An Algebraic Geometry Primer for physicists

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Memo: This note is approximately a write-up version of a series of informal lectures for (not so much math-inclined) particle-theory major graduate students at U. Tokyo in 2016. Primary objectives of the series was for those students to get acquainted with minimum basic concepts in algebraic geometry (so that they can use algebraic geometry, or at least they do not do knee-jerk rejection against it).

Given those objectives in that year, priority was not to intimidate those students too much. Materials were therefore kept almost to the absolute minimum; applications to arithmetic geometry are entirely thrown away, and the ground field is fixed to $\mathbb{C}$. We neither assumed that the students were already familiar with homology algebra, nor tried to step too much into the subject. The latter must be regarded as an important omission by those interested in short-distance behaviour of string theory. This informal lecture series in 2016 was for [1 hour] $\times$ [11 weeks]; in a re-run in 2019 ([1.5hour] x [11 weeks]), materials in addenda sections (§1.5 + §3.5) were also included partially, as students were more math lovers. Elliptic functions are treated only lightly in the appendix, because string students are already familiar to some extent. Section 7 is intended to fill the gap between basic principles of algebraic geometry and many existing literatures on the toric technique, which are often addressed to those who wish to use, than to understand.

This note gives priority to motivations, intuitions, and typical examples, than to precision, logic, rigorousness, definitions, and pathological examples. <u>jargon</u> is a jargon to be defined shortly after in the text. *jargon* is a jargon being defined there. Definitions will not be given to jargons$_*$ with $_*$ in this lecture note.

Many apologies for scientific and grammatical errors in this note.

1 Basic Concepts

1.1 Ring–Geometry Correspondence

Ring–geometry correspondence is one of key concepts in mathematics. When a geometry is given, $\mathbb{C}$-valued functions on that geometry forms a ring, so one can work out the ring of functions of that geometry. The ring-geometry correspondence is an observation that, morally speaking, this ring of functions retains all the information of the original geometry. This observation is stated in a more precise manner in this section 1.1 for Affine varieties.

Definition 1.1.1. When a geometry $X \subset \mathbb{C}^n$ is given by a finite number of polynomial equations on $\mathbb{C}^n$, it is called an *Affine variety.*

Definition 1.1.2. Definitions for *group*, *ring*, *field* are omitted. We never consider a ring or field that is not commutative in this lecture note.
An *ideal* of a ring R is a subring I of R where arbitrary $r \in R$ and $x \in I$ satisfy $r \cdot x \in I$.

Example 1.1.3. i) In a ring $R = \mathbb{Z}$, the set of all the integers divisible by a non-zero integer m, that is, $I := \{n \in \mathbb{Z} \mid m|n\} \subset R$, forms an ideal.
ii) More generally, for any ring R and a set of its element $x_1, \cdots, x_k \in R$, the subset $\{\sum_i^{\text{finite}} r_i \cdot x_i \mid r_i \in R\}$ is an ideal of R, and is denoted by $(x_1, \cdots, x_k)$. The set of elements $x_1, \cdots, x_k$ are called *generator*s. An ideal generated by a single element is callded a *principal ideal.*
iii) In the ring $R = \mathbb{Z}$, the integers divisible by m is regarded as an ideal (m) in this notation. For a set of integers $x_1, \cdots, x_k$ we can define their GCD; as ideals in $R = \mathbb{Z}$, $(x_1, \cdots, x_k) = (\mathrm{GCD}(x_1, \cdots, x_k))$. All the ideals in $R = \mathbb{Z}$ are principal ideals for that reason.
iv) The ring of $\mathbb{C}$-coefficient polynomials with n independent variables is denoted by $\mathbb{C}[x_1, \cdots, x_n]$. When there is just one variable x_1, we can define division of one polynomial by another, and also the remainder of the division. Euclidean algorithm can be used to find a GCD then. So, all the ideals in the ring $R = \mathbb{C}[x_1]$ are also principal ideals. Rings with this property are called *principal ideal ring.* The rings $\mathbb{C}[x_1, \cdots, x_n]$ with $n > 1$ are the typical examples of non-principal ideal rings. An ideal $(x - a, y - b)$ of a ring $\mathbb{C}[x, y]$, for example, is not a principal ideal.
v) for more information, see Addenda 1.5.1.

Functions in the ring $\mathbb{C}[x_1, \cdots, x_n]$ are called *regular functions* on $\mathbb{C}^n$. Exponential and trigonometric functions are holomorphic everywhere in $\mathbb{C}^n$, but are not included in this ring. It is the ring of regular functions, rather than that of holomorphic functions, that we pay attention to in algebraic geometry; those functions are not "liked" in algebraic geometry, because their singularity at infinity is not in the form of a simple pole. It does not invite too much troubles in many situations, though, by thinking that regular functions and holomorphic functions are much the same thing. $\mathbb{C}(x_1, \cdots, x_n)$ is the field of rational functions with n-variables; here, only a function obtained as a ratio of two regular functions is regarded as a *rational function* on $\mathbb{C}^n$.

1.1.4. Let $X \subset \mathbb{C}^n$ be an Affine variety, and $\{f_1, \cdots, f_k\}$ be a set of polynomial equations defining X. Then $I_X := \{g_1 f_1 + \cdots + g_k f_k \mid g_i \in \mathbb{C}[x_1, \cdots, x_n]\}$ forms an ideal of $\mathbb{C}[x_1, \cdots, x_n]$. I_X is the set of all the regular functions on $\mathbb{C}^n$ that vanish entirely on X. Now, two regular functions on $\mathbb{C}^n$, φ_1 and φ_2, become the same after restricted to $X \subset \mathbb{C}^n$, if and only if $(\varphi_1 - \varphi_2) \in I_X$. So, we declare that the ring of regular functions on an Affine variety $X \subset \mathbb{C}^n$ is the quotient ring $\mathbb{C}[x_1, \cdots, x_n]/I_X$. The ring of regular functions on an Affine variety X is denoted by $\mathbb{C}[X]$. So, $\mathbb{C}[X] = \mathbb{C}[x_1, \cdots, x_n]/I_X$.

Note that the definition of regular functions on an Affine variety X involves how X is embedded into $\mathbb{C}^n$. Such spaces as $\mathbb{C}^n$ in this case to which X is embedded and which is used to define the ring of regular functions are called *ambient space*s. Does this mean that the ring of regular functions "on X" depends on how it is embedded into a subvariety of some $\mathbb{C}^m$'s? We will come back to this question later (see 1.2.4).

1.1.5. Suppose that an Affine variety $X_1 \subset \mathbb{C}^n$ is the zero locus of a polynomial $f_1 \in \mathbb{C}[x_1, \cdots, x_n]$; $X_1 = \{x \in \mathbb{C}^n \mid f_1(x) = 0\}$. Similarly, let another Affine variety X_2 be the zero locus of another polynomial $f_2 \in \mathbb{C}[x_1, \cdots, x_n]$. Then $\mathbb{C}[X_1] = \mathbb{C}[x_1, \cdots, x_n]/(f_1)$ and $\mathbb{C}[X_2] = \mathbb{C}[x_1, \cdots, x_n]/(f_2)$. Now, what is the ring of regular functions of their intersection $X_1 \cap X_2$ and that of their union $X_1 \cup X_2$? As for $X_1 \cap X_2$, we can think of $\{f_1, f_2\}$ as the set of defining polynomials of $(X_1 \cap X_2) \subset C^n$, so $I_{X_1 \cap X_2} = (f_1, f_2)$, and $\mathbb{C}[X_1 \cap X_2] = \mathbb{C}[x_1, \cdots, x_n]/(f_1, f_2)$. As for $X_1 \cup X_2$, we can use $f_1 f_2$ as its defining polynomial. So, $I_{X_1 \cap X_2} = (f_1 f_2)$, and $\mathbb{C}[X_1 \cup X_2] = \mathbb{C}[x_1, \cdots, x_n]/(f_1 f_2)$.

Let I_1 and I_2 be ideals of a ring R. Then we can introduce two ideals of R:

$$I_1 I_2 := \{r x_1 x_2 \mid r \in R,\ x_1 \in I_1,\ x_2 \in I_2\}, \tag{1}$$

$$I_1 + I_2 := \{r_1 x_1 + r_2 x_2 \mid r_{1,2} \in R,\ x_1 \in I_1,\ x_2 \in I_2\}. \tag{2}$$

Using those notations, one can state that, for two Affine varieties X_1 and X_2 of $\mathbb{C}^n$, not necessarily codimension$_{\mathbb{C}} = 1$, the ring of regular functions of their intersection and union is given by $\mathbb{C}[x_1, \cdots, x_n]/(I_{X_1} + I_{X_2})$ and $\mathbb{C}[x_1, \cdots, x_n]/I_{X_1} I_{X_2}$, respectively.

1.1.6. When an Affine variety $X \subset \mathbb{C}^n$ is given as the zero locus of a polynomial $f_1 f_2$, which is not an irreducible element of $\mathbb{C}[x_1, \cdots, x_n]$, X consists of two pieces $\{f_1 = 0\}$ and $\{f_2 = 0\}$. Such an Affine variety is called *reducible*, and those that are not *irreducible*. When X is irreducible, the corresponding ideal $I_X \subset R = \mathbb{C}[x_1, \cdots, x_n]$ is a prime ideal of R, and R/I_X is a domain.

Definition 1.1.7. A proper ideal P of a ring R is *prime* (already $P = R$ is excluded), when the following condition is satisfied: Whenever $x, y \in R$ satisfy $xy \in P$, either $x \in P$ or $y \in P$.
In a ring R, a non-zero element $a \in R$ is a *zero divisor*, if there exists a non-zero element $b \in R$ so that $ab = 0$. (an example: $R = \mathbb{C}[x, y]/(xy)$; $x, y \in R$ are zero divisors.)
A ring R is a *domain*, if it does not have a zero divisor.

Suppose that the ideal I_X of an Affine variety $X \subset \mathbb{C}^n$ is not prime. This then means that there exists regular functions $f_1, f_2 \in R = \mathbb{C}[x_1, \cdots, x_n]$ so that $f_1 f_2 \in I_X$ but neither f_1 nor f_2 vanishes entirely over X. Now we can define two ideals $I^1 := (f_1, I_X)$ and $I^2 := (f_2, I_X)$ and consider two Affine varieties X_1 and X_2 given as the vanishing locus of all the elements in I^1 and I^2, respectively. X_1 and X_2 must be a proper subset of X, and X_1 and X_2 are distinct. So, X is not irreducible. [This means that, when X is irreducible, I_X is prime: justification for 1.1.6]

Exercise 1.1. A proof of the statement that "the quotient ring R/I is a domain if and only if I is prime" is left as an exercise.

1.1.8. A point in $\mathbb{C}^n$ corresponding to the coordinate $(x_1, \cdots, x_n) = (a_1, \cdots, a_n) \in \mathbb{C}^n$ is also regarded as an irreducible Affine variety in $\mathbb{C}^n$. The corresponding ideal is $I_a := (x_1 - a_1, \cdots, x_n - a_n)$. The ring of regular functions on this Affine variety is $\mathbb{C}[x_1, \cdots, x_n]/I_a \cong \mathbb{C}$; only the $\mathbb{C}$-values of polynomials at that point remain to be relevant information at that subvariety. More generally, an Affine variety is a point if and only if the corresponding ideal is a maximal ideal.

Definition 1.1.9. An ideal $\mathfrak{m}$ of a ring R is a *maximal ideal*, if any ideal of R containing $\mathfrak{m}$ as a proper subset is R itself.

Exercise 1.2. A proof of a statement that "the quotient ring R/I is a field if and only if I is a maximal ideal" is also left as an exercise.

1.1.10. Let $X \subset \mathbb{C}^n$ be an Affine variety, and $\mathbb{C}[X] = \mathbb{C}[x_1, \cdots, x_n]/I_X$ the ring of regular functions on X. Then its irreducible Affine subvarieties of X and prime ideals of $\mathbb{C}[X]$ are in one-to-one correspondence. As a part of this correspondence, there is also one-to-one correspondence between points in X and maximal ideals of $\mathbb{C}[X]$. So, the ring $\mathbb{C}[X]$ has full geometric information of X.

Definition 1.1.11. For an irreducible Affine variety X, $\mathbb{C}(X)$ denotes the field of fractions of a domain $R = \mathbb{C}[X]$. Any element in $\mathbb{C}(X)$ is called a *rational function* on X.

1.2 Contravariant Nature

We have seen how geometry of a given Affine variety is reflected in algebra of the ring of its regular functions. Let us now see how geometry of maps between Affine varieties is described in algebra of betrween the corresponding rings of regular functions.

