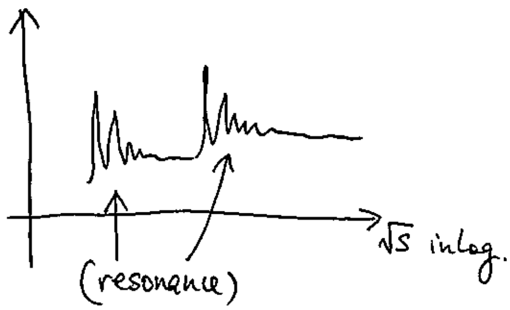
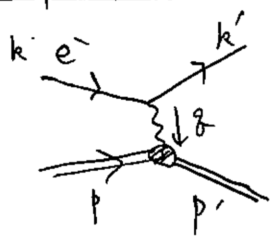
§11. Parton Distribution and Factorization

§11.1 Deep Inelastic Scattering

- $e^+ + e^- \to$ anything (total σ), jet cross section..
- $e + p \to$
- $p + (\bar{p}$ or $p) \to$.
  or π ...



$e^- + p^+ \to e^- + p^+$ elastic scattering



$p'_\mu = (p+q)_\mu$. but $p^2 = m_p^2$ & $(p')^2 = m_p^2$

$\Rightarrow 2p\cdot q + q^2 = 0. \to \boxed{\frac{-q^2}{2p\cdot q} = 1.}$

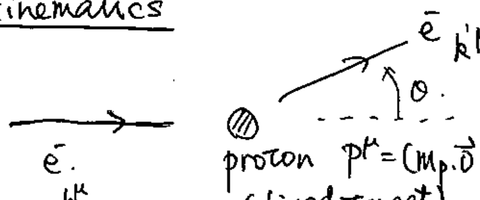
$e^- + p^+ \to$ inelastic scattering via resonance

if $(q+p)^2 > m_p^2$? $\gamma^{(*)} + p^+ \to$ (resonance) $\to$ anything. (incl. p^+ or n)

$$\Leftrightarrow \quad 0 \le \frac{(q+p)^2 - m_p^2}{(-q^2)} = \frac{2p\cdot q + q^2}{(-q^2)} = \omega - 1 = \frac{1}{x} - 1.$$

$\Leftrightarrow 1 \le \omega$. $x \le 1$. inelastic if $x < 1$.

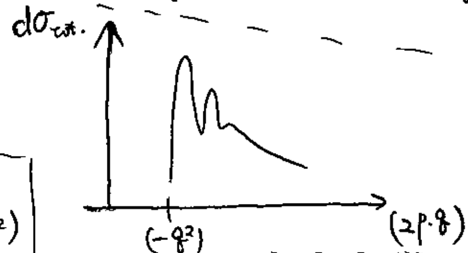
kinematics

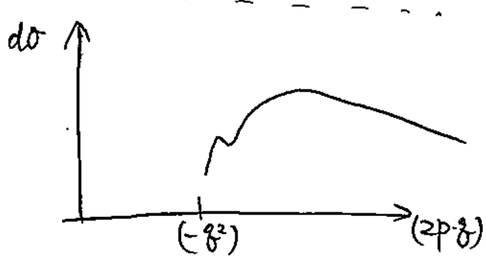


for a given energy of e^- $(k\cdot p)$ (center of mass energy.)

$\left\{\begin{array}{l}\text{scattering angle } \theta \\ \text{energy loss } (k^0 - k'^0)\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}(q^2) \\ (p\cdot q)\end{array}\right.$

At a given $(k\cdot p)$
fixed $(-q^2)$





$q^2 < 0$.
$\because (k-q)^2 = m_e^2 = k^2 \to q^2 = 2k\cdot q$. $\to$ (negative)
At rest frame of k^μ: $(m_e, \vec{0})$
$k^\mu = k'^\mu + q^\mu$; $k'^0 > m_e \Rightarrow q^0 < 0$

experimental result : $\dfrac{(d^2\sigma)}{d(q^2)(d\omega)}$ almost $(-q^2)$-indep. fcn of $\omega \geq 1$
[for large $(-q^2)$] (or $x \leq 1$)

Bjorken scaling.

Deep. Inelastic Scattering.

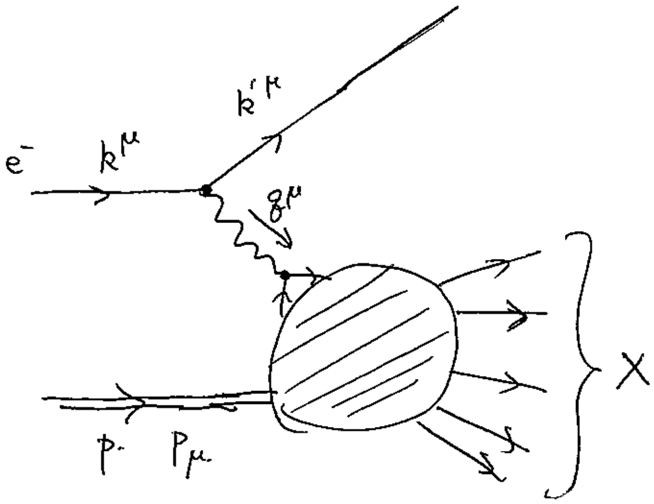
$\Uparrow$ large $(-q^2)$.

if we ignore m_e & m_p.

$s \simeq 2k\cdot p$. $\quad y \equiv \dfrac{2p\cdot q}{2p\cdot k} \quad x \equiv \dfrac{-q^2}{2p\cdot q} \Rightarrow \underline{s\cdot x\cdot y \simeq (-q^2)}$.

fixed target $\hookrightarrow \dfrac{(k^0 - k'^0)}{k^0} = \begin{pmatrix} e^- \text{ energy loss} \\ \text{fraction} \end{pmatrix}$

DIS : inclusive. process.



§11.2 DIS Structure Functions

$$\boxed{d\sigma_{DIS} \simeq \frac{1}{4k\cdot p} \int \frac{d^3\vec{k}'}{(2\pi)^3} \frac{1}{2E_{k'}} \int d\Pi_X \quad (2\pi)^4 \delta^4(p_X - p - q). \quad \sum_s \left| \left[\bar{u}_s(k')\gamma^\nu u_r(k)(ie)\right]\left(\frac{-i}{q^2+i\varepsilon}\right)(ie)\langle X | J^\nu(0) | \vec{p}\rangle \right|^2}$$

(Ignore m_e, m_p.)

note.

$$\int d^4y \; e^{-iq\cdot y} \langle X | J^\nu(y) | \vec{p}\rangle = \int d^4y \; e^{-iq\cdot y} \langle X | \overset{\text{translation operator}}{e^{i\hat{P}\cdot y}} J^\nu(0) e^{-i\hat{P}\cdot y} | \vec{p}\rangle$$

$$= \int d^4y \; e^{-iq\cdot y} e^{i(p_X - p)\cdot y} \langle X | J^\nu(0) | \vec{p}\rangle.$$

$$= (2\pi)^4 \delta^4(p_X - p - q). \langle X | J^\nu(0) | \vec{p}\rangle.$$

n-particle state $\langle X|$

$$\ni d\Pi_X \equiv \prod_{i=1}^{n} \left(\frac{d^3\vec{p}_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right).$$

- $$\sum_s \left[\bar{u}_r(k)\gamma^\mu u_s(k')\right]\left[\bar{u}_s(k')\gamma^\nu u_r(k)\right] = \mathrm{Tr}\left[\gamma^\mu \left\{\sum_s u_s(k')\bar{u}_s(k')\right\} \gamma^\nu \left\{u_r(k)\bar{u}_r(k)\right\}\right]$$

$$= \mathrm{Tr}\left[\gamma^\mu (\not{k}' + m_e)\gamma^\nu \left\{u_r(k)\bar{u}_r(k)\right\}\right] \xrightarrow[\text{spin average } \left(\frac{1}{2}\sum_r\right)]{} \frac{1}{2}\mathrm{Tr}\left[\gamma^\mu(\not{k}'+m_e)\gamma^\nu(\not{k}+m_e)\right]$$

$$\shortparallel$$

$$\boxed{2\left[k'^\mu k^\nu + k^\mu k'^\nu - \eta^{\mu\nu}(k\cdot k')\right]}$$

- $$(2\pi)^4\delta^4(p_X - p - q) \quad \langle \vec{p} | J_\mu(0) | X\rangle \langle X | J_\nu(0) | \vec{p}\rangle$$

$$= \int \langle \vec{p} | J_\mu(0) | X\rangle\langle X | J_\nu(y) | \vec{p}\rangle \, e^{-iq\cdot y} \, d^4y$$

$$\Rightarrow \boxed{d\sigma_{DIS} \simeq \frac{1}{4k\cdot p}\int \frac{d^3\vec{k}'}{(2\pi)^3}\frac{1}{2E_{k'}}\, 2\left[k'_\mu k_\nu + k_\mu k'_\nu - \eta_{\mu\nu}(k\cdot k')\right]\frac{e^4}{(q^2)^2} \cdot \underbrace{\int d\Pi_X \int \langle\vec{p}|J^\mu(0)|X\rangle\langle X|J^\nu(y)|\vec{p}\rangle\, e^{-iq\cdot y}\, d^4y}_{\shortparallel}}$$

$$\int d^4y\ \langle\vec{p}|J^\mu(0)J^\nu(y)|\vec{p}\rangle\, e^{-iq\cdot y}.$$

if $p\cdot q > 0. \rightarrow$ $$= \int d^4y\ \langle\vec{p}|[J^\mu(0), J^\nu(y)]|\vec{p}\rangle\, e^{-iq\cdot y}.$$

$\because \int d^4y \langle\vec{p}|J^\nu(y)\, e^{-iq\cdot y} = 0$

(imagine $\gamma^*(q^\mu) + X \rightarrow p$ at the rest frame of p. $\Rightarrow \boxed{q^0 < 0.\ \text{or}.\ M.E. = 0}$)

$$= G^{\text{retard.}}_{J^\mu - J^\nu}(q) - G^{\text{advanced.}}_{J^\mu - J^\nu}(q)$$

$$= \mathrm{Re}\left[G^{\text{retard}}_{J^\mu - J^\nu}\right] - \mathrm{Re}\left[G^{\text{advanced}}_{J^\mu - J^\nu}\right]$$

if $p\cdot q > 0.$ $\longrightarrow$ $$= 2\,\mathrm{Re}\left[G^{\text{time}}_{J^\mu - J^\nu}\right]$$

$$\equiv 2\,\mathrm{Re}\left[\int d^4y\ \langle\vec{p}|T\{J^\mu(0)J^\nu(y)\}|\vec{p}\rangle\, e^{-iq\cdot y}\right]$$

$$\boxed{T^{\mu\nu} \equiv i\int d^4y\ \langle\vec{p}|T\{J^\mu(0)J^\nu(y)\}|\vec{p}\rangle\, e^{-iq\cdot y}}$$

From gauge invariance of QED.

$$q_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} q_\nu = 0.$$

$$\Rightarrow \boxed{T^{\mu\nu} = (4\pi)\left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)T_1 + \frac{1}{(p\cdot q)}\left[p^\mu - \frac{(p\cdot q)}{q^2}q^\mu\right]\left[p^\nu - \frac{p\cdot q}{q^2}q^\nu\right]T_2\right\}}$$

parametrized by T_1, T_2.

$$\boxed{d\sigma_{DIS}} \simeq \frac{1}{4p\cdot k}\int\frac{d^3\vec{k}'}{(2\pi)^3}\frac{1}{2E_{k'}}\,2\left[k'_\mu k_\nu + k_\mu k'_\nu - \eta_{\mu\nu}k\cdot k'\right]\frac{e^4}{(q^2)^2}$$

$$\times(4\pi)\left\{\left(-\eta^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)F_1 + \frac{1}{p\cdot q}\left[p^\mu - \left(\frac{p\cdot q}{q^2}\right)q^\mu\right]\left[p^\nu - \left(\frac{p\cdot q}{q^2}\right)q^\nu\right]F_2\right\}$$

$$\begin{cases} 2\,\mathrm{Im}\,T_1 \equiv F_1 \\ 2\,\mathrm{Im}\,T_2 \equiv F_2 \end{cases}$$

$$[\ ]\cdot\{\ \}$$

$$= \frac{1}{(4p\cdot k)}\underbrace{\int\frac{d^3\vec{k}'}{(2\pi)^3}\frac{1}{2E_{k'}}}\,8\pi\frac{e^4}{q^4}\overbrace{\frac{2p\cdot q}{y^2}\left[xy^2F_1+(1-y)F_2\right]}$$

$$= \frac{1}{(4p\cdot k)}\overbrace{\frac{1}{4(2\pi)^2}\int dQ^2\,dy}\,\frac{8\pi}{Q^2}\frac{e^4}{xy^2}\left[xy^2F_1+(1-y)F_2\right]$$

$$= \boxed{\frac{dQ^2\,dy}{Q^4\,y}\,4\pi\alpha_e^2\left[xy^2F_1+(1-y)F_2\right]}$$

$$= \boxed{\frac{dx\,dy}{\cancel{Q}}\,\frac{4\pi\alpha_e^2 s}{Q^4}\left[xy^2F_1+(1-y)F_2\right]}$$

$$= \boxed{dx\,dQ^2\,\frac{4\pi\alpha_e^2}{x\,Q^4}\left[xy^2F_1+(1-y)F_2\right]}$$

$F_1\,(x;Q^2)$
$F_2\,(x;Q^2)$ structure functions.

Bjorken scaling: F_1, F_2 depend primarily on x. not on Q^2.

§11.3 Evaluation of $T^{\mu\nu}$ by OPE

$$iT^{\mu\nu} \equiv i\int d^4y\; e^{-iq\cdot y} \langle \vec p\,|T\{J^\mu(0)\,J^\nu(y)\}|\,\vec p\rangle \qquad \text{w/ } \underset{\text{spacelike.}}{q^2<0}$$

(instead of $\quad \Pi^{\mu\nu} \equiv i e^2 \int d^4y\; e^{-iq\cdot y} \langle \Omega|T\{J^\mu(0)\,J^\nu(y)\}|\Omega\rangle$

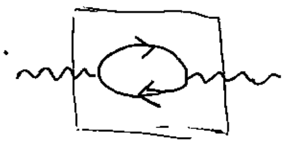
(vacuum polarization).

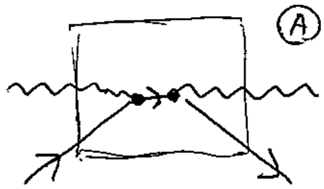
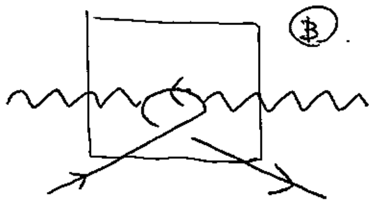
time-like $q^2>0$. $\Rightarrow \mathrm{Im}\left[\Pi^{\mu\nu}(q^2)\right] \neq 0$.

☆ $i\int d^4y\; e^{-iq\cdot y}\; T\{J^\mu(0)\,J^\nu(y)\}$

- begins with. $(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi_{\text{ren}}(q^2+i\varepsilon)\,\mathbb{1}$..

  But $\mathrm{Im}\left[\Pi(q^2+i\varepsilon)\right] = 0$. if $q^2<0$.



-  + 

(A)

$$i\int d^4(x-y)\; e^{+iq\cdot(x-y)} \int \frac{d^4k}{(2\pi)^4}\; \bar\psi(x)\,\gamma^\mu\, \frac{i(\not k+\not q)}{(k+q)^2+i\varepsilon}\,\gamma^\nu\,\psi(y)\, e^{-i(k+q)\cdot(x-y)}$$

$$= (-)\int d^4(x-y)\int\frac{d^4k}{(2\pi)^4}\; e^{-i(x-y)\cdot k}\; e^{\frac{(x-y)}{2}(\overleftarrow{\partial}-\vec\partial)} \left[\bar\psi(\bar x)\,\frac{\gamma^\mu(\not q+\not k)\gamma^\nu}{(q+k)^2}\,\psi(\bar x)\right]$$

$$\Rightarrow \delta^4\!\left(k-\tfrac{i}{2}(\overleftarrow{\partial}-\vec\partial)\right)(2\pi)^4$$

$$= (-)\left[\bar\psi\, \frac{\gamma^\mu(\not q+\frac{i}{2}\overleftrightarrow{\not D})\gamma^\nu}{(q+\frac{i}{2}\overleftrightarrow{D})^2}\,\psi\right]_{\text{at }\frac{x+y}{2}} \quad \text{local operator.}$$

$\Rightarrow$ expand in $\dfrac{q\cdot\frac{i}{2}\overleftrightarrow{D}}{q^2}$

(B). $$i\int d^4(x-y)\; e^{iq\cdot(x-y)}\int\frac{d^4k}{(2\pi)^4}\; \bar\psi(y)\,\gamma^\nu\,\frac{i(\not k-\not q)}{(k-q)^2+i\varepsilon}\,\gamma^\mu\,\psi(x)\; e^{-i(k-q)\cdot(y-x)}$$

$$= \cdots = (+)\left[\bar\psi\,\frac{\gamma^\nu(\not q-\frac{i}{2}\overleftrightarrow{\not D})\gamma^\mu}{(q-\frac{i}{2}\overleftrightarrow{D})^2}\,\psi\right]_{\text{at }\frac{x+y}{2}}$$

OPE

$$i\int d^4(x-y)\; e^{iq\cdot(x-y)}\; T\{J^\mu(x)\,J^\nu(y)\}$$ contains.

$$\begin{cases} \cdot\ (q^2\eta^{\mu\nu}-q^\mu q^\nu)\,\Pi_{ren}(q^2)\cdot \mathbb{1} \\ \cdot\ \sum_{j=1} C^{\mu\nu}{}_{\lambda_1\cdots\lambda_j}(q)\left[\bar\psi\,\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_j}\psi\right] \\ \quad + (\mu\leftrightarrow\nu)\ \text{anti-symmetric part.} \\ \cdot\ \cdots \end{cases}$$

$$\begin{array}{l} \text{naive dim} \Rightarrow (2+j) \\ -)\ \text{spin} \Rightarrow j \\ \hline \boxed{\underbrace{(\text{dim}-\text{spin})}_{\text{twist.}} = 2.} \end{array}$$

$T^{\mu\nu}$: insert OPE. in proton. $\langle\vec p|$ and $|\vec p\rangle$.

Assume. $$\langle\vec p|\,\bar\psi\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_j}\psi\,|\vec p\rangle = p^{\lambda_1}\cdots p^{\lambda_j}\,A_j.$$

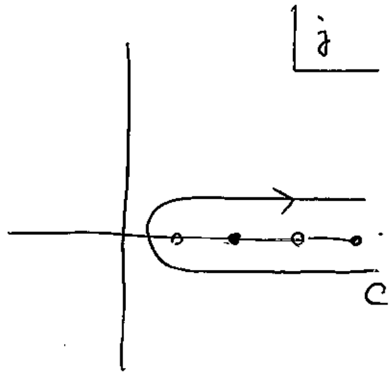
non-perturbative information.

(hw) $\Rightarrow$
$$\boxed{\begin{aligned} T^{\mu\nu}_{\left[\substack{(\mu\nu\,\mathrm{sym})\\ \text{twist } 2}\right]} &= \left\{\left[-\eta^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right]+\frac{1}{p\cdot q}\left[p^\mu-\frac{p\cdot q}{q^2}q^\mu\right]\left[p^\nu-\frac{p\cdot q}{q^2}q^\nu\right](2x)\right\}\times \\ &\quad \left[\sum_{j=1}^{\infty}\left[1+(-)^j\right]\left(\frac{1}{x}\right)^j\left(-\frac{A_j}{2}\right)\right] \end{aligned}}$$

$x\equiv\dfrac{-q^2}{2p\cdot q}$. $\begin{cases} \star\ \text{gauge invariant.} \\ \star\ (2\,\mathrm{Im}\,T_2)=(2x)(2\,\mathrm{Im}\,T_1) \end{cases}$ $\longleftrightarrow$ manifestation of fermionic parton. (hw).

$$☆\quad T_1 = \frac{-1}{8\pi}\sum_{j=1}^{\infty}\left[1+(-1)^j\right]\frac{1}{x^j}A_j$$

$$= \frac{1}{8\pi}\int_C \frac{dj}{2i}\,\frac{1+e^{-\pi i j}}{\sin(\pi j)}\,\frac{1}{x^j}A_j^{(+)}$$



$$A_j^{(+)}\Big|_{j=\text{even}} = A_j$$

$\mathbb{C}$ holomorphic fcn of j

$$\boxed{2\,\mathrm{Im}\,T_1 = \frac{1}{4\pi}\int_C\frac{dj}{2i}\,\frac{1}{x^j}A_j^{(+)}}$$

Mellin transform. for $\varphi(x)$ $\quad(x\in[0,\infty])$

$$\tilde{\varphi}(j) = \int_0^{+\infty}dx\,x^{j-1}\varphi(x). = \int_0^{+\infty}\frac{dx}{x}\,x^j\varphi(x) = \int_{-\infty}^{+\infty}d\ln(1/x)\,e^{-j\ln(1/x)}\varphi(x).$$

Inverse Mellin transform. for $\tilde{\varphi}$

$$\varphi(x) = \int_{-i\infty}^{+i\infty}\frac{dj}{2\pi i}\left(\frac{1}{x}\right)^j\tilde{\varphi}(x).$$

Fourier transformation. between. $\underline{\ln(1/x)}$ and. $\underline{(j/i)}$

<u>structure functions</u>

given by inverse Mellin transform of $\begin{pmatrix}\text{twist-2 spin-}j\text{ op.}\\ \text{matrix element}\end{pmatrix}$

§11.4 Parton Distribution Function.

$$\boxed{f_q(x) \equiv \frac{1}{4\pi}\,\frac{1}{2}\int_{-\infty}^{+\infty}dk\; e^{ikx}\,\langle \vec{p}\,|\left[\bar{\psi}(-\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(\tfrac{\bar{n}}{2}k)\right]|\vec{p}\rangle} \Leftarrow \begin{pmatrix}\text{quark}\\ \text{PDF}\end{pmatrix}$$

$$\boxed{f_{\bar{q}}(x) \equiv \frac{1}{4\pi}\,\frac{-1}{2}\int_{-\infty}^{+\infty}dk\; e^{ikx}\,\langle \vec{p}\,|\left[\bar{\psi}(\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(-\tfrac{\bar{n}}{2}k)\right]|\vec{p}\rangle}$$

$$\boxed{= \frac{1}{4\pi}\,\frac{1}{2}\int_{-\infty}^{+\infty}dk\; e^{ikx}\,\langle \vec{p}\,|\left[\bar{\psi}^c(-\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(\tfrac{\bar{n}}{2}k)\right]|\vec{p}\rangle} \quad \begin{pmatrix}\text{anti-quark}\\ \text{PDF}\end{pmatrix}$$

$$\bar{n}^\mu \equiv \frac{q^\mu}{(p\cdot q)}$$

one can show $\boxed{f_{\bar{q}}(x) = -f_q(-x).}$

$\boxed{\text{For even } \underline{\underline{j}}}$

$$\int_0^{+\infty}dx\; x^{j-1}\left\{f_q(x)+f_{\bar{q}}(x)\right\} = \frac{1}{2}\int_{-\infty}^{+\infty}dx\; x^{j-1}\left\{f_q(x)+f_{\bar{q}}(+x)\right\}$$

$$= \frac{1}{2}\,\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{dx}{2}\int_{-\infty}^{+\infty}dk\left[\left(-i\frac{\partial}{\partial k}\right)^{j-1}e^{ikx}\right]\left(\begin{array}{l}\langle \vec{p}\,|\,\bar{\psi}(-\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(\tfrac{\bar{n}}{2}k)|\vec{p}\rangle \\ -\langle \vec{p}\,|\,\bar{\psi}(\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(-\tfrac{\bar{n}}{2}k)|\vec{p}\rangle\end{array}\right)$$

$$= \frac{1}{4}\,\frac{1}{4\pi}\left.\left(\left(i\frac{\partial}{\partial k}\right)^{j-1}\langle \vec{p}\,|\left\{\begin{array}{l}\bar{\psi}(-\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(\tfrac{\bar{n}}{2}k) \\ -\bar{\psi}(\tfrac{\bar{n}}{2}k)\,\bar{\not{n}}\,\psi(-\tfrac{\bar{n}}{2}k)\end{array}\right\}|\vec{p}\rangle\right)\right|_{k=0}.$$

$$= \frac{1}{4\pi}\,\frac{1}{4}\cdot\langle \vec{p}\,|\left[\bar{\psi}\,\bar{\not{n}}\left(\frac{\bar{n}}{2}\cdot i\overleftrightarrow{D}\right)^{j-1}\cdot\psi\right]|\vec{p}\rangle \cdot \times\left[1+(-)^j\right]$$

$$= \frac{1}{4\pi}\left(\frac{1}{2}\right)\cdot A_j.$$

$$\Rightarrow \boxed{\left[2\,\mathrm{Im}\,T_1 = F_1(x)\right] \Longleftrightarrow \frac{1}{2\pi}\left[f_q(x)+f_{\bar{q}}(x)\right]}$$

q-PDF & $\bar{q}$-PDF contribute to the str. fcn.