Summary of the previous lecture (useful formulae)

$$\boxed{(i T^{\mu\nu}) \equiv \int d^4y \; e^{iq\cdot(x-y)} \langle h(\vec{p}) | T\{ iJ^\mu(x) \; iJ^\nu(y)\} | h(\vec{p})\rangle}$$

(definition)

$$J^\mu(x) = Q_q (\bar{\psi}_q \gamma^\mu \psi_q)$$

Q_q : charge $\begin{cases} 2/3 & \text{for u-type} \\ -1/3 & \text{for d-type} \end{cases}$

$$T^{\mu\nu} = (4\pi)\left\{ \left[-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right] T_1 + \frac{1}{(p\cdot q)}\left[p^\mu - \frac{(p\cdot q)}{q^2} q^\mu\right]\left[p^\nu - \frac{(p\cdot q)}{q^2} q^\nu\right] T_2 \right\}$$

(gauge-inv parametrization)

$$F_1 \equiv 2\,\mathrm{Im}\,T_1, \qquad F_2 \equiv 2\,\mathrm{Im}\,T_2.$$

$$\frac{d^2\sigma_{DIS}}{dQ^2\,dx} = \frac{4\pi\alpha_e^2}{Q^4}\,\frac{1}{x}\left[x y^2 F_1 + (1-y) F_2\right]$$

$F_2 = 2x F_1$ (for fermion parton at tree level.)

$$\longrightarrow \boxed{\frac{d^2\sigma_{DIS}}{dQ^2\,dx} = \frac{4\pi\alpha_e^2}{Q^4}\left[1+(1-y)^2\right] F_1(x)}$$

$$\boxed{\langle h(\vec{p}) | \left[\bar{\psi}\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{D}^{\lambda_2}\right)\cdots\left(\tfrac{i}{2}\overleftrightarrow{D}^{\lambda_j}\right)\psi\right]_{\text{sym. traceless}} | h(\vec{p})\rangle \equiv p^{\lambda_1}\cdots p^{\lambda_j} A_j}$$

(def. of hadron matrix element A_j)

$A_j^{(+)}$: hol. fcn of j s.t. $A_j^{(+)}\big|_{j\in 2\mathbb{N}} = A_j$

☆ tree-level OPE of $T^{\mu\nu}$ $\Rightarrow$ $\boxed{\text{Mellin transform of } (F_1 = 2\,\mathrm{Im}\,T_1) = \tfrac{1}{4} A_j Q_q^2}$

☆ $$\boxed{f_q(x) \equiv \int_{-\infty}^{+\infty} \frac{dk}{4\pi}\, e^{ikx} \langle h(\vec{p}) | \bar{\psi}(\bar{x} - \tfrac{\bar{n}}{2}k)\, \bar{\not{n}}\, \psi(\bar{x} + \tfrac{\bar{n}}{2}k) | h(\vec{p})\rangle}$$

$$\boxed{f_{\bar{q}}(x) \equiv -\int_{-\infty}^{+\infty} \frac{dk}{4\pi}\, e^{ikx} \langle h(\vec{p}) | \bar{\psi}(\bar{x} + \tfrac{\bar{n}}{2}k)\, \bar{\not{n}}\, \psi(\bar{x} - \tfrac{\bar{n}}{2}k) | h(\vec{p})\rangle}$$

(def. of "PDF")

$\begin{cases} \bar{n}^\mu = \dfrac{q^\mu}{(p\cdot q)} \\ \bar{x}^\mu : \text{arbitrary} \end{cases}$

$$\Rightarrow \boxed{\text{Mellin transform of } (f_q(x) + f_{\bar{q}}(x)) = \tfrac{1}{2} A_j}$$

$$\longrightarrow \boxed{F_1(x) = \tfrac{1}{2} Q_q^2 \left[f_q(x) + f_{\bar{q}}(x)\right]} \qquad \boxed{F_2(x) = Q_q^2\, x \left[f_q(x) + f_{\bar{q}}(x)\right]}$$

$$\longrightarrow \boxed{\frac{d^2\sigma_{DIS}}{dQ^2\,dx} = \frac{2\pi\alpha_e^2}{Q^4}\left[1+(1-y)^2\right] Q_q^2 \left[f_q(x) + f_{\bar{q}}(x)\right]}$$