1.2.1. Let us first think of a map from $\mathbb{C}^n = \{(x_1, \cdots, x_n)\}$ to $\mathbb{C}^m = \{(y_1, \cdots, y_m)\}$. In the geometric intution, a map $\phi : \mathbb{C}^n \to \mathbb{C}^m$ is dictated by specifying how y_i (for $i = 1, \cdots, m$) depends on $(x_1, \cdots, x_n)$. In algebraic geometry, we restrict our attention only to polynomial dependence on $(x_1, \cdots, x_n)$. Namely, we consider a class of maps realized by

$$\phi : \mathbb{C}^n \ni (x_1, \cdots, x_n) \longmapsto (\phi_1(x), \cdots, \phi_m(x)) = (y_1, \cdots, y_m) \in \mathbb{C}^m \tag{3}$$

for m polynomials $\phi_i \in \mathbb{C}[x_1, \cdots x_n]$ $(i = 1, \cdots, m)$. Now, we can think of pulling back regular functions on $\mathbb{C}^m$ to $\mathbb{C}^n$ by ϕ:

$$\phi^* \ : \ \mathbb{C}[y_1, \cdots, y_m] \ni f \longmapsto (f \circ \phi) \in \mathbb{C}[x_1, \cdots, x_n]. \tag{4}$$

This map ϕ^* is realized by sending the generators of $\mathbb{C}[y_1, \cdots, y_m]$ by

$$\phi^* \ : \ y_i \longmapsto \phi_i \in \mathbb{C}[x_1, \cdots, x_n]. \tag{5}$$

ϕ^* is a homomorphism between the ring of regular functions, and the direction of the arrow is oppsite from that of ϕ. Conversely, if ψ^* is a homomorphism from $\mathbb{C}[y_1, \cdots, y_m]$ to $\mathbb{C}[x_1, \cdots, x_n]$, then ψ^* must send each one of y_i's to some polynomials denoted by $\psi_i \in \mathbb{C}[x_1, \cdots, x_n]$. These ψ_i's $(i = 1, \cdots, m)$ define a map from $\mathbb{C}^n$ to $\mathbb{C}^m$.

1.2.2. Consider two Affine varieties $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$. We hope that something we call a <u>regular map</u> ϕ from X to Y is such that the pull-back of a regular function of Y under ϕ^* is also a regular function of X. So, this means that a *regular map* is in one-to-one correspondence with a ring homorphism

$$\phi^* \ : \ \mathbb{C}[y_1, \cdots, y_m]/I_Y = \mathbb{C}[Y] \longrightarrow \mathbb{C}[X] = \mathbb{C}[x_1, \cdots, x_n]/I_X. \tag{6}$$

Put more intuitively, a regular map is specified by picking up m polynomials $\phi_i \in \mathbb{C}[x_1, \cdots, x_n]$ modulo I_X; ϕ_i's $(i = 1, \cdots, m)$ are used map the ambient space $\mathbb{C}^n$ of X into the ambient spacee $\mathbb{C}^m$ of Y; the umbiguity in the choices of ϕ_i's (modulo I_X) does not introduce any ambiguity in where in $\mathbb{C}^m$ the subvariety X is sent; we have to make sure, however, that the image of X is within the vanishing locus of I_Y (i.e., the subvariety Y).

1.2.3. When a regular map $\phi : X \longrightarrow Y$ is an injection (embedding), $\phi^* : \mathbb{C}[Y] \longrightarrow C[X]$ is surjective. When a regular map $\phi : X \longrightarrow Y$ is a surjection, $\phi^* : \mathbb{C}[Y] \longrightarrow \mathbb{C}[X]$ is injective. One can use the simple injection $\phi : \mathbb{C} \hookrightarrow \mathbb{C}^2$ ($\phi^* : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x]$ by $\phi^*(x) = x$, $\phi^*(y) = 0$) and the simple projection $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$ ($\phi^* : \mathbb{C}[x] \hookrightarrow \mathbb{C}[x, y]$ by $\phi^*(x) = x$) to see that.

1.2.4. Two Affine varieties $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ are regarded the same (*isomorphic*) in algebraic geometry, if and only if there is a pair of regular maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ so that $g \circ f = \mathbf{id}_X$ and $f \circ g = \mathbf{id}_Y$. This is the way algebraic geometry sees geometry;[1] this is how varieties are distinguished from one another. In particular, a way a variety is embedded into an ambient space (including the holomorphic coordinates in the ambient space, and the polynomial equations for the variety) is an important part of the property (identity) of the variety. So, in algebraic geometry, we do not take a perspective that there would be an abstract geometry X a priori without referring to how X is embedded into an ambient space and characterized by a set of polynomial equations.

1.3 Zariski Topology and Structure Sheaf

1.3.1. Fix an irreducible Affine variety $X \subset \mathbb{C}^n$. Let $R = \mathbb{C}[X]$ be its ring of regular functions, and $\mathbb{C}(X)$ the field of rational functions. For $0 \neq f \in R$, let V_f be the subvariety of X specified as the zero locus of f, and $U_f := X \backslash V_f$. Then any function of the form (regular function)$/f^n$ for some $n \geq 0$ is, certainly a rational function on X, and moreover, its pole locus is contained within V_f so that it remains regular within U_f. Conversely, one can see that any rational function on X that may have pole only within V_f and remain regular on U_f is in this form for some $n \geq 0$. So, based on the definition provided below, we see that all the rational functions of X that remain regular in U_f is given by the ring of fractions $(S_f)^{-1}R$ associated with a multiplicatively closed subset $S_f := \{f^n \mid n = 0, 1, \cdots ,\}$.

Definition 1.3.2. For a ring R and its multiplicatively close subset S in which $0 \notin S$, the *ring of fractions* $S^{-1}R$ is defined by the set of $\{r/s \mid r \in R, s \in S\}$; (r/s) and (r'/s') are regarded as the same element in $S^{-1}R$, when (r/s) can be reduced to (r'/s') or vice versa. Addition, subtraction and multiplication laws are introduced just as we do to $\mathbb{Q}$.
When R is a domain, its *field of fractions* is obtained by choosing $S = R\backslash\{0\}$. (i.e., $\mathbb{Q}$ for $R = \mathbb{Z}$).

1.3.3. In algebraic geometry, rational functions have poles along subvarieties, and they remain

[1] Note also that we have not introduced metric on those geometry. When we introduce two different metrics on a givene geometry X, they will be regarded distinct in the category of Riemann geometry. But we throw away (ignore) such information associated with the choice of metric, when we deal with geometries in the category of algebraic geometry.

regular in the complement of those subvarieties. So, it is tempting to give a special status to those subvarieties / complements of subvarieties. So here comes a

Definition (axiom): To *introduce a structure of topological space to a set*[2] X is either a) to specify a set $\mathcal{U} = \{U_i\}_{i\in I}$ of subsets $U_i \subset X$ so that the following three conditions are satisfied:

O-1 Intersection of finite number of members of $\mathcal{U}$ is also a member of $\mathcal{U}$. Namely, $(\cap_{a\in A} U_a)$ for any $A \subset I$ is a member of $\mathcal{U}$ if $|A| < \infty$.

O-2 Union of any members of $\mathcal{U}$ is also a member of $\mathcal{U}$. Namely, $(\cup_{a\in A} U_a)$ for any $A \subset I$ is a member of $\mathcal{U}$.

O-3 X itself as a subset of X, and the empty subset $\phi \subset X$ are also members of $\mathcal{U}$,

or b) to specify a set $\mathcal{V} = \{V_i\}_{i\in J}$ of subsets $V_i \subset X$ so that the following conditions are satisfied:

C-1 Union of finite number of members of $\mathcal{V}$ is also a member of $\mathcal{V}$. Namely, $(\cup_{a\in A} V_a)$ for any $A \subset J$ is a member of $\mathcal{V}$ if $|A| < \infty$.

C-2 Intersection of any members of $\mathcal{V}$ is also a member of $\mathcal{V}$. Namely, $(\cap_{a\in A} V_a)$ for any $A \subset J$ is a member of $\mathcal{V}$.

C-3 The empty subset $\phi \subset X$, and X itself as a subset of X, are also members of $\mathcal{V}$.

When such a set of subsets $\mathcal{U}$ [resp. $\mathcal{V}$] is given, one can always find a corresponding set of subsets $\mathcal{V}$ [resp. $\mathcal{U}$] satisfying the conditions C-1 3 [resp. O-1 3] by $\mathcal{V} := \{V_i := X\backslash U_i\}_{i\in I}$ [resp. $\mathcal{U} := \{U_j := X\backslash V_j\}_{j\in J}$].

Implicit in this definition is an observation that even when a set of points X is fixed, there can be more than one ways to specify a set of subsets $\mathcal{U}$ [resp. $\mathcal{V}$] of X satisfying the conditions O-1–3 [resp. C-1–3]; a given set of points X may admit multiple structures of topological space. So, when a structure of topological space is introduced to a set of points X, we call any member of $\mathcal{U}$ as an *open subset* of X, and any member of $\mathcal{V}$ as an *closed subset* of X under the structure of topological space introduced to X.

Consider a manifold M; by definition, it has a local neighbourhood to which a set of local coordinates is given. Implicitly a structure of topological space is chosen in a manifold; a subset $U \subset M$ is regarded as an open subset of M if and only if for any point $p \in U$, one can find an open

[2] When we throw away information of metric on manifolds (structure of Riemann geometry), one can still talk of whether two manifolds are diffeomorphic or not; even when we throw away choices of local coordinates from manifolds (structure of manifold), one can still retain which subsets are regarded as open/closed subsets and which subsets are not—(*); this remaining information (*) on top of a set as a collection of points is called the *structure of topological space*.

disc of small enough radius ϵ around p so that the disc is entirely contained in U—(*). The set of open subsets M satisfies the conditions O-1 3 above; the conditions O-1 3 have been abstracted from various properties of open subsets of M. Similarly, the conditions C-1 3 have been abstracted from various properties of closed subsets of M. The structure of topological space (*), which relies on local coordinates and discs with small enough raidi, is called *analytic topology*. Whenever we think of a manifold in the category of manifolds, we implicitly assume that we use the analytic topology as the structure of topological space. This still allows us to introduce another structure of topological space to one and the same set of points as M, when we think of M in another category (such as the category of algebraic varieties).

In the category of algebraic geometries, it is customary to introduce the structure of topological space in the following way. Suppose that X is an Affine variety. Any subsets of the form $V_f \subset X$ and their arbitrary intersections are registered as members of $\mathcal{V}$, and they are all the members of $\mathcal{V}$. Then the conditions C-1 3 are satisfied.[3] The structure of topological space introduced in this way is called *Zariski topology* (as opposed to analytic topology). Any member of $\mathcal{V}$ are called *Zariski-closed subset* of X. So, the set of Zariski-open subsets $\mathcal{U}$ of an Affine variety X consists of any sets of the form $U_f = X \backslash V_f$ and their arbitrary unions.[4]

Much smaller subsets of X is registered as an open subset in Zariski topology than in analytic topology. Under Zariski topology, an Affine variety is not even Hausdorff. An advantage of introducing Zariski topology (rather than sticking to the analytic topology available on a manifold) is that any rational function on an algebraic variety is expected to have singularity only at a Zariski-closed subvariety of X, and still remains regular on the complement, which is Zariski-open subset of X. By keeping track of the ring of functions that remain regular only at individual members of the Zariski-open subsets $\mathcal{U}$ of X, we can have a full grasp of which rational functions have singularity where. In a more mathematical language, this observation is summarized as follows.

1.3.4. Structure sheaf of an Affine variety X: for any Zariski-open subset $U \in \mathcal{U}$ of X, there is a corresponding ring of functions that remain regular everywhere in U. This ring is denoted by $\mathcal{O}_X(U)$. Whenever one Zariski-open subset U_1 is fully contained in another Zariski-open subset

[3] Consder an infinite closed subsets $V_n := \left\{(x, y) \in \mathbb{C}^2 \mid y = nx\right\} \subset \mathbb{C}^2$. Intersection of all of them $\cap_{n=0,\cdots,\infty} V_n = \{0\} \subset \mathbb{C}^2$ is still regarded as a closed subset of $\mathbb{C}^2$. A union of finite number of them $\cup_{i \in A \subset \mathbb{N}} V_i$ for $|A| < \infty$ is still a closed subset, but the union of all of them, $\cup_{i \in \mathbb{N}} V_i$ is not. This subset is not contained in the list of Zariski closed subsets because $\prod_{n=1}^{\infty}(y - nx)$ is not regarded as a polynomial on $\mathbb{C}^2$.

[4] Here is a side remark. **Def**: A map $f : X \to Y$ between two topological spaces is *continuous*, if and only if $f^{-1}(U')$ is an open subset of X for any open subset U' of Y. With this definition, here is a well-defined exercise:

Exercise 1.3. When we see two Affine varieties X and Y as topological spaces under the Zariski topology, one can see that a regular map $f : X \to Y$ is a continuous map in the standard definition above; a proof of this statement is left as an exercise.

U_0, any regular function on U_0 defines a corresponding regular function on U_2 by just ristricting the range of definition from U_0 to U_1. This is always a ring homomorphism $\rho_{U_1U_0} : \mathcal{O}_X(U_0) \longrightarrow \mathcal{O}_X(U_1)$. Furthermore, if there are two Zariski-open subsets U_1 and U_2 contained in U_0, then there are two series of ring homomorphisms ($i = 1, 2$), $\rho_{U_iU_0} : \mathcal{O}_X(U_0) \to \mathcal{O}_X(U_i)$ and $\rho_{U_*U_i} : \mathcal{O}_X(U_i) \to \mathcal{O}_X(U_*)$, where $U_* := U_1 \cap U_2$ is another Zariski open subset of X. By definition, $\rho_{U_*U_1} \circ \rho_{U_1U_0}$ and $\rho_{U_*U_2} \circ \rho_{U_2U_0}$ are the same ring homomorphism from $\mathcal{O}_X(U_0)$ to $\mathcal{O}_X(U_*)$ determined by simple restriction of the range of definition of regular functions from U_0 to $U_* \subset U_0$.

1.3.5. Discussions 1.3.1–1.3.4 are written for a ring $R = \mathbb{C}[X]$ of an Affine variety, where $\mathbb{C}[X]$ is the quotient ring of $\mathbb{C}[x_1, \cdots, x_n]$. A completely parallel discussion can be repeated for a more general class of rings R, rings that appear in arithmetic set-ups for examples, however. See Addenda 1.5.3 and 1.5.4.

The observation above is a motivation for introducing the following definition.

Definition 1.3.6. Let $(X,\mathcal{U})$ be a topological space; a structure of topological space is introduced by specifying a set $\mathcal{U}$ of open subsets of a set of points X. A *sheaf of rings*[5] $\mathcal{F}$ on $(X,\mathcal{U})$ is a collection of the following information: For every open subset $U \in \mathcal{U}$, a ring denoted by $\mathcal{F}(U)$ is given, and for every pair of open subsets satisfying $U_1 \subset U_0$, a ring homomorphism $\rho_{U_1U_0} : \mathcal{F}(U_0) \to \mathcal{F}(U_1)$ is specified in such a way that $\rho_{U_*U_1} \circ \rho_{U_1U_0} = \rho_{U_*U_2} \circ \rho_{U_2U_0}$.

1.3.7. So, $\mathcal{O}_X$ for an Affine variety X is an example of sheaves of rings on X with Zariski topology. In fact, we can think of a sheaf of regular functions on an algebraic variety X that is not necessarily an Affine variety. A *projective variety* X is a subset of a projective space $\mathbb{P}^n$ given by the common zero locus of a finite set of homogeneous polynomials on $\mathbb{P}^n$. In introducing Zariski topology to X, the role played by V_f's is carried by V_F's, where F is a homogeneous function on X. In a Zariski open subset $U_F = X \backslash V_F$, regular functions are in the form of [homog. fcns on X]/[some power of F]. The sheaf of rings of regular functions on an Affine variety X or a projective variety X is always denoted by $\mathcal{O}_X$, and is called the *structure sheaf* of X.

[5] What is written here is in fact the definition of a *presheaf* of rings, rather than that of a *sheaf* of rings. For a presheaf to be regarded as a sheaf, two more conditions need to be satisfied. The two conditions will be found in any math textbooks explaining sheaves, so we omit them in this note. The two conditions are an abstraction of two properties that continuous functions have: i) a function on an open set U can be constructed, if its definition is given in each one of the patches of an open covering $U = \cup_i U_i$, and the definitions on individual open patches U_i are mutually consistent over their overlaps $U_i \cap U_j$, and ii) two functions are identical, if they are identical in each one of the patches. Since we only deal with sheaves of functions or things that are similar in this note, we do not emphasize the two extra conditions for a presheaf to be regarded as a sheaf. A typical example of a presheaf that is not a sheaf is a constant sheaf (rather than a locally constant sheaf) on a topological space X that contains an open subset that is not connected.

The structure sheaf is precisely the machinary (language) where we can keep track of the ring–geometry correspondence. If we were to deal only with Affine varieties, we would not have to introduce a notion "sheaf;" we just have to know $R := \mathbb{C}[X]$, and the ring of functions regular in an open subset U can be derived from R systematically as a ring of fractions $S^{-1}R$ for some $S \subset R$. When X is a projective (compact) variety, however, $R = \mathbb{C}[X]$ consists only of constant valued functions, and hence $R \cong \mathbb{C}$. We canNOT start from this ring R and construct the ring of regular functions of all the Zariski-open subsets of X as its rings of fractions $S^{-1}R$ for some $S \subset R$ (though we can in the case of Affine varieties). This observation may well be regarded as one of advantages of introducing the notion "sheaf."

1.4 Introduction to Sheaf

1.4.1. The structure sheaf $\mathcal{O}_X$ of an algebraic variety X is an example of sheaves of rings on X, but there are more examples. Let us have a look at a few of them. For a subvariety Y of X, we can define the *ideal sheaf* of Y, denoted by $\mathcal{I}_Y$, by setting $\mathcal{I}_Y(U)$ as the subring of $\mathcal{O}_X(U)$ that vanish entirely on $Y \cap U$. We can also define another sheaf $i_*(\mathcal{O}_Y)$ on X, by setting $i_*(\mathcal{O}_Y)(U)$ for an open subset U of X as $\mathcal{O}_Y(Y \cap U)$; the latter sheaf is referred to as the push-forward of the structure sheaf of Y.

As a generalization of sheaves of rings on a topological space, we introduce sheaves of Abelian groups:

Definition 1.4.2. Let $(X,\mathcal{U})$ be a topological space. A *sheaf of Abelian groups* $\mathcal{F}$ on $(X,\mathcal{U})$ is a collection of the following information: For every open subset $U \in \mathcal{U}$ of X, an Abelian group denoted by $\mathcal{F}(U)$ is given, and for every pair of open susets satisfying $U_1 \subset U_0$, a homomorphism $\rho_{U_1U_0} : \mathcal{F}(U_0) \to \mathcal{F}(U_1)$ is specified in such a way that $\rho_{U_*U_1} \circ \rho_{U_1U_0} = \rho_{U_*U_2} \circ \rho_{U_2U_0}$, whenever $U_{1,2} \subset U_0$ and $U_* := U_1 \cap U_2$.

When X is an algebraic variety (with Zariski topology), examples of a sheaf of Abelian groups include i) the sheaf $\mathcal{E}$ of regular sections of a holomorphic vector bundle E on X, where $\mathcal{E}(U)$ is a space of sections of E that remain regular on $U \subset X$, ii) the sheaf $\mathcal{H}om(E_1, E_2)$ of homomorphisms between two distinct vector bundles E_1 and E_2 on X. For a manifold M with the analytic topology, $\mathcal{A}^p$ denotes the sheaf of p-forms, where $\mathcal{A}^p(U)$ for an open subset $U \subset M$ is the space of p-forms that remain smooth in $U \subset M$. In all the examples here, $\mathcal{E}(U)$, $\mathcal{H}om(E_1, E_2)(U)$ and $\mathcal{A}^p(U)$ may be regarded Abelian groups, but they do not have a structure of a ring. They are examples of a sheaf of Abelian groups, but not a sheaf of rings. Any sheaf of rings $\mathcal{F}$ on a topological space can also be regarded as a sheaf of Abelian groups, by simply ignoring the multiplication laws in the rings $\mathcal{F}(U)$.

It is customary to use $\mathcal{F}$ and something similar (such as $\mathcal{E}$ and $\mathcal{G}$) as a notation for a sheaf, because the French word *faisceaux* corresponds to *sheaf* in English. The homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ for open sets $V \subset U$ is often denoted by ρ, because $\mathcal{F}(U)$ as the ring of functions or differential forms on U is a prototypical examples of sheaves, when the homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is the $\underline{R}$estriction of the region of definition of those functions / differential forms.

1.4.3. Poincare lemma states that any closed p-form on a topological space $(X, \mathcal{U})$ can be regarded as an exact p-form at least locally, though not necessarily globally on M. In this subtle difference lives topology of X. Let us translate the property into the language of sheaves of Abelian groups. We begin with generalizing the d operation on p-forms, and then translate the Poincare lemma into a definition of exact sequence of sheaves of Abelian groups.

Definition 1.4.4. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of Abelian groups on a topological space $(X, \mathcal{U})$. A *map* $\phi : \mathcal{F} \to \mathcal{G}$ between the two sheaves of Abelian groups is a collection of homomorphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for open subsets $U \in \mathcal{U}$ that are compatible with the restriction homomorphisms in $\mathcal{F}$ and $\mathcal{G}$, i.e., $\phi_{U_1} \circ \rho^{(\mathcal{F})}_{U_1 U_0} = \rho^{(\mathcal{G})}_{U_1 U_0} \circ \phi_U$, for any two open subsets satisfying $U_1 \subset U_0$.

Certainly the d operation (taking exterior derivative) induces a map $d : \mathcal{A}^p \to \mathcal{A}^{p+1}$. The property that the map $d \circ d : \mathcal{A}^p \to \mathcal{A}^{p+2}$ vanishes motivates to introduce

Definition 1.4.5. When a chain of maps between sheaves of Abelian groups $\phi_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ satisfies the property that $\phi_{i+1} \circ \phi_i : \mathcal{F}_i \to \mathcal{F}_{i+2}$ vanishes for all $i \in \mathbb{Z}$, the chain

$$\to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to \tag{7}$$

is called a *chain complex of sheaves of Abelian groups.*

Poincare's lemma for the chain complex of sheaves of differential forms

$$0 \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots \to \mathcal{A}^n \to 0 \tag{8}$$

on a real n-dimensional manifold M is translated into a statement that this chain complex of sheaves of Abelian groups is exact.

Definition 1.4.6. Consider a chain complex of sheaves of Abelian groups on a topological space $(X, \mathcal{U})$. Suppose that for any $s \in \mathcal{F}_i(U)$ such that $\phi_i(s) = 0 \in \mathcal{F}_{i+1}(U)$ and a point $x \in U$, one can find a small enough neighbourhood $U_{x,s} \in \mathcal{U}$ of x, $U_{x,s} \subset U$, so that $\rho_{U_{x,s}U}(s)$ is contained in $\mathrm{Im}\,(\phi_{i-1} : \mathcal{F}_{i-1}(U_{x,s}) \to \mathcal{F}_i(U_{x,s}))$. Then the chain complex of sheaves of Abelian groups is called *exact.*[6]

[6] In this lecture note, we avoid introducing such technical terms as stalk, direct limit, local ring or residue field in this lecture note. We keep exposure to algebra and also homology algebra minimum.

Chain complexes that are exact are also called *exact sequence*s.

Here are a few examples of exact sequences of sheaves of Abelian groups on algebraic varieties. First, for an algebraic variety X and its subvariety Y,

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_*(\mathcal{O}_Y) \to 0 \tag{9}$$

is an exact sequence. This will be obvious when X is an Affine variety; it is enough to remember that $\mathbb{C}[Y] = \mathbb{C}[X]/I_Y$, and its corresponding versions in the rings of fractions. Even when X is a projective variety, we can choose a neighbourhood $U_x \in \mathcal{U}$ of $x \in X$ so that U_x is an Affine variety. In fact, even for $U = X$, the chain complex of Abelian groups

$$0 \to [\mathcal{I}_Y(X) \cong 0] \to [\mathcal{O}_X(X) \cong \mathbb{C}] \to [i_*(\mathcal{O}_Y)(X) = \mathcal{O}_Y(Y) \cong \mathbb{C}] \to 0 \tag{10}$$

is exact.

As in the example of the chain complex of differential forms on a manifold, however, an exact sequence of sheaves of Abelian groups evaluated at $X \in \mathcal{U}$ is not always an exact sequence of Abelian groups. Just like

$$0 \to \mathcal{A}^0(M) \to \mathcal{A}^1(M) \to \cdots \to \mathcal{A}^n(M) \to 0 \tag{11}$$

is still a chain complex but not necessarily an exact sequence of Abelian groups,

$$\to \mathcal{F}_{i-1}(X) \to \mathcal{F}_i(X) \to \mathcal{F}_{i+1}(X) \to \tag{12}$$

is still a chain complex, but not necessarily an exact sequence of Aelian groups.

Here is such an example. Let $X = E$ be an elliptic curve, with $e \in E$ the origin of $E = \mathbb{C}/\mathbb{Z}^{\oplus 2}$ (infinity point in the Weierstrass model). Let $\mathcal{F}$ be the sheaf[7] of rational fuctions on E that are allowed to have a pole of order unity at e and remain regular elsewhere; $\mathcal{F}(U)$ is therefore the same as $\mathcal{O}_E(U)$ if $e \notin U$, while $\mathcal{F}(U)$ is slightly larger than $\mathcal{O}_E(U)$, if $e \in U$. Then

$$0 \to \mathcal{O}_E \to \mathcal{F} \to i_*(\mathcal{O}_e) \to 0 \tag{13}$$

is an exact sequence of sheaves of Abelian groups on the elliptic curve (see Exercise 1.4). In this example,

$$0 \to [\mathcal{O}_E(E) = \mathbb{C}] \to [\mathcal{F}(E) = \mathbb{C}] \to [i_*(\mathcal{O}_e)(E) = \mathbb{C}] \to 0 \tag{14}$$

is still a chain complex of Abelian groups, but is not exact, because the homomorphism to $[i_*(\mathcal{O}_e)(E) \cong \mathbb{C}]$ is a zero map, not surjective.

Exercise 1.4. A proof that (13) is an exact sequence is left as an exercise.

[7] In the notation that is introduced in section 3, this sheaf $\mathcal{F}$ is $\mathcal{O}_E(e)$.

1.5 Addenda

1.5.1. Rings, with more special properties, or with more general properties: fields can be regarded as a class of rings that have a special property, and similarly, PID's are also a class of rings that have a special property. Schematically,

$$
\begin{tikzcd}
 & & \text{Dedekind ring} \arrow[d] \arrow[dr] & & & \\
\text{field} \arrow[r] & \text{PID} \arrow[ur] \arrow[dr] & \text{\&} \arrow[l] & \text{normal ring} \arrow[r] & \text{domain} \arrow[r] & \text{ring,} \\
 & & \text{UFD} \arrow[u] \arrow[ur] & & &
\end{tikzcd}
\tag{15}
$$

where rings with more special properties are placed toward the left, and those with more general properties toward the right. It is known that a Dedekind ring that is also a UFD (unique factorization domain$_*$) is a PID (as indicated in (15)). A normal ring R is also said to be an *integrally closed domain*; a domain R is a *normal ring*, if all the elements of the field of fractions of R that are integrally closed over R are contained already within R. A normal ring R is a *Dedekind ring*, if its Krull dimension$_*$ is 1 (an alternative definition: any ideal can be factorized into a product of prime ideals, and the prime ideal decomposition is unique).

PID's that are not fields include $R = \mathbb{Z}$, $\mathbb{Z}[\sqrt{-1}]$, and $k[x_1]$ with any field k.

UFD's that are not PID's include $R = k[x_1, x_2, \cdots, x_n]$ and $R = \mathbb{Z}[x_1, \cdots, x_n]$, both with $n > 1$ (being multiple dimensions is the essence).

Dedekind rings that are not PID's include $R = \mathbb{Z}[\sqrt{-5}]$. (memo: we have stated above that a ring that is both Dedekind and UFD is a PID; this means that a Dedekind ring that is not a PID should not be a UFD; $6 = 2 \cdot 3 = (1 - \sqrt{-5}) \cdot (1 + \sqrt{-5})$ is an example of factorization in the ring $R = \mathbb{Z}[\sqrt{-5}]$ that is not unique) Behind this non-unique factorization is the following fact: in the ring $R = \mathbb{Q}[x]/(f(x))$, $P([x]) \cdot Q([x]) = P'([x]) \cdot Q'([x])$ as an element in R if $P(x)Q(x) \equiv P'(x)Q'(x)$ in $\mathbb{Q}[x]$ mod $(f(x))$.

$R = k[x, y, z, w]/(xy - zw)$ is a normal ring, but it is not a UFD, or a Dedekind ring, either.

$R = k[x, y]/(y^2 - x^3)$ is a domain, but it is not a normal ring.

$R = k[x, y]/(xy)$ is a ring, but it is not a domain.

1.5.2. Local rings (localization), stalk. Let X be an Affine variety X; $\mathcal{O}_X(X) = \mathbb{C}[X] =: R$ is its ring of regular functions. We have seen before that $\mathcal{O}_X(U)$ is the ring of functions that remain regular at any point in $U \subset X$; those functions may (or may not) have pole along the subvariety $X \backslash U$; this is a statment for a fixed subvariety Y of X and arbitrary points in the complement of Y. One may also be interested, however, in the ring of functions that are regular at one specified point $p \in X$ that may (or may not) have a pole along any subvariety $Y \subset X$ that does not pass through p ($p \notin Y$). The ring in question is known to be $R_{\mathfrak{m}_p}$ (notation explained below), where $\mathfrak{m}_p$ is the ideal sheaf of the point p.

For a prime ideal $\mathfrak{p}$ of a ring R, $R_{\mathfrak{p}}$ stands for the ring of fractions $S^{-1}R$ with $S := R \backslash \mathfrak{p}$ (the complement of a prime ideal is always multiplicatively closed (almost by def)). Such a ring $R_{\mathfrak{p}}$ is said to be the *localization* of R at (along) $\mathfrak{p}$.

The idea behind associating $R_{\mathfrak{m}_p}$ to the point $p \in X$ is that $\mathfrak{m}_p$ is the collection of functions in R (over X) that vanish at p; so $S = R \backslash \mathfrak{m}_p$ is the collection of functions that do not vanish at p. Those functions in S may vanish along subvarieties $Y \subset X$, but those Y never pass through p. For each element $r \in R_{\mathfrak{m}_p}$, therefore, there must be a pole locus $Y_r \subset X$, and r can be regarded as a regular function over $(X \backslash Y_r) \ni p$; r can be regarded as an element of $\mathcal{O}_X(X \backslash Y_r) = S_{Y_r}^{-1} R$. For this reason, the localization $R_{\mathfrak{m}_p}$ is the collection of functions each one of which remains regular at least in some Zariski-open neighbourhood of $p \in X$.

In the language of sheaf theory, the ring $R_{\mathfrak{m}_p}$ corresponds to the stalk$_*$ of $\mathcal{O}_X$ at $p \in X$.

The localization ring $R_{\mathfrak{p}}$ of a prime ideal $\mathfrak{p}$ (not necessarily a maximal ideal) has a similar interpretation. Let $Y_{\mathfrak{p}}$ be the irreducible subvariety of an Affine variety X where $R = \mathbb{C}[X]$ and $I_{Y_{\mathfrak{p}}} = \mathfrak{p}$. Functions in $S = R \backslash \mathfrak{p}$ are those that are non-zero at least somewhere on $Y_{\mathfrak{p}}$. For each element in $r \in R_{\mathfrak{p}}$, there must be a pole locus $Y_r \subset X$ satisfying $(X \backslash Y_r) \cap Y_{\mathfrak{p}} \neq \phi$, and r can be regarded as a regular function over $(X \backslash Y_r)$. r can be regarded as an element of $\mathcal{O}_X(X \backslash Y_r)$.

residue field$_*$: ex. i) for an Affine variety X with $\mathbb{C}[X] = \mathbb{C}[x_1, \cdots, x_n]/I_X$, the residue field is $\mathbb{C}$ for a point p (corresponding to a maximal ideal $\mathfrak{m}_p$), and the residue field is $\mathbb{C}(Y_{\mathfrak{p}})$ for an irreducible subvariety $Y_{\mathfrak{p}} \subset X$ (corresponding to a prime ideal $\mathfrak{p}$). ex. ii) for $R = \mathbb{Q}[x]/(x^2+x+1)$, R is itself a field. It has just one maximal ideal $\mathfrak{m} = \{0\}$, where $R_{\mathfrak{m}} = R$, and the residue field is R itself.

1.5.3. Let R be a ring. $\mathrm{Spec}(R)$, when regarded as a set, is the set of all the prime ideals of R; it contains (as its subset) the set of all the maximal ideals of R. More structure is given to $\mathrm{Spec}(R)$ shortly, so $\mathrm{Spec}(R)$ is not just a set.

Ex.1: for the ring of regular functions $R = \mathbb{C}[x_1, \cdots, x_n]/I_X = \mathbb{C}[X]$ of an Affine variety X, $\mathrm{Spec}(R)$ as a set has a one-to-one correspondence with the set that consists of all the points of X, all the irreducible subvarieties of X of dimension 2, 3, ..., $\dim_{\mathbb{C}} X$. When X is irreducible, the zero ideal $\{0\} \subset R$ corresponds to the dimension-($\dim_{\mathbb{C}} X$) subvariety of X—X itself. When X is not irreducible, the zero ideal $\{0\} \subset R$ is not a prime ideal; there must be multiple prime ideals instead that correspond to the irreducible components of X.

Ex.2: for $R = \mathbb{Q}[x, y]$, an ideal $(x^2 + y^2)$ is prime, but $(x^2 - y^2) = (x - y) \cdot (x + y)$ is not.

Ex.3: for $R = \mathbb{Z}$, the set $\mathrm{Spec}(\mathbb{Z})$ consists of prime ideal (2), (3), (5), ... and $\{0\}$.

Ex.4: for $R = \mathbb{Z}[i]$, $\mathrm{Spec}(R)$ consists of $(1+i)$, (3), $(1+2i)$, $(1-2i)$, ..., $\{0\}$.

In all those examples, elements of R should be regarded as regular functions on $\mathrm{Spec}(R)$. Prime ideals $\mathfrak{p}$ of R can be regarded as irreducible "subvarieties" of $\mathrm{Spec}(R)$, even in Ex.3 and Ex.4. See below for more.

1.5.4. The set $\mathrm{Spec}(R)$ for a ring R is endowed with the Zariski topology to be a topological space. The choice of this specific topology is always implicit, whenever we refer to $\mathrm{Spec}(R)$ of a ring R. Moreover, the structure sheaf on the topological space $\mathrm{Spec}(R)$ is given precisely in the same as before; rings of fractions of R are assigned to Zariski-open subsets of $\mathrm{Spec}(R)$.

For a prime ideal $\mathfrak{p}$ of a ring R, the localization $R_{\mathfrak{p}}$ is the colleciton of things each one of which can be regarded as a regular functions on a Zariski-open neighbourhood of $\mathfrak{p} \in \mathrm{Spec}(R)$. The residue field at $\mathfrak{p}$ is the field of functions on the subvariety corresponding to $\mathfrak{p}$.

ringed space, *scheme*..

1.5.5. base scheme: In 1.2.1–1.2.3, we implicitly considered only ring homomorphism $\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ between the ring of regular functions of two Affine varieties X and Y that preserve scalar multiplications by complex numbers. They are the ring homomorphisms where the diagram

$$\begin{tikzcd} \mathbb{C}[Y] \arrow[rr, "\phi^*"] & & \mathbb{C}[X] \\ & \mathbb{C} \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{16}$$

commutes; equivalently they are homomorphisms as algebras over $\mathbb{C}$. One could have considered a class of more general ring homomorphisms such as those as algebra over $\mathbb{R}$ (by thinking of $\mathbb{C}[Y]$ and $\mathbb{C}[X]$ as algebras over $\mathbb{R}$), or those as algebra over $\mathbb{Z}$ (i.e., all the ring homomorphisms). When one think of such questions as the set of regular maps between varieties (or schemes) and modulo-isomorphism classification of varieties, one needs to specify this information: scalar multiplication over which ring we require in the homomorphisms between the ring of functions.

The inclusion homomorphism $\mathbb{C} \hookrightarrow \mathbb{C}[X]$ and $\mathbb{C} \hookrightarrow \mathbb{C}[Y]$ in the diagram above corresponds to $X \to \mathrm{Spec}(\mathbb{C})$ and $Y \to \mathrm{Spec}(\mathbb{C})$; the regular maps $\phi : X \to Y$ obtained as homomorphism between the algebras over $\mathbb{C}$ are the regular maps where the diagram

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{Spec}(\mathbb{C}) & \end{tikzcd} \tag{17}$$

commutes. $\mathrm{Spec}(\mathbb{C})$ in the diagram above may well be replaced by $\mathrm{Spec}(\mathbb{R})$ or $\mathrm{Spec}(\mathbb{Z})$, when a broader class of regular maps are considered. Note that $X \to \mathrm{Spec}(k)$ should not be regarded just as a map between two point sets; even when k is a field, Spec(k) as a point set consists of just one point, but the structure sheaf of Spec(k) assigns the ring k to the unique non-empty open set of Spec(k), which is Spec(k) as a whole. The regular map (*morphism*) $X \to \mathrm{Spec}(k)$ includes a ring homomorphism $k \to \mathcal{O}_X(X)$.

The chosen Spec of a ring is called the *base scheme*.

1.5.6. fibre product in gemometry corresopnds to tensor product in (ring):

$$\begin{tikzcd} X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} R_X \otimes_{R_B} R_Y & R_Y \arrow[l] \\ R_X \arrow[u] & R_B \arrow[l] \arrow[u] \end{tikzcd} \tag{18}$$

where both R_X and R_Y are algebras over a ring R_B, and X, Y, B are Spec of R_X, R_Y, and R_B, respectively. This statement could have been placed in section 1.2

2 Divisor, Linear Equivalence, Intersection Number

2.1 Weil Divisors and Linear Equivalence

Definition 2.1.1. A *Weil divisor* on an algebraic variety X is a formal finite sum of the form

$$D = \sum_i n_i D_i, \tag{19}$$

where $n_i \in \mathbb{Z}$ and D_i is a codimension-1 irreducible subvariety of X. Weil divisors form an Abelian group under the addition of the $\mathbb{Z}$-valued coefficients; this group is called *divisor group* and denoted by Div(X).

This is a very large group. In the case of $X = \mathbb{P}^1$, for example, any point $z \in \mathbb{P}^1$ is regarded as a codimension-1 irreducible subvariety, so $D = \sum_z n_z z$ is a Weil divisor, if n_z is a non-zero integer for a finite number of points in $\mathbb{P}^1$. There are infinitely many, not even countablly many, choices of such $\{n_z\}_{z\in\mathbb{P}^1}$.

This Div(X) group in algebraic geometry corresponds to the Abelian group of real $(2n-2)$-dimensional cycles Z_{2n-2} of a real $2n$-dimensional simplicial complex in algebraic topology. A real $2n$-dimensional simplicial complex consists of p-simplices with $0 \leq p \leq 2n$, and the Abelian group of p-chains C_p is the free Abelian group

$$C_p := \left\{ \sum_{i\in\{p \text{ simplices}\}} n_i [p \text{ simplex}]_i \right\} = \oplus_{i\in\{p \text{ simplices}\}} \mathbb{Z} \left\langle [p \text{ simplex}]_i \right\rangle . \tag{20}$$

Z_p is the subgroup of C_p characterized by the absence of boundary. $\mathrm{Div}(X)$ is similar to Z_{2n-2} in that their general elements are in the form of a formal sum of real $(2n-2)$-dimensional geometric objects without a boundary.

The homology group H_p of a simplicial complex is obtained by introducing an equivalence relation in Z_p in algebraic topology; two p-cycles are regarded equivalent if and only if their difference corresponds to the boundary of a $(p+1)$-chain. So, we also introduce the following equivalence relation with which we take a quotient of the divisor group $\mathrm{Div}(X)$ in algebraic geometry.

Definition 2.1.2. For a given non-zero rational function $\varphi \in \mathbb{C}(X)$ on an algebraic variety X, we can read out its zero loci and pole loci. Let D_i $(i \in I)$ be the irreducible components of the zero loci, with n_i the order of vanishing along D_i. Let D'_j $(j \in J)$ be the irreducible components of the pole loci, with m_j the order of the poles along D'_j. With those data, we can define a Weil divisor $\mathrm{div}(\varphi) := \sum_{i\in I} n_i D_i + \sum_{j \in J} m_j D'_j$. Weil divisors of this form are called *principal divisors*, and forms a subgroup of $\mathrm{Div}(X)$.

Definition 2.1.3. When two Weil divisors D and D' on an algebraic variety X are different only by a principal divisor, $D = D' + \mathrm{div}(\varphi)$, ${}^\exists\varphi \in \mathbb{C}(X)$, then we say that D and D' are linearly equivalent, and write $D \sim D'$.

Definition 2.1.4. By taking a quotient of $\mathrm{Div}(X)$ by the linear equivalence relation, we define $\mathrm{Cl}(X) := \mathrm{Div}(X)/\sim$, the *divisor class group* of X. The linear equivalence class of a Weil divisor D is often denoted by $[D]$.

In the case of $X = \mathbb{P}^1$, with $[X_0 : X_1]$ the homogeneous coordinates, any two distinct points are regarded different Weil divisors, but are linearly equivalent. Let $z := X_1/X_0$ be the homogeneous coordinates, and also (the Weil divisor that consists of) the point whose coordinate is z; then $z' = z + \mathrm{div}((zX_0 - X_1)/(z'X_0 - X_1)) \sim z$. This means that $\sum_z n_z z \sim \left(\sum_z n_z\right) z_0$, and $\mathrm{Cl}(\mathbb{P}^1) \cong \mathbb{Z}$. More generally, a Weil divisor on $X = \mathbb{P}^n$ is in the form of

$$D = \sum_F n_F D_F \tag{21}$$

where D_F is the zero locus of a homogeneous function F (with some homegeneous degree) on $X = \mathbb{P}^n$, and $n_F \in \mathbb{Z}$. When F is a homogeneous function of degree k_F, $D_F = k_F D_{X_0} + \mathrm{div}(F/(X_0)^{k_F}) \sim k_F D_{X_0}$. So, any Weil divisor is linearly equivalent to an integer multiple of D_{X_0}. We have learned that $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$. The generator of this divisor class group is represented by the zero locus of a hyperplane $\mathbb{P}^1$, such as $\{X_0 = 0\}$, $\{X_1 = 0\}$ $\{\sum_{i=0}^n a_i X_i = 0\}$. This divisor (and its divisor class) is often denoted by H, is called the *hyperplane divisor (class)*.

Whereas $\mathrm{Cl}(X)$ tells us the classification of divisors modulo linear equivalence, we may sometimes be interested also in the set of divisors within a given linear equivalence class.

Definition 2.1.5. A Weil divisor (19) is *effective*, if all the n_i's are positive. A notation $D \geq 0$ means that D is effective. For a Weil divisor D_0 that is effective, let $|D_0|$ be the set of all the effective Weil divisors D linearly equivalent to D_0.

$$|D_0| = \left(\{\mathbb{C}(X) \ni \varphi \neq 0 \mid D_0 + \mathrm{div}(\varphi) \geq 0\}\right)/\mathbb{C}^\times, \tag{22}$$

because two rational functions φ and φ' that are different only by a non-zero complex multiple give rise to the same effective divisor $D \sim D_0$. $|D_0|$ is called the *complete linear system* of D_0.

When $X = \mathbb{P}^2$, for example, $|dH|$ is a family of degree-d curves in $\mathbb{P}^2$; H is a hyperplane divisor of the projective space X, as before. When we choose H to be the $X_0 = 0$ hyperplane, elements of $|dH|$ is in one to one correspondence with rational functions

$$\frac{\text{homogen. fcn of deg } d}{(X_0)^d} \tag{23}$$

modulo $\times\mathbb{C}^\times$. So, they form a space with ${}_{d+2}C_2 - 1$ dimensions. Curves in a given family, say $|dH|$, are all topologically the same; when $d = 1$ and $d = 2$, they are all genus-zero curves ($\mathbb{P}^1$); when $d = 3$, they are all genus-1 curves. $2g - 2 = d(d-3)$ is the formula between the degree d and the genus g. We will derive this formula later in section 6.2.

When $X = \mathbb{P}^{n-1}$, it is known that general members of $|nH|$ is a Calabi–Yau $(n-2)$-fold (cf section 4.4). The $n = 3$ case corresponds to the $g = 1$ curve (elliptic curve), $(3) \subset \mathbb{P}^2$, as we have stated above. The $n = 4$, $n = 5$ and $n = 6$ cases are the quartic K3, quintic Calabi–Yau three-folds and sextic Calabi–Yau four-folds. They come in the form of a family, $|nH|$.

2.1.6. Codimension-1 subvarieties are called "divisors". What do they "divide"? See 3.5.4, where we touch upon the divisor class group $\mathrm{Cl}(X)$ of $X = \mathrm{Spec}(R)$ for R in number-theory settings.

2.2 Linear Equivalence, Algebraic Equivalence

2.2.1. Although we introduced linear equivalence relation among divisors in an algebraic variety as an analog of modulo-boundary indentification of closed cycles in a simplicial complex, the divisor class group $\mathrm{Cl}(X)$ provides a little finer classification than the topological classification $H_{2n-2}(X;\mathbb{Z})$ of complex codimension-1 cycles in X. Certainly $\mathrm{Cl}(X) \cong \mathbb{Z} \cong H^2(X;\mathbb{Z})$ for $X = \mathbb{P}^n$, as we saw above, but they are not necessarily the same.

The best example where they are not the same is for X to be an elliptic curve E. Let $D = \sum_i n_i p_i$ be a Weil divisor on $X = E$; p_i's are points (complex codimension-1 subvarieties in E).

Let the point $e \in E$ be the origin of $E = \mathbb{C}/(\mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle \tau\rangle)$, which plays the role of the identity 0 element in the Abelian group law on E. The field of rational functions on E, $\mathbb{C}(E)$, is nothing other than the field of elliptic functions on E. When arbitrary two points p_1 and p_2 in E are given, it is always possible to find an elliptic function on E so that it i) has two zeros precisely at p_1 and p_2, nowhere else, and ii) also has two poles of order 1, with one of the two poles at $e = 0$; the remaining one pole must be at $(p_1 \boxplus p_2)$ in $E = \mathbb{C}/(\mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle \tau\rangle)$, where the $\boxplus$ is meant to be the sum of the group law in $E = \mathbb{C}/(\mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle \tau\rangle)$, rather than in the divisor group $\mathrm{Div}(E)$. See Exercise A.1 for more. Thus, as divisors, two divisors on E are linearly equivalent, $p_1 + p_2 \sim (p_1 \boxplus p_2) + e$. Using this argument multiple times, one can always find a linear equivalence relation of the form

$$D = \sum_i n_i p_i \sim \left(-1 + \sum_i n_i\right) e + q, \qquad q = \boxplus_i p_i \in E. \tag{24}$$

Now let us focus on the kernel $\mathrm{Cl}(E)^0$ of

$$\mathrm{Cl}(E) \ni \left[\sum_i n_i p_i\right] \longmapsto \sum_i n_i \in \mathbb{Z}. \tag{25}$$

For a general element of the kernel, one can always find a representative of the form $D = q - e$ for some $q \in E$, and in fact, the kernel is in one-to-one correspondence $\mathrm{Cl}^0(E) \cong E$ in this way. A divisor q is linearly equivalent to the divisor e, if and only if $q = e$; this follows from the nature of elliptic functions (ie, the nature of $\mathbb{C}(E)$). In $H^2(E;\mathbb{Z}) \cong \mathbb{Z} \cong H_0(E;\mathbb{Z})$, we do not distinguish two points in E, wherever they are. $[e] \in H_0(E;\mathbb{Z})$ is the same as $[q] \in H_0(E;\mathbb{Z})$. So, $\mathrm{Cl}(E)$ retains finer information than $H_0(E;\mathbb{Z}) \cong H^2(E;\mathbb{Z})$ in the case the variety in question is $X = E$. We will come back to this point again in 4.2.3.

Definition 2.2.2. Suppose that $D_1, D_2 \in \mathrm{Div}(X)$. D_1 and D_2 are *algebraically equivalent*, if there exists a curve S, two points $s_1, s_2 \in S$, and a divisor $D \in \mathrm{Div}(S \times X)$ such that $\pi^{-1}(s_1) \cdot D = D_1$ and $\pi^{-1}(s_2) \cdot D = D_2$; here, $\pi : S \times X \to S$ is the simple projection.

Whenever $D_1 \sim D_2$, then the two divisors are also algebraically equivalent. To see this, note that $D_2 = D_1 + \mathrm{div}(\varphi)$ for some $\varphi \in \mathbb{C}(X)$. Let $F = 0$ be the defining equation for the divisor D_1 in X. Then we can use $F \cdot (1 + z\varphi) = 0$ to define a divisor D in $\mathbb{P}^1 \times X$, where z is an inhomogeneous coordinate on $\mathbb{P}^1$. $\pi^{-1}(z = 0) \cdot D$ is D_1, and $\pi^{-1}(z = \infty) \cdot D$ is D_2 then.

Definition 2.2.3. On an algebraic variety X, $\mathrm{Div}^a(X)$ denotes the divisors that are algebraically equivalent to the trivial divisor $0 \in \mathrm{Div}(X)$. The quotient $\mathrm{Div}(X)/\mathrm{Div}^a(X)$ by the algebraic equivalence relation is called Neron–Severi group, and are denoted by $\mathrm{NS}(X)$.

Since the algebraic equivalence relation is broader than the linear equivalence relation, classification by the linear equivalence is finner. There is a well-defined homomorphism $\mathrm{Cl}(X) \to \mathrm{NS}(X)$.

In the example of $X = E$, all the divisor classes in $\mathrm{Cl}^0(E)$ are trivial under the algebraic equivalence relation. To see this, just note that we can use the diagonal divisor $\Delta \subset E \times E$. Any two points $p_1, p_2 \in E$ define divisors that are mutually algebraically equivalent, although they are not linearly equivalent. Thus, $\mathrm{NS}(E) \cong \mathbb{Z} \cong H_0(E;\mathbb{Z}) \cong H^2(E;\mathbb{Z})$.

2.2.4. There is one more equivalence relation that can be introduced among divisors on X. Two divisors $D_1, D_2 \in \mathrm{Div}(X)$ are *numerically equivalent*, if $D_1 \cdot C = D_2 \cdot C$ for any curve C in X.

It is known that two divisors are always numerically equivalent, if they are algebraically equivalent. So, the quotient of $\mathrm{Div}(X)$ by the numerical equivalence provides even coarser classification than $\mathrm{NS}(X)$. It is known that the classification by the numerical equivalence drops the torsion components in $\mathrm{NS}(X)$ and retains the free Abelian part.

2.3 Cartier Divisors

Definition 2.3.1. A *Cartier divisor* $D = \{(U_i, f_i)\}$ on an algebraic variety X is an open covering $\mathcal{U} = \{U_i | i \in I\}$ of X and a rational function $f_i \in \mathbb{C}(U_i)$ satisfying the condition that the rational function f_i/f_j on $U_i \cap U_j$ neither has zero or pole in $U_i \cap U_j$ $(i \neq j)$.

Any Cartier divisor D on X can be regarded as a Weil divisor; the corresponding Weil divsor D is the one that looks in $U_i \subset X$ as $\mathrm{div}(f_i)$. Whenever two open set U_i and U_j overlaps, theh condition in the definition of a Cartider divisor implies that $\mathrm{div}(f_i) = \mathrm{div}(f_j)$, so there is no disagreement.

Definition 2.3.2. Cartier divisor $\{(U_i, f_i)\}$ and $\{(U_i, f'_i)\}$ are *linearly equivalent*, if and only if there is a rational function $\varphi \in \mathbb{C}(X)$ so that $f_i = f_i \varphi|_{U_i}$.

This definition of linear equivalence between a pair of Cartier divisors agrees with that of linear equivalence between the corresponding pair of Weil divisors.

Definition 2.3.3. The *Picard group* $\mathrm{Pic}(X)$ of an algebraic variety X is the the group of Cartier divisors on X modulo linear equivalence relation among them.

The sum of $\{(U_i, f_i)\}$ and $\{(U_i, g_i)\}$ is $\{(U_i, f_i g_i)\}$.

2.4 Weil vs Cartier

Cartier divisors on X can always be regarded as Weil divisors, but the converse is not always true.

Theorem 2.4.1. When an algebraic variety X is non-singular, then any Weil divisor is regarded as a Cartier divisor.

This is because, for a Weil divisor $D = \sum_a n_a D_a$, one can always find a defining equation of D_i's locally in a non-singular variety X; let $\mathcal{U} = \cup_i U_i$ be such an open patches covering the entire X, and let $D_a|_{U_i}$ be given by $f_{a,i} = 0$. Then $\{(U_i, \prod_a f_{a,i}^{n_a})\}$ is the corresponding description as a Cartier divisor.

When an algebraic variety X is not non-singular, however, Weil divisors do not always have a description as Cartier divisors. Here is an

Example 2.4.2. Let X be a three-dimensional (non-compact) variety given by $\{(x, y, z, w) \in \mathbb{C}^4 \mid xy - zw = 0\}$. This three-fold is called a *conifold*. This variety has a singularity at $(x, y, z, w) = (0, 0, 0, 0)$. A divisor given by a codimension-1 subvariety $\{(x, y, z, w) \in X \mid x = z = 0\}$ is an example of Weil divisor that is not Cartier. Can you find a rational function in a neighbourhood of the singular point whose zero and pole locus agrees precisely with that Weil divisor?

2.5 Intersection Number

When we regard a Cartier divisor D on a non-singular algebraic variety X as an element in $H_{2n-2}(X; \mathbb{Z}) \cong H^2(X; \mathbb{Z})$, the intersection number is defined in topology for n Cartier divisors $D_1, D_2, \cdots, D_n$ on X. It is denoted by $D_1 \cdot D_2 \cdot \cdots \cdot D_n \in \mathbb{Z} \cong H_0(X; \mathbb{Z})$. The intersection number does not change when one of the divisors in question, say, D_1, is replaced by another divisor D'_1 that is linear equivalent to the original D_1, because D_1 and D'_1 are both in the same topological class $H_{2n-2}(X; \mathbb{Z})$.

The fundamental theorem of algebra states that any polynomial $f(t)$ of degree n has n solutions to $f(t) = 0$ in $\mathbb{C}$. We can state this theorem in terms of algebraic geometry as follows. Let D_1 and D_2 be divisors on $X = \mathbb{P}^2$ given by $U^{n-1}S - U^n f(T/U) = 0$ and $S = 0$, respectively, where $[T : S : U]$ are the homogeneous coordinates of $X = \mathbb{P}^2$. Then $D_1 \cdot D_2 = n$ in X, independent of the choice of the degree-n polynomial $f(t)$. In this case, $D_1 \sim nH$ and $D_2 \sim H$, where H is the hyperplane divisor of $\mathbb{P}^2$. It is a more general version of the fundamental theorem of algebra that a degree d_1 curve $C_1 \in |d_1 H|$ and a degree d_2 curve $C_2 \in |d_2 H|$ in $X = \mathbb{P}^2$ intersect at $d_1 d_2$ points (when multiplicity included).

The intersection number $D_1 \cdot D_2 \cdot \cdots \cdot D_n$ is still defined, when some of the divisors among $D_1, \cdots, D_n$ are the same, eg., $D_1 = D_2$. In topology, we do so by replacing the cycle D_2 by a cycle $D_2' \neq D_2 = D_1$ that is still topologically the same as D_2. In algebraic geometry, we replace the divisor D_2 by another divisor $D_2' \neq D_2$ that is still linearly equivalent to D_2.

As an example, consider the case $X = \mathbb{P}^2$ and $D_1 = D_2 = \{U = 0\}$, where U is one of the homogeneous coordinates $[T : S : U]$ of $X = \mathbb{P}^2$. We can define $D_1 \cdot D_2$, although $D_1 = D_2$, by replacing the divisor D_2 by $D_2' = \{S = 0\}$, which is still linear equivalent to D_2 (use the rationa function $\varphi = S/U \in \mathbb{C}(X)$ for linear equivalence). The intersection number $D_1 \cdot D_2 = D_1 \cdot D_2'$ is $+1$, because D_1 and D_2' intersect transversely at just one point $[T : S : U] = [* : 0 : 0] \in \mathbb{P}^2$.

Intersection of k divisors, $D_1, \cdots, D_k$ with $k \leq n := \dim_{\mathbb{C}} X$, defines an algebraic cycle, in general. The algebraic cycle so defined is a formal sum of complex codimension-k subvarieties with integer coefficients. When $k = n$, algebraic cycles defined by intersection are collection of points in X with integer coefficients for individual points; the net number of points is the intersection number. We do not explore in this lecture further on various equivalence relations among algebraic cycles.

More subtlties are involved when one wishes to discuss intersection numbers of divisors on a singular variety. cf a book "*Intersection Theory*" by W. Fulton, Springer.

3 Resolution of Singularity

3.1 Singularity

Definition 3.1.1. Let X be an $(n-k)$-dimensional Affine variety given by k polynomial equations $f_a = 0$ $(a = 1, \cdots, k)$ in an ambient space $\mathbb{C}^n = \{(x^1, \cdots, x^n)\}$. X is *singular* at its point $z \in X$, if the $k \times n$ matrix $(\partial f_a / \partial x^i|_z)_{ai}$ has a rank less than k. A point of X where X is singular is called *singularity*. Sincularity of X is not necessarily a collection of isolated points. When X is not singular anywhere in X (i.e., free of singularity), we say that X is *non-singular* (or also *smooth*).

In the case X is a hypersurface (given by $k = 1$ defining equation $f = 0$), then X is singular at $z \in X$ (i.e., $f(z) = 0$) if and only if $\partial f(z)/\partial x^i = 0$ for all of $i = 1, \cdots, n$.

3.1.2. Examples: in an ambient space $\mathbb{C}^2$, define a curve X by $f(x, y) = x^2 - y^2$, or $f(x, y) = x^2 - y^3$. Curves X define that way are singular at the point $(x, y) = (0, 0) \in X \subset \mathbb{C}^2$. One can also verify that the conifold is singular at $(x, y, z, w) = (0, 0, 0, 0)$.

3.2 Blow-up and Proper Transform

Definition 3.2.1. *Resolution of a singularity* of a singular variety X is a process of finding another variety $\widetilde{X}$ and a map $\nu : \widetilde{X} \to X$ where ν is regular and ν^{-1} rational, satisfying certain properties (which we elaborate more later), as well as the variety $\widetilde{X}$ itself. Let $X_{\rm singl} \subset X$ be the collection of singular points in X; X is non-singular in $X \backslash X_{\rm singl}$. At the least, we require that the map

$$\nu|_{\nu^{-1}(X \backslash X_{\rm singl})} : \; \nu^{-1}(X \backslash X_{\rm singl}) \longrightarrow (X \backslash X_{\rm singl}) \tag{26}$$

is an isomorphism (regular map in both directions and their composition is identify on both). The spirit here is to replace X by $\tilde{X}$ so that $\tilde{X}$ remains the same as X wherever X is not singular and differs from X only at $X_{\rm singl}$. When $\tilde{X}$ still has singularity (but less singular than X), then we say that $(\tilde{X}, \nu)$ is a *partial resolution* of the singularity of X.

[8]

Fortunately, mathematicians have proved that one can always find $(\tilde{X}, \nu)$ so that $\tilde{X}$ is non-singular, so far as we work on $\mathbb{C}$, and do not step into the world of characteristic $p > 0$.

3.2.2. We wish to know practical procedures of finding/constructing such a resolution. Since we need to construct a new variety $\tilde{X}$, at least we always need to construc/specify an ambient space of $\tilde{X}$. So, the process of replacing X by $\tilde{X}$ begins with a process of replacing the ambient space

[8] Has a rational map been defined in this lecture note?

of X by another ambient space (for $\tilde{X}$). The simplest case of resolution of singularity is an Affine variety X in an ambient space $\mathbb{C}^n$ where X has singularity at $\vec{0} \in X \subset \mathbb{C}^n$. In this case, one tries to replace the ambient space $\mathbb{C}^n$ by the following variety:

Definition 3.2.3. $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ is an n-dimensional subvariety of $\mathbb{C}^n \times \mathbb{P}^{n-1}$ given by

$$\mathrm{Bl}_{\vec{0}}\mathbb{C}^n = \left\{ ((x_1, \cdots, x_n), [\xi_1 : \cdots : \xi_n]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid x_i \xi_j = x_j \xi_i \right\}. \tag{27}$$

A map $\nu : \mathrm{Bl}_{\vec{0}}\mathbb{C}^n \to \mathbb{C}^n$ is obtained by forgetting $[\xi_1 : \cdots : \xi_n]$ and projecting on to $(x_1, \cdots, x_n) \in \mathbb{C}^n$.

For any point in $\mathbb{C}^n$ other than $(x_1, \cdots, x_n) = \vec{0}$, a point $[\xi_1 : \cdots : \xi_n] \in \mathbb{P}^{n-1}$ is uniquely determined; $[\xi_1 : \cdots : \xi_n] = [x_1 : \cdots : x_n]$. The inverse image of $\vec{0} \in \mathbb{C}^n$, on the other hand, consists of full $\mathbb{P}^{n-1}$. The map $\nu : \mathrm{Bl}_{\vec{0}}\mathbb{C}^n \to \mathbb{C}^n$ is regular, because the map ν specifies the coordinates $(x_1, \cdots, x_n)$ of $\mathbb{C}^n$ by regular functions x_1, x_2, $\cdots, x_n$ of $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$.

This variety $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ can be covered by n open patches; the open subspace U_i $(i = 1, \cdots, n)$ of $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ characterized by the condition $\xi_i \neq 0$ is isomorphic to $\mathbb{C}^n$; we can take $(\xi_j/\xi_i)_{j \in \{1, \cdots, n\} \setminus i}$'s and x_i as the set of Affine coordinates of $U_i \cong \mathbb{C}^n$; the coordinates x_j in U_i are given by $x_j = x_i(\xi_j/\xi_i)$ for $j \neq i$. The map

$$\nu^{-1} : \mathbb{C}^n \supset \nu(U_i) \to U_i \subset \mathrm{Bl}_{\vec{0}}\mathbb{C}^n \tag{28}$$

is rational (and hence $\nu^{-1} : \mathbb{C}^n \to \mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ is rational (by def)), because the coordinates of U_i, x_i and (ξ_j/ξ_i)'s $(j \neq i)$ are expressed by rational functions of $\nu(U_i) \subset \mathbb{C}^n$, x_i and x_j/x_i's, respectively. The inverse map ν^{-1} fails to be regular (but remains rational) only at $\cup_i (\nu(U_i) \cap \{x_i = 0\}) = \vec{0} \in \mathbb{C}^n$.

Definition 3.2.4. The variety $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ is called the *blow-up of* $\mathbb{C}^n$ *centered at* $\vec{0} \in \mathbb{C}^n$. More generally, there is a variety denoted by $\mathrm{Bl}_Z Y$, where Z is a subvariety of a variety Y, and called *blow-up of* Y *centred at* Z, and $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ is a special case $Y = \mathbb{C}^n$ and $Z = \vec{0}$.

The inverse image of the center of blow-up $\nu^{-1}(\vec{0})$ is, as a subvariety of $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$, isomorphic to $\mathbb{P}^{n-1}$. This is a divisor, whose defining equation is $x_i = 0$ in U_i. This Cartier divisor, $E = (U_i, x_i)$, is called the *exceptional divisor of the blow-up.* Moregenerally, the inverse image of the center of blow-up, $\nu^{-1}(Z) \subset \mathrm{Bl}_Z Y$, is used to define a divisor, and those divisors are called exceptional divisors of the blow-up.

3.2.5. By using a map $\nu : \mathrm{Bl}_{\vec{0}}\mathbb{C}^n \to \mathbb{C}^n$ where ν is regular and ν^{-1} rational (non-regular only at $\vec{0} \in \mathbb{C}^n$), we can (partially) resolve singularity of an Affine variety X that has singularity only at $\vec{0} \in X \subset \mathbb{C}^n$, as follows.

Definition 3.2.6. $\nu^*(X) \subset \mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ is called the *total transform of* X. Here, $\nu^*(X)$ is obtained by pulling back defining equations of X by $\nu : \mathrm{Bl}_{\vec{0}}\mathbb{C}^n \to \mathbb{C}^n$. $\nu^*(X)$ is the same as $\nu^{-1}(X) \subset \mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ as a set of points, but we retain the information of multiplicity. The closure of $\nu^*\left(X\backslash\vec{0}\right)$ in $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ is denoted by $\overline{X}$ and called the *proper transform of* X. String theorists often call $\overline{X}$ the *blow-up of* X in their dialect, however. Restricting the map $\nu : \mathrm{Bl}_{\vec{0}}\mathbb{C}^n \to \mathbb{C}^n$ to $\overline{X}$, we obtain a map $\bar{\nu} : \overline{X} \to X$.

Often the variety $\overline{X}$ is less singular than X, so $(\overline{X}, \bar{\nu})$ serves for the purpose of (albeit often partial) resolution of X. Let us see how it works in concrete examples.

3.2.7. Let X be a curve given by $\{f(x,y) := x^2 - y^2 = 0\} \subset \mathbb{C}^2$, which is singular at $Z = \vec{0} \subset Y = \mathbb{C}^2$. This singularity is called a *double point singularity.* Now, we use $\mathrm{Bl}_Z Y = \mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ as the new ambient space, instead of $\mathbb{C}^2$. In the U_1 patch, with the Affine coordinates $(x, (\eta/\xi))$, the pull-back of the defining equation is $\nu^*(f) = x^2 - (x(\eta/\xi))^2 = x^2(1-(\eta/\xi)^2)$, so the total transform $\nu^*(X) \cap U_1$ consists of three irreducible piecies, $x = 0$ (with multiplicity 2), $1-(\eta/\xi) = 0$ and $1 + (\eta/\xi) = 0$. Similar analysis indicates that $\nu^*(X) \cap U_2$ consists of three irreducible pieces, $y = 0$ (multiplicity 2), $(\xi/\eta) - 1 = 0$ and $(\xi/\eta) + 1 = 0$. Overall, the total transform $\nu^*(X)$ consists of three irreducible pieces; two of them are curves $C_\pm$ given by $\xi = \pm\eta$, and the other is the exceptional curve E in $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$, which comes with multiplicity 2. As a divisor in $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$,

$$\nu^*(X) = 2E + C_+ + C_-. \tag{29}$$

The proper transoform, on the other hand, is

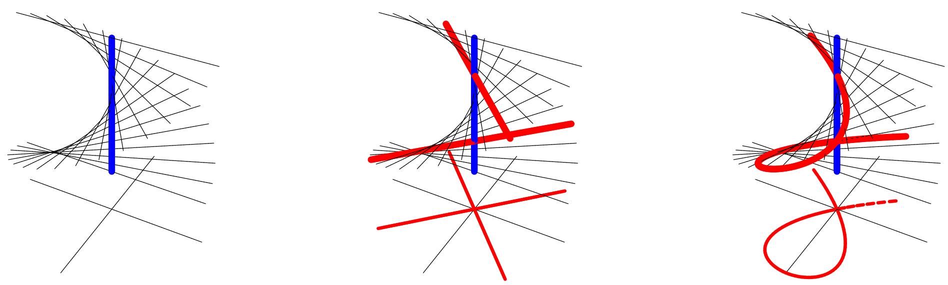
$$\overline{X} = C_+ + C_- \tag{30}$$

in this example. To see this, one only needs to see that $\nu^{-1}(X\backslash\vec{0})$ is $(C_+ + C_-) \cap (\mathrm{Bl}_{\vec{0}}\mathbb{C}^2\backslash E)$. Thus, C_+C_- is enough for the closure of this geometry in $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$; we do not need to include E as a part of $\overline{X}$. Now, $\overline{X} = C_+ + C_-$ consists of just two disjoint irreducible curves; even in the fibre of the center of the blow-up, $\nu^{-1}(\vec{0}) = E = \mathbb{P}^1$, the two curves C_+ and C_- pass through different points $[\xi : \eta] = [1 : 1]$ and $[\xi : \eta] = [1 : -1]$. So, $\overline{X}$ is free from singularity. $\bar{\nu} : \overline{X} \to X$ is a resolution, with $\overline{X}$ completely non-singular. See Figure 1.

3.2.8. Much the same procedure also resolves the singularity of a curve

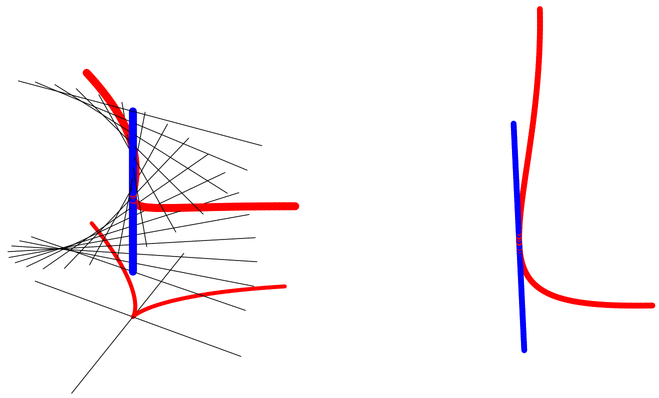
$$X = \left\{f(x,y) := x^2 - y^3 = 0\right\} \subset \mathbb{C}^2; \tag{31}$$

this singularity is called a cusp. We use $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ as a new ambient space, where the defining equation of the total transform $\nu^*(X)$ is $x^2(1-x(\eta/\xi)^3) = 0$ in the patch $U_1 \subset \mathrm{Bl}_{\vec{0}}\mathbb{C}^2$, and $y^2((\xi/\eta)^2 - y) = 0$



(a) (b) (c)

Figure 1: (a) Geometry of $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ and its projection $\nu : \mathrm{Bl}_{\vec{0}}\mathbb{C}^2 \to \mathbb{C}^2$ is illustrated by extracting the real locus. The real locus of $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$, where $x, y, \xi/\eta \in \mathbb{R}$, is like a spiral staircase; projection of the staircase in (a) to the ground, the real locus $\mathbb{R}^2 \subset \mathbb{C}^2$, provides the 1-to-1 correspondence with $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ and $\mathbb{C}^2$; the axis of the spiral (thick blue line)—the real locus of the exceptional locus—is projected to the center of the blow-up, $\vec{0} \in \mathbb{C}^2$, however. This $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ is used as an ambient space in (b) to resolve a double point singularity of a curve X (thin red lines). The proper transform $\overline{X}$ consists of two irreducible components C_+ and C_- (thick red lines), while the total transform $\nu^{-1}(X)$ also contains the exceptional curve E (thick blue line). $\overline{X}$ is not singular anymore. (c) shows the proper transform $\overline{X}$ (thick red curve) for the curve $X = \{x^2 = y^2(y+1)\}$.



(a) (b)

Figure 2: A curve $X \subset \mathbb{C}^2$ with a cusp singularity $x^2 - y^3 = 0$ (thin red curve) is lifted into a new ambient space $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ to be $\overline{X}$ (thick red curve), where the singularity is resolved (a); the total transform $\nu^*(X)$ also contains the exceptional curve E (blue line). (b) is a zoom-up picture of the geometry of curves E and $\overline{X}$ near the point of their intersection. $\overline{X}$ (thick red curve) is non-singular indeed, and touches with E with multiplicity 2.

in $U_2 \subset \mathrm{Bl}_{\vec{0}}\mathbb{C}^2$. The total transform consists of two irreducible components, one of which is the exceptional curve E in $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$, and the other the proper transoform. $\nu^*(X) = 2E + \overline{X}$. The proper transform $\overline{X}$ is non-singular in this example; in the U_2 patch, for example, we can use (ξ/η) as the local coordinate of $\overline{X}$, and solve the other coordinate y in terms of (ξ/η); $y = (\xi/\eta)^2$. The intersection number $E \cdot \overline{X}$ in $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ is $+2$, because the defining equation of E, $y = 0$, appears quadratic $y = (\xi/\eta)^2$ in the local coordinate on $\overline{X}$. See Figure 2.

3.2.9. A curve $X = \{f(x,y) := x^2 - y^n = 0\} \subset \mathbb{C}^2$ with $n \geq 4$ has a singularity at $(x,y) = \vec{0}$; the proper transform $\overline{X}$ in a new ambient space $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ is not completely resolved yet, because $\overline{X}$ is given by $(\xi/\eta)^2 - y^{n-2} = 0$ in $\{((\xi/\eta), y)\} \cong U_2 \subset \mathrm{Bl}_{\vec{0}}\mathbb{C}^2$, with $n - 2 \geq 2$. Replacing the ambient space $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ by yet another blow-up of $\mathrm{Bl}_{\vec{0}}\mathbb{C}^2$ centered at the singularity point of $\overline{X} \subset \mathrm{Bl}_{\vec{0}}\mathbb{C}^2$, and replacing $\overline{X}$ by its proper transform, singularity becomes less severe. After repeating this process for $n/2$ times or so, we will arrive at a curve $\widetilde{X}$ without a singularity, from which there is a regular and birational map to the original X.

Definition 3.2.10. At least among string theorists, a subvariety Z of $\widetilde{X}$ in a resolution of singularity, $\nu : \widetilde{X} \to X$, is called an *exceptional locus* if $\dim_{\mathbb{C}} Z > \dim_{\mathbb{C}}(\nu(Z))$; when Z is a divisor of

$\widetilde{X}$, it is also called an *exceptional divisor*.

Exercise 3.1. It is a good excersize to find a resolution of a singularity of a surface $X = \{(x, y, z) \mid xy = z^N\} \subset \mathbb{C}^3$. This singularity is called A_{N-1} singularity. It is known that there is a resolution $\nu : \widetilde{X} \to X$ where the exceptional locus consists of $N-1$ irreducible curves C_a ($a = 1, \cdots, N-1$), each one of them is isomorphic to $\mathbb{P}^1$; their intersection numbers, when presented in the form of an $(N-1) \times (N-1)$ matrix, is (-1) times the Cartan matrix of A_{N-1}.

3.3 More on blow-up

3.3.1. The variety $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ can be regarded as a line bundle on $\mathbb{P}^{n-1}$. The open covering $\cup_{i=1}^n U_i = \mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ provides local trivialization; projection to the base space $\mathbb{P}^{n-1}$ is given by

$$\nu|_{U_i} : \; U_i \cong \mathbb{C}^n = \{(x_i, (\xi_j/\xi_i)_{j\neq i})\} \to \{(\xi_j/\xi_i)_{j\neq i}\} = \mathbb{C}^{n-1} \subset \mathbb{P}^{n-1}, \tag{32}$$

and x_i plays the role of the fibre coordinate. The transition function of this line bundle is $g_{ji} = x_j/x_i = (\xi_j/\xi_i)$ on $U_j \cap U_i$, so that $x_j = g_{ji}x_i$.

3.3.2. Let us compute the self-intersection of the exceptional divisor $E = (U_i, x_i)$ of the variety $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$. To do so, we need to find a divisor E_{i_0} that is linearly equivalent to, but not the same as, E. For any $i_0 \in \{1, \cdots, n\}$, let E_{i_0} be a Cartier divisor $(U_i, x_i/x_{i_0})$; x_{i_0} is a rational function of $\mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ that establishes the linear equivalence $E \sim E_{i_0}$. Since $x_i/x_{i_0} = 1$ in the patch $U_{i=i_0}$, $E_{i_0} \cap U_{i_0} = \phi$; in other patches $U_{i\neq i_0}$, $E_{i_0} = \mathrm{div}(\xi_i/\xi_{i_0}) = -\mathrm{div}(\xi_{i_0}/\xi_i)$. Thus, $E \cdot E_{i_0}$ is an algebraic cycle given by the $\xi_{i_0} = 0$ hyperplane in $E \cong \mathbb{P}^{n-1}$ with multiplicity (-1). The intersection number E^n is obtained by $E \cdot E_{i_0=1} \cdot E_{i_0=2} \cdot \cdots \cdot E_{i_0=n-1} = -1$. $(-1)^{n-1}$?

3.4 Canonical Divisor, Crepant Resolution, and Minimal Model

3.4.1. Suppose that X is a variety in an ambient space Y, and X has singularity. When there is a resolution $\nu : \widetilde{X} \to X$, where $\widetilde{Y}$ is the ambient space of $\widetilde{X}$, there are infinitely many other resolutions of the singularity of X. For example, for any point $p \in \widetilde{X} \subset \widetilde{Y}$, we can replace $\widetilde{Y}$ and $\widetilde{X}$ by $\mathrm{Bl}_p\widetilde{Y}$ and the proper transform of $\widetilde{X}$. We can choose arbitrary non-singular point $p \in \widetilde{X}$, and furthermore, we can repeat this process arbitrary number of times. For a given singular variety X over $\mathbb{C}$, resolution $(\widetilde{X}, \nu)$ is far from unique; just its existence is guaranteed.

The sequence of blow-ups beyond minimal necessity for resolving singularity does not do anything in the case X is a curve, in fact. A resolved non-singular variety $\widetilde{X}$ may be replaced by another non-singular variety $\widetilde{X}'$ by one more (unnecessary) blow-up of the ambient space, but $\widetilde{X}'$

turns out to be isomorphic to $\widetilde{X}$. In the case $\dim_{\mathbb{C}} X > 1$, however, $\widetilde{X}'$ is not always isomorphic to $\widetilde{X}$.

This observation motivates to impose some conditions that reflects the idea of avoiding unnecessary blow-ups of the ambient space. As a preparation, we introduce

Definition 3.4.2. For a non-singular variety X, one can always difine a divisor denoted by K_X and referred to as *canonical divisor* of X, as follows. Let $\cup_i U_i$ be local coordinate patches, with which X is covered entirely. K_X as a Cartier divisor is given by setting $f_{i_0} = 1$ for one of the coordinate patches U_{i_0}, and set $f_{i \neq i_0} = |\partial\{x\}_{i_0}/\partial\{x\}_i|$, using the Jacobian of the two sets of local coordinates, $\{x\}_{i_0}$ in the patch U_{i_0} and $\{x\}_i$ in U_i.

Example 3.4.3. Since $\mathbb{C}^n$ is covered by a single coordinate patch, its canonical divisor is trivial, $K_{\mathbb{C}^n} = 0$. Explicit computations following the definition above leads to $K_X = -(n+1)H_{\xi_{i_0}=0}$ for $X = \mathbb{P}^n$; when $X = \mathrm{Bl}_{\vec{0}}\mathbb{C}^n$, $K_X = (n-1)E$.

When X is singular, its canonical divisor K_X cannot be defined in the way described above. A possible way to go[9] is to think of a complex structure deformation X_{dfm} so that X_{dfm} is non-singular; K_X is "defined" as a limit of $K_{X_{\mathrm{dfm}}}$; one has to verify whether such deformation exists, and whether or not such a limit remains independent of the choice of deformation. There are two other ways to define a canonical divisor for a variety with singularity; see 3.5.8 and 3.5.9.

Definition 3.4.4. A resolution $(\widetilde{X}, \nu)$ of a singular variety X is a *crepant resolution*, when $\widetilde{X}$ is non-singular and $K_{\widetilde{X}} = \nu^*(K_X)$. The word "crepant" came from the condition, which can be read as absence of *dis*crepancy between $K_{\widetilde{X}}$ and $\nu^*(K_X)$.

When $(\widetilde{X}, \nu)$ is a crepant resolution, and $\nu_* : \widetilde{X}' \to \widetilde{X}$ is an unnecessary extra blow-up, $(\widetilde{X}', \nu \circ \nu_*)$ is still a resolution, but usually not crepant. Imposing the condition "crepant" therefore reduces possible choices of resolution for a given singular variety X. For some variety X over $\mathbb{C}$, there is no crepant resolutions, although there are non-crepant resolutions. Even for some variety X that has a crepant reslution, there can be more than one crepant resolutions that are mutually non-isomorphic. We just have to live with these facts in mathematics.

3.4.5. The idea of avoiding unnecessary extra blow-ups may also be stated in terms of the minimal model program. Statements in the case of $\dim_{\mathbb{C}} X = 2$ are particularly simple. Here is

Theorem 3.4.6 (Castelnuovo)**.** For a non-singular algebraic surface X which contains an irreducible curve C satisfying $C \cdot C = -1$ and $C \cong \mathbb{P}^1$, there is a regular and birational map $\nu : X \to \underline{X}$

[9] Here is an example of this procedure: for a singular variety X given by (compactification of) $\{(x, y) \in \mathbb{C}^2 \mid y^2 = x^2(x-1)\}$, we can think of a deformation $X_\epsilon = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x^2 - \epsilon)(x-1)\}$ with a small parameter $\epsilon \in \mathbb{C}$. Since $K_{X_\epsilon} = 0$ for $\epsilon \neq 0$, it is reasonable to claim that $K_X = 0$.

where a surface $\underline{X}$ remains non-singular. Conversely, if a non-singular algebraic surface X has a regular and birational map $\nu : X \to \underline{X}$ to a non-singular surface $\underline{X}$, and if the exceptional locus is a curve C that is isomorphic to $\mathbb{P}^1$, then it satisfies $C \cdot C = -1$.

The exceptional locus that emerges after the resolution of A_1 singularity is a curve $C \cong \mathbb{P}^1$ where $C \cdot C = -2$. There is a regular and birational map from the resolved geometry to the A_1 singularity, but $C \cdot C \neq -1$ and the image of the map is not non-singular either. So, this example does not contradict against the theorem of Castelnuovo.

3.4.7. omission: small resolution, terminal/canonical singularity, global aspects.

3.5 Addenda to Sections 2 and 3

We begin with materials to be added to section 2.

Definition 3.5.1. Let R be a ring, and $\mathfrak{p}$ its prime ideal. When one exhausts all the strictly decreasing chain of prime ideals of R, $R \supset \mathfrak{p} \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r$, the length of the longest such chains $\max(r)$ is said to be the *height of* $\mathfrak{p}$.

In $R = \mathbb{C}[x, y]$, $\mathrm{hgt}(\mathfrak{p}) = 2$ for $\mathfrak{p} = (x) + (y)$, $\mathrm{hgt}(\mathfrak{p}) = 1$ for $\mathfrak{p} = (x)$, and $\mathrm{hgt}(\mathfrak{p}) = 0$ for $\mathfrak{p} = (0)$.

When $R = \mathbb{C}[X]$, and a prime ideal $\mathfrak{p} \subset R$ is the defining ideal of an irreducible subvariety $Y \subset X$, then $\mathrm{hgt}(\mathfrak{p})$ is the codimension of Y in X.

For a ring R, its *Krull dimension* $\dim R$ is the largest of all $\mathrm{hgt}(\mathfrak{p})$ of prime ideals $\mathfrak{p}$ of R.

3.5.2. For an irreduicible algebraic variety X defined over a field k (such as $\mathbb{C}$), its field of rational functions $k(X)$ is an extension field over k. The transcendental degree of the extension $k(X)/k$, $\mathrm{tr.deg}(k(X)/k)$ may also be used as a definition of the irreducible variety X.

For an irreducible algebraic variety X over a field k, the Krull dimension of the ring $\mathcal{O}_X(U)$ of its Affine open patch U is known to be the same as $\mathrm{tr.deg}(k(X)/k)$.

Not all the rings can be regarded as the coordinate ring of an Affine open patch of an algebraic variety defined over a field k, however, as seen below.

Example 3.5.3. Think of the ring $R = \mathbb{Z}$. Then its Krull dimension is $\dim \mathbb{Z} = 1$, because $\mathrm{hgt}(p) = 1$ (remember $(p) \supset (0)$) for all the prime numbers p. So, a prime ideal (p) corresponds to an irreducible codimension-1 subvariety of $\mathrm{Spec}(\mathbb{Z})$. So, prime ideals (p) are hight-1 in $R = \mathbb{Z}$, and hence are regarded as codimension-1 subvarieties of $\mathrm{Spec}(\mathbb{Z})$.

3.5.4. More generally, let K be a finite dimensional extension field over $\mathbb{Q}$. For any element $\alpha \in K$, there must be a $\mathbb{Q}$-coefficient polynomial $f_\alpha(x)$ (of finite degree) such that $f_\alpha(x = \alpha) \in K$ is equal to $0 \in K$. The subset $\mathcal{O}_K \subset K$ is the set of all the elements in K for which we can choose

$f_\alpha(x)$ to be monic (one can choose $f_\alpha(x)$ so that all the coefficients are in $\mathbb{Z}$, not $\mathbb{Q}$, and yet the top degree coefficient is 1). Examples: 1) $K = \mathbb{Q}(\sqrt{-1})$, where $\mathcal{O}_K = \mathbb{Z}[\sqrt{-1}]$. 2) $K = \mathbb{Q}(\sqrt{-3})$, where $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{-3})/2]$, and 3) $K = \mathbb{Q}(\sqrt{-5})$, where $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$.

It is known that $\mathcal{O}_K$ forms a subring in K, is a Dedekind ring, and any ideals of $\mathcal{O}_K$ can be decomposed uniquely into a product of prime ideals of $\mathcal{O}_K$, and $\mathrm{hgt}(\mathfrak{p}) = 1$ for all the prime ideals of $\mathcal{O}_K$ (similarly to $\mathcal{O}_K = \mathbb{Z}$ for $K = \mathbb{Q}$). So, in the ring $\mathcal{O}_K$, the Krull dimension is 1, and the prime ideals $\mathfrak{p}$ are all regarded as codim-1 subvarieties of $\mathrm{Spec}(\mathcal{O}_K)$.

For $X = \mathrm{Spec}(\mathcal{O}_K)$, a general element of $\mathrm{Div}(X)$ is of the form $\sum_\mathfrak{p} n_\mathfrak{p} D_\mathfrak{p}$, where $\mathfrak{p}$ are prime ideals of $\mathcal{O}_K$, and $n_\mathfrak{p} \in \mathbb{Z}$, so $\mathrm{Div}(X)$ is the same as the ideal group of K. The linear equivalence relation among those divisors are generalted by principal ideals. So, the quotient $\mathrm{Cl}(X)$ of $X = \mathrm{Spec}(\mathcal{O}_K)$ is precisely what is known as the ideal class group Cl_K in algebraic number theory.

Codimension-1 subvarieties are "divisors" indeed, and such notations as $\mathcal{O}_K$ and $\mathrm{Cl}(X)$ are shared between algebraic number theory and algebraic geometry.

....

Next, here we have a few memo's on definition of "dimensions" in algebraic geometry. We have already introduced Krull dimension for a ring, but there are a few other characterizations of dimension of an algebraic variety.

3.5.5. For an irreducible algebraic variety X defined over a field k, the field of rational functions $k(X)$ on X is an extension field over k. The transcendence degree of the extension $k(X)/k$ is another characterization of the dimension of X.

3.5.6. For a domain R finitely generated over a field k, the Krull dimension of R and the transcendence degree of the field of fractions of R are known to be the same. In other words, the Krull dimension of an irreducible Affine variety X defined over a field k is the same as the transcendence degree of $k(X)/k$.

Another characterization is through <u>system of parameters</u> (s.o.p.).

3.5.7. For a local ring $(R, \mathfrak{m})$, a set $\{x_1, x_2, \cdots, x_r\} \subset \mathfrak{m}$ is a *system of parameters*, if the ideal $(x_1, \cdots, x_r)$ is not contained in any one of prime ideals $\mathfrak{p} \subset \mathfrak{m}$ that is not $\mathfrak{m}$.

The number of elements r of a system of parameters can also be used as definition of a local ring. So, an idea here is to measure at least how many functions are necessary in order to pin down a given point (that the local ring corresponds to) within the variety. An example is, for $R = k[[x,y]]/(xy)$ and $\mathfrak{m} = (x) + (y)$, choose $x_{1=r} = x + y$.

It is known that this dimension is the same as two other definitions (Krull dimension and transcendental degree), if the variety in question is irreducible. The definition here, however, has an advantage in that the definition is given in terms of local geometry (local ring), and whether the variety is reducible or irreducible does not take a center stage.

....

The canonical divisor is well-defined for a non-singular variety X, but we cannot ask whether a resolution of singluarity $\nu : \tilde{X} \to X$ is crepant or not if the canonical divisor K_X of a singular variety X is not defined in the first place. There are two classes of not-necessarily non-singular algebraic varieties where the canonical divisor is well-defined.

3.5.8. When X is a *normal variety* (def. the local ring is normal (integrally closed domain) at every point in X), the canonical bundle is a well-defined line bundle on $X \backslash X_{\rm singl}$, so a Cartier divisor D_X on $X \backslash X_{\rm singl}$ is assigned (next section), so D_X can also be regarded as a Weil divisor on $X \backslash X_{\rm singl}$. The property that $\mathrm{codim}_{\mathbb{C}} X_{\rm singl} \geq 2$ in a normal variety X implies that $\mathrm{Cl}(X) \twoheadrightarrow \mathrm{Cl}(X \backslash X_{\rm singl})$ is an isomoprhism in fact. So, there is a unique Weil divisor on X that corresponds to $D_X \in \mathrm{Cl}(X \backslash X_{\rm singl}$. The Weil divisor on a normal variety X determined in this way is denoted by K_X, and called the canonical divisor on X.

For a Noetherian local ring,

$$\text{regular} \to UFD \to \text{normal (integrally closed domain)}. \tag{33}$$

Examples: $X = \mathbb{C}^2/\mathbb{Z}_2$ is normal but not UFD, or regular.

$X = \mathbb{C}^3/\mathbb{Z}_2$, $\mathbb{C}[X] = \mathbb{C}[x^2, y^2, z^2, xy, yz, zx]$ (this is not a complete intersection). The Weil divisor K_X is known to be $\mathbb{Q}$-Cartier, but not Cartier in this case.

As seen above, one approach to defining the canonical divisor K_X is to use the canonical sheaf Ω_X; this approach was used for a normal variety X. Another approach is to use the dualizing complex$_*$; this approach is to be used for <u>Gorenstein</u> varieties, as follows.

3.5.9. It is almost the definition of a *Cohen–Macaulay* variety X for the dualizing complex to be represented by just one sheaf (rather than a chain complex of sheaves), so the corresponding sheaf is called the dualizing sheaf. It is almost the definition of a *Gorenstein* variety X for the dualizing sheaf of a Cohen–Macaulay variety X to be locally free. So (next section), when X is Gorenstein, the dualizing sheaf is regarded as $\mathcal{O}_X(K_X)$ with some appropriately chosen Cartier divisor K_X.

For a local ring, the arrows of implication is

$$\text{regular} \to \text{compl. intersect'n} \to \text{Gorenstein} \to \text{Cohen-Macaulay}. \tag{34}$$

4 Vector Bundle, Locally Free Sheaf, and Divisor

4.1 Preparation

Definition 4.1.1. A sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ is *locally free*, if one can find an open neighbour $U_x \ni x$ for any one point $x \ni X$ so that $\mathcal{F}|_{U_x}$ is isomorphic to $\oplus_{i=1}^r \mathcal{O}_X$.

Definition 4.1.2. A sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ is *torsion free*, if the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ does not have any torsion element[10] for any open subset $U \subset X$.

The value of r of a locally free sheaf $\mathcal{F}$ remains the same anywhere in X if X is connected. This value r is called *rank* of $\mathcal{F}$.

Example 4.1.3. An ideal sheaf $\mathcal{I}_Y$ for a divisor Y of X is locally free, but an ideal sheaf $\mathcal{I}_Z$ for a codimension$_{\mathbb{C}}$-two subvariety Z is not locally free. Such $\mathcal{I}_Z$ is still torsion free. This difference reflects the fact that $R = \mathbb{C}[x]$ (the ring of $\mathbb{C}$-coefficient polynomials in one variable) is a *principal ideal domain* (PID), where one can define "division" operation, so that any ideal of the ring R can be expressed in the form of (f) for some element $f \in R$. On the other hand, the ring $\mathbb{C}[x_1, x_2, \cdots, x_n]$ (the ring of $\mathbb{C}$-coefficient polynