}{2s}\times\left(\frac{1}{i}[\bar{\mathcal{M}}-\bar{\mathcal{M}}^\dagger]\right) = \frac{e^4}{12\pi s}\times\#\left(\begin{matrix}\text{light}\\ \text{Dirac}\\ \text{ferm}\end{matrix}\right) = \frac{4\pi\alpha_e^2}{3s}\times\#\left[\begin{matrix}\text{light}\\ \text{Dirac}\\ \text{fermion}\end{matrix}\right.$$

going to higher order

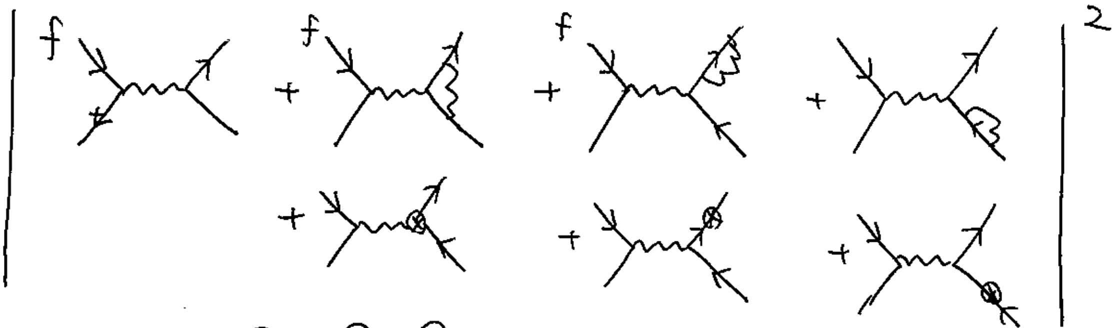
$$\frac{1}{i}\left[\mathcal{M}_{\alpha\alpha}-\mathcal{M}^*_{\alpha\alpha}\right] = \sum_{\gamma} \left|\mathcal{M}_{\gamma\alpha}\right|^2 = \sum_{\gamma} \prod_{i\in\gamma}\left(\int\frac{d^3\vec{p}_i}{(2\pi)^3}\frac{1}{2E_{\vec{p}_i}}\right)(2\pi)^4\delta^4(p_{in}-p_{out})\left|\mathcal{M}_{\gamma\alpha}\right|^2$$

(LHS) =



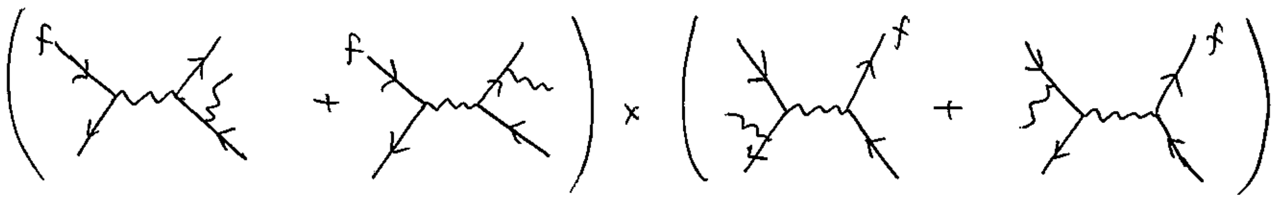
(RHS)

☆ $\gamma = \psi+\bar{\psi}$



$\Rightarrow$ a + b + c

☆ $\gamma = \psi+\bar{\psi}+[\gamma$ (photon) or gluon.]



$\Rightarrow$ c + b + a.

$$\left[a+b+c+\cdots\right]_{LHS} \propto \left[\sigma(f+\bar{f}\to\psi+\bar{\psi}) + \sigma(f+\bar{f}\to\psi+\bar{\psi}+\gamma\,(\text{or } g))\right]^{RHS}$$

@ $\mathcal{O}(e^6)$ or $\mathcal{O}(e^4 g^2)$.

cancellation of soft & collinear divergence at $O(e^6$ or $e^4 g^2)$

in total cross section.