$\boxed{\text{parton model.}}$ (idea)

replace $\langle h(\vec{p})| \;\cdots\; |h(\vec{p})\rangle$

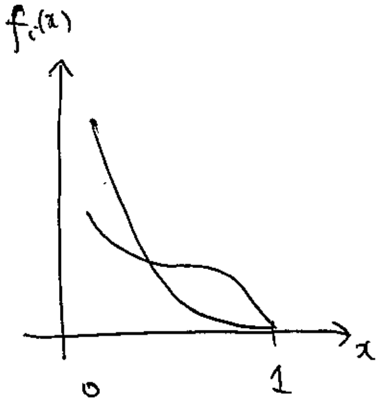
by $\displaystyle\sum_i \int_0^1 \frac{dz}{z} f_{q_i}(z) \langle q_i(z\vec{p})| \;\cdots\; |q_i(z\vec{p})\rangle$

$q_i = u, d, \cdots$
$\bar{u}, \bar{d} \cdots$

$\Longrightarrow$ direct computation of $2\,\mathrm{Im}\,T_1 \equiv F_1$, $2\,\mathrm{Im}\,T_2 \equiv F_2$

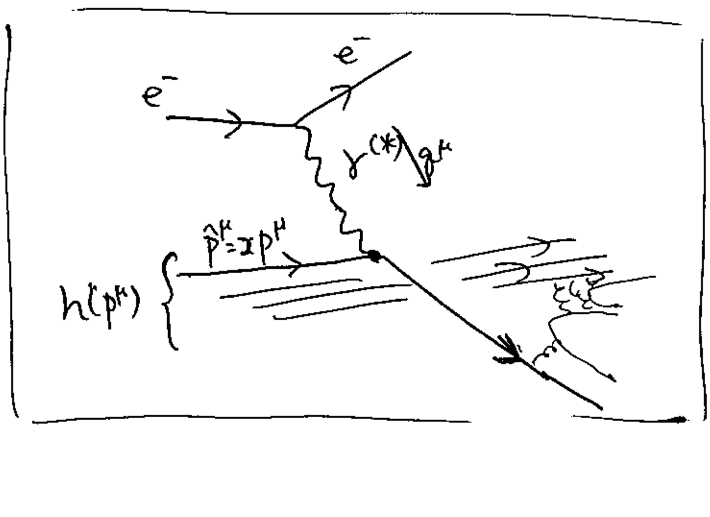
$\rightarrow F_1 = \frac{1}{2} Q_q^2 \left[f_q(x) + f_{\bar{q}}(x) \right]$

(homework).



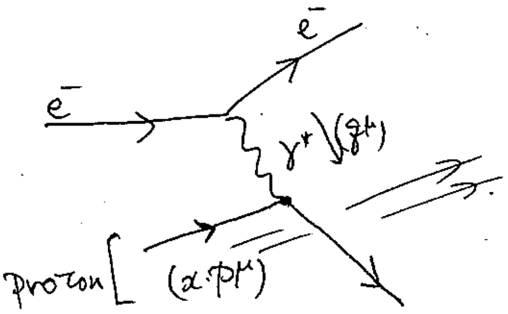
hadron : parton
w/ longitudinal momentum fraction. x.
$f_{q_i}(x)$

other partons : remain unimportant.
if $(-q^2) \gg \Lambda_{QCD}^2$

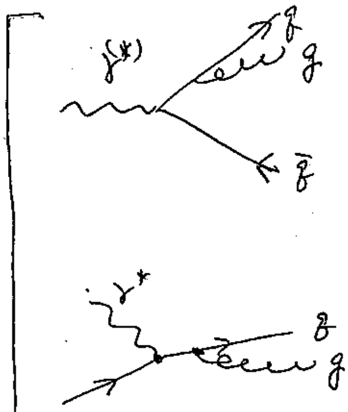


§11.5 Initial State Radiation (ISR)

(cf. FSR)



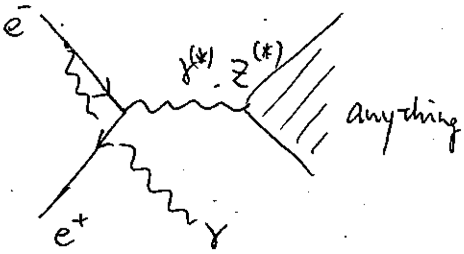
$\boxed{\text{DIS}}$

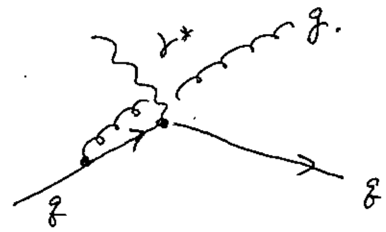


FSR
↓
soft γ / collinear divergence.

ISR

(s-channel) (t-channel)





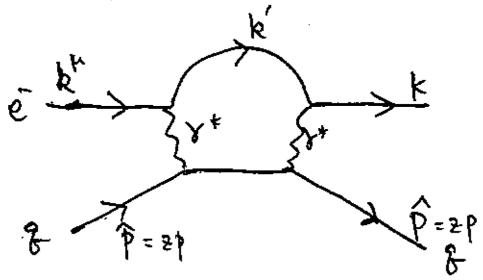
- ☆ divergence for the same reason as in FSR.
- ☆ un-observable (as in FSR) because the radiation goes down the beam pipe.

Does divergence cancel in observables in DIS?

- QED: see Peskin-Schroder. §6.5
- QCD: ? confinement. hadron...

$\boxed{\S 11.6 \quad \text{DGLAP equation.}}$

< Dokshitzer – Gribov – Lipatov – Altarelli – Parisi >



$$(2\,\mathrm{Im}\,\mathcal{M}) = \int d\Pi\, (\mathcal{M}^\dagger \mathcal{M})$$

unitarity.

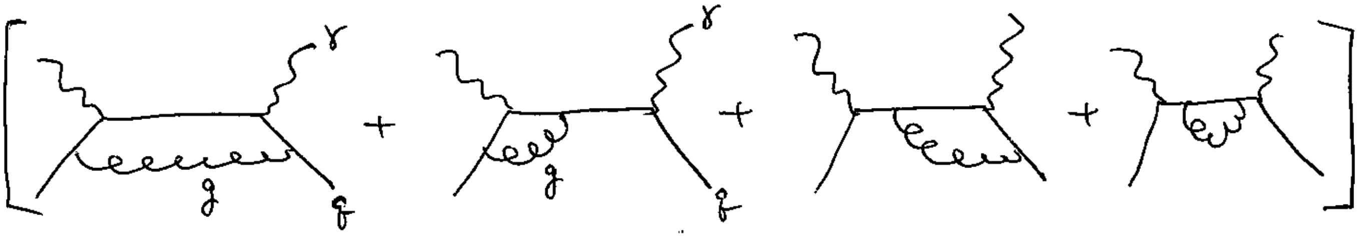
(total inelastic cross section)

Im [forward amplitude.].

Higher-order loop correction?

< ~~ignore~~ omit. lepton line. >

$[\gamma^* + q \to \gamma^* + q]$



$\Downarrow$

$\boxed{\text{real gluon emission.}}$

$\gamma^* + q \to q + g$

$g + q \to q + \gamma^*$.

$$\Delta(2\,\mathrm{Im}\,T^{\mu\nu}) = \underbrace{\int_0^1 \frac{dz}{z} f_q(z)}_{\text{parton model.}} \times \left[(-) \times \left[i\mathcal{M}^{\mu\nu}(\gamma^* + q \to \gamma^* + q) \right]^{(\text{spin-average})} \right]$$

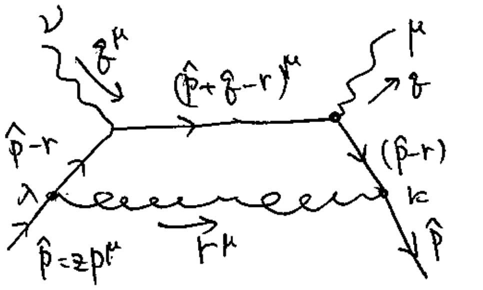
with on-shell propagators $\boxed{\dfrac{i}{p^2 - m^2 + i\varepsilon} \Rightarrow 2\pi\delta(p^2 - m^2)}$

Cutkosky rule

Peskin – Schröder §7.3.

e.g. <u>1st graph.</u>

$$\Delta^{(1)}(2\,\mathrm{Im}\,T^{\mu\nu}) = \int_0^1 \frac{dz}{z}\, f_q(z) \times (-)\int \frac{1}{2}\sum_r \bar{u}_r(\hat{p})\left[(-igt^a\gamma^\kappa)\frac{i(\hat{\not p}-\not r)}{(\hat{p}-r)^2+i\varepsilon}(iQ_q\gamma^\mu)\frac{i(\hat{\not p}+\not q-\not r)}{(\hat{p}+q-r)^2+i\varepsilon}\right.$$



$$\left.(iQ_q\gamma^\nu)\frac{i(\hat{\not p}-\not r)}{(\hat{p}-r)^2+i\varepsilon}(-igt^b\gamma^\lambda)\right]u_r(\hat{p})$$

$$\times\left(\frac{-i\,\eta_{\kappa\lambda}\,\delta_{ab}}{r^2+i\varepsilon}\right)\frac{d^4r}{(2\pi)^4}$$

with
$$\begin{cases}\dfrac{i}{(\hat{p}+q-r)^2+i\varepsilon} \Rightarrow (2\pi)\,\delta\left((\hat{p}+q-r)^2\right)\\[2ex] \dfrac{i}{r^2+i\varepsilon} \Rightarrow (2\pi)\,\delta(r^2).\end{cases}$$

spin average... $\mathrm{Tr}\left[\sum_r\left(u_r(\hat{p})\,\bar{u}_r(\hat{p})\right)\cdots\right] = \mathrm{Tr}\left[(\hat{\not p})\cdots\right].$

Feynman gauge.

If we were to use the tree level result.

$$\eta_{\mu\nu}(2\,\mathrm{Im}\,T^{\mu\nu}) = \frac{2\pi}{x}(F_2 - 6xF_1) \to -(4\pi)\,Q_q^2\left[f_q(x)+f_{\bar q}(x)\right]$$

$$\Rightarrow \boxed{-\frac{1}{4\pi}\,\eta_{\mu\nu}\,\Delta(2\,\mathrm{Im}\,T^{\mu\nu})}$$ how is it like?

$$\int\frac{d^4r}{(2\pi)^4}(2\pi)^2\delta(r^2)\,\delta\left((\hat{p}+q-r)^2\right) \Rightarrow$$

set an axis.
$$\begin{cases}\hat{p} = z(p, p, \vec{0})\\ q^\mu = (Q^0, Q^3, \vec{0})\end{cases}$$

$r^\mu = (r^0, r^3, \vec{r}_T).$

- $|\vec{r}_T|$ set by $\delta(r^2)$
- focus on a region. $r^\mu \not\parallel p^\mu$ $(r^0 \sim r^3) \Leftrightarrow (r^+, r^-, \vec{r}_T) \sim (1, \lambda^2, \lambda).$

$\Rightarrow (\hat{p}+q-r)^2 \simeq \chi z(2p\cdot q) + q^2$ if $(\Delta r)^\mu$ small.

$r^\mu = \vec{r}_T + z(1-\chi)p^\mu + (\Delta r)^\mu$ $\Rightarrow \boxed{z\chi - x \simeq 0}$

$$\boxed{-\frac{1}{4\pi}\left(2\,\mathrm{Im}\,T^{\mu\nu}\right)\eta_{\mu\nu}} \simeq Q_q^2 f_q(x) +$$

$$Q_q^2 \int \frac{dz}{z} f_q(z) \frac{[-g^2 C_2(R)]}{(-4\pi)} \int \frac{d(p\cdot r)}{(p\cdot q)} \frac{1}{8\pi} \times \left\{ 4\frac{r\cdot q}{(\hat{p}\cdot r)} + 8\frac{(\hat{p}\cdot q)\,(\hat{p}-r)\cdot q}{(\hat{p}\cdot r)(2\hat{p}\cdot q + q^2)} \times 2 \right\}$$

① ②+③

apart from $\frac{1}{(\hat{p}\cdot r)}$. use $r_\mu = (1-x)\hat{p}_\mu$.

$$\frac{(\hat{p}\cdot q)}{(\hat{p}\cdot r)} \times 4\left\{(1-x) + \frac{2x}{(1-x)}\right\} = \left\{\frac{1+x^2}{1-x}\right\}$$

$$\Rightarrow \text{"}(Q_q^2)\text{"}\,\text{"}[f_q(x)]\text{"} \cong Q_q^2 f_q(x) + Q_q^2 \frac{\alpha_s}{2\pi} C_2(R) \int \frac{d(p\cdot r)}{(p\cdot r)} \int_x^1 \frac{dz}{z} f_q(z) \left\{\frac{1+(x/z)^2}{1-(x/z)}\right\}$$

pure logarithmic divergence.
from a region w/ small $(p\cdot r)$.

Just like renormalization

$$\frac{4\pi}{g^2(p^2)} = \frac{4\pi}{g_0^2} + \frac{b}{2\pi}\ln\left(\frac{p^2}{\Lambda^2}\right) \Rightarrow \left[\frac{4\pi}{g_0^2} + \frac{b}{2\pi}\ln\left(\frac{\mu_R^2}{\Lambda^2}\right)\right] + \frac{b}{2\pi}\ln\left(\frac{p^2}{\mu_R^2}\right)$$

$$\left[\ldots\right] \equiv \left(\frac{4\pi}{g^2(\mu_R)}\right)$$

physical. observable parameter

renormalized coupling.

$$\boxed{\text{"}f_q(x;Q^2)\text{"} = f_q(x;\mu_F^2) + \frac{\alpha_s}{2\pi} C_2(R) \int_{\mu_F^2}^{Q^2} \frac{d(p\cdot r)}{(p\cdot r)} \int_x^1 \frac{dz}{z} f_q(z;\mu_F^2) \frac{1+(x/z)^2}{1-(x/z)}}$$

observable "renormalized"

μ_F: factorization scale.

$(-2\hat{p}\cdot r) \sim (\hat{p}-r)^2$: virtuality of parton q.

radiative corr. with $\boxed{\text{virtuality} \lesssim \mu_F^2}$: swept under the carpet. $f_q(z;\mu_F^2)$

near light cone.

- Wilson's interpretation of renormalization:

  momenta above μ_R : renormalized into $g(\mu_R)$

- (fluctuation / distribution) close to the light cone than μ_F^2 :

  swept into / taken into account in $f_q(x;\mu_F^2)$

$$\boxed{\frac{\partial f_q(x;\mu_F^2)}{\partial \ln(\mu_F^2)} = \frac{\alpha_s}{2\pi} C_2(R) \int_x^1 \frac{dz}{z} \frac{1+(x/z)^2}{1-(x/z)} f_q(z;\mu_F^2)}$$

DGLAP eq

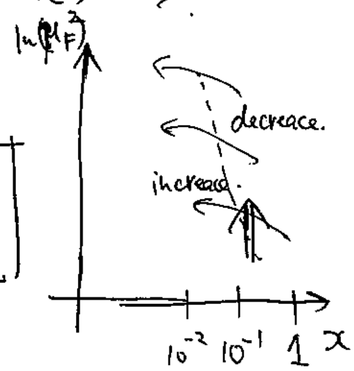
[weakly depend on μ_F

$\frac{1+(x/z)^2}{1-(x/z)} = P(x/z)$ splitting function.

convolution form.

< * extra contribution at $\delta(1-x/z)$ >

Mellin transform.



$$\boxed{\frac{\partial \tilde{f}_q(j;\mu_F^2)}{\partial \ln(\mu_F^2)} = \frac{\alpha_s}{2\pi} C_2(R)\, \gamma(j)\, \tilde{f}_q(j;\mu_F^2)}$$

$$\boxed{\tilde{f}_q(j) + \tilde{f}_{\bar{q}}(j) = \frac{1}{2} A_j}$$

OPE

operator M.E.

$$\int T\{J^\mu(x) J^\nu(y)\} e^{+iq\cdot(x-y)} d^4y \Rightarrow \sum_j C_j(q^2;\mu_R^2) \left[\bar{\psi}\gamma^\mu (\overleftrightarrow{D})^{j-1}\psi\right]_{\mu_R^2}$$

$$\Downarrow$$

$$\sum_j C_j(q^2;\mu_R^2) \langle h | [\quad]_{\mu_R^2} | h \rangle.$$

take care of UV DOF first

"IR" DOF later.

Parton model

$dz\, f_q(z;\mu_F^2)$

take care of collinear DOF first

$\Downarrow$

$dz\, f_q(z;\mu_F^2)\, \cancel{P(x/z)}\, \hat{\sigma}(z)$

hard scatter later.

the same thing.

Factorization into (hard part) × (non-perturbative part)

OPE makes it clear in the case of DIS.