$\Rightarrow$ use optical theorem. and see the absence of divergence in $[\mathrm{Im}\, M_{\alpha\alpha}]$

eg. an amplitude given by

$$\prod_a \int \frac{d^4 k_a}{(2\pi)^4} \prod_i \frac{1}{[l_i^2 - m_i^2 + i\varepsilon]} = \prod_{i=1}^{(\#\mathrm{prop})} \prod_a \int \frac{d^4 k_a}{(2\pi)^a} \int_{\left\{\substack{0 \le x_i \le 1 \\ \sum_i x_i = 1}\right\}} dx_i \; \frac{\Gamma(\#(\mathrm{prop}))}{\left(\sum_i x_i [l_i^2 - m_i^2]\right)^{(\#\mathrm{prop})}}$$

loop momentum. propagator

l_i : linear comb. of k_a's and ext. momenta.

$\boxed{\text{step 1}}$ possible places in $d^4k_a\, dx_i$ where IR divergence might arises (pinch surface)

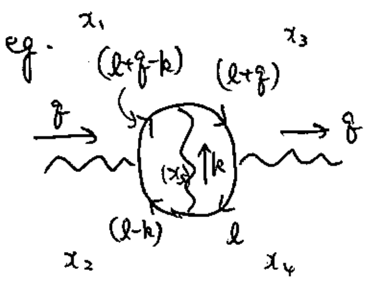
☆ for each $\underline{i} \in \{1, 2, \cdots, \#(\mathrm{prop})\}$.
impose either $x_i = 0$ or $[l_i^2 - m_i^2] = 0$.

☆ for each $\underline{a} \in \{1, 2, \cdots, \#(\mathrm{Loop})\}$.
impose $\sum_i x_i \frac{\partial}{\partial k_a^\mu}(l_i^2) = 0$. ——— $(*)$

$\boxed{\text{step 2}}$ study the behavior of $\left\{\begin{array}{l}\text{measure } d^4k_a\, dx_i \\ \text{denominator } \left(\sum x_i [l_i^2 - m_i^2]\right)^{(\#\mathrm{prop})}\end{array}\right\}$

in the direction transverse to the pinch surface.

eg.



$$\text{denominator} = \Big(x_1 (l+q-k)^2 + x_2 (l-k)^2 + x_3 (l+q)^2 + x_4 (l^2) + x_5 (k^2)\Big).$$

eg. $\alpha_1 \simeq 0$, $\alpha_2 \simeq 0$. $\alpha_3 \simeq 0$, $\alpha_4 \simeq 0$. $k^2 = 0$ $(\alpha_5 \neq 0)$.

$\Rightarrow (* k^\mu) \because k^\mu \simeq 0$.

$k^\mu \sim \mathcal{O}(\lambda)$, $x_{1,2,3,4} \sim \mathcal{O}(\lambda^2)$.

measure $d^4k\, d^4l\, dx_1\, dx_2 dx_3 dx_4 \sim \mathcal{O}(\lambda^4 \cdot \lambda^{2\cdot 4} = \underline{\underline{\lambda^{12}}})$.

denominator $[\lambda^2]^5 \sim \underline{\underline{\lambda^{10}}}$

$\Rightarrow \mathcal{O}(\lambda^2)$ finite.

eg $\alpha_1 \simeq 0$ $x_2 \simeq 0$. $(l+q)^2 \simeq 0$ $l^2 \simeq 0$ $k^2 \simeq 0$.

$\Rightarrow$ (* for k^μ) $k^\mu \simeq 0$

$\Rightarrow$ (* for l^μ) $x_3 (l+q)^\mu + x_4 l^\mu \simeq 0$.

not compatible.

Absence of IR divergence can be shown in this way for ⓐ, ⓑ and ⓒ.

$\Rightarrow$ meaning cancellation of divergence between $\sigma(f\bar{f} \to \psi\bar{\psi})$ and $\sigma(f\bar{f} \to \psi\bar{\psi}\gamma)$.

From 2-loop calculation..... of

$$\sigma_{tot} \simeq \frac{4\pi \alpha_e^2}{3S} \times (3 Q_q^2) \times \left[1 + \frac{\alpha_s(\mu^2)}{\pi} + \cdots\right]$$

[relativistic $e^+e^- \to q\bar{q}\,(+g)$. total σ]

$$\sigma(\to q\bar{q})_{cut} \sim \frac{4\pi\alpha_e^2}{3S}(3Q_q^2)\left[1 - \alpha_s \text{ double Log}\right]$$

$$+)\ \sigma(\to q\bar{q}g) \sim \frac{4\pi\alpha_e^2}{3S}(3Q_q^2)\left[\quad + \alpha_s \text{ double Log}\right]$$

$$\left[1 + \frac{\alpha_s}{\pi}\right]$$ finite positive correction (K-factor).

$\to$ IR-safe.

$\boxed{\text{Observables w/o IR divergence.}}$

☆ Total σ. : safe! $\to$ single scale problem. S.

☆ $q\bar{q}$ prod. σ +$q\bar{q}g$ w/ cut $\Rightarrow$ need $\boxed{\text{careful}}$ definition of jet $\to$ safe.

$\to$ multi scale problem S, k_T cut.

need resummation

§ 10.3 OPE and. Non-perturbative Corrections.

$$\left\{\begin{array}{l} e^+e^- \to q\bar{q}g \cdots \text{etc.} \Rightarrow \text{hadron production.} \\ \qquad \text{Is perturbative QCD approach justified if } s \gg \Lambda_{QCD}^2 \text{ ?} \end{array}\right\}$$

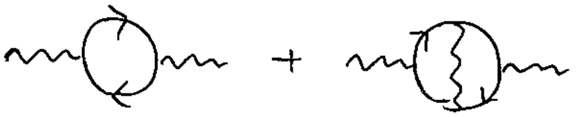
Optical theorem : $\left\{\begin{array}{l}\text{unitarity of } S\text{-matrix} \\ \text{insertion of a complete system of the Hilbert space.}\end{array}\right.$

$$\boxed{\sigma_{tot} \propto \frac{1}{i}\left[\bar{\mathcal{M}} - \bar{\mathcal{M}}^\dagger\right] \cong \frac{(-e^2)}{i}\left[\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)\right]}$$

$$\iint (ie)^2 \langle \Omega | T\{J^\mu(x) J^\nu(y)\} | \Omega \rangle e^{iq' \cdot x} e^{-iq \cdot x} d^4x\, d^4y$$
$$= (2\pi)^4 \delta^4(q'-q)\; i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2+i\varepsilon)$$

But!!

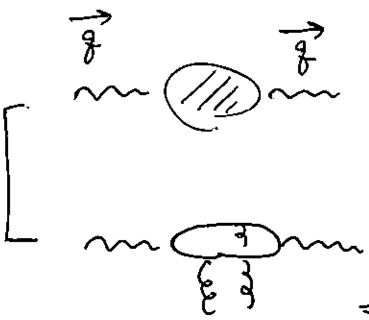
perturbative calculation.



$\Rightarrow \Pi^{\text{pert.}}$ not Π.

In OPE

$$\int (ie)^2\, T\{J^\mu(x) J^\nu(y)\}\, e^{iq\cdot(x-y)} d^4y = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\, \Pi^{\text{pert}}(q^2+i\varepsilon)\cdot \mathbb{1}$$
$$+ \qquad = \times\left[\sim \frac{1}{(q^2)^2}\right] \times \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})$$
$$+ \qquad \cdots$$



$\Rightarrow \mathrm{tr}[F_{\rho\sigma}F^{\rho\sigma}]$

$$\langle \Omega | \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) | \Omega \rangle \sim O(\Lambda_{QCD}^4).$$

$\Rightarrow$ non-perturbative corrections.

relatively $\boxed{\times\left(\frac{\Lambda_{QCD}^4}{s^2}\right)}$

$$\boxed{\Pi(s) = \Pi^{\text{pert}}(s) + O\left(\Lambda_{QCD}^4/s^2\right)}$$ good approx. if $\Lambda_{QCD}^2 \ll s$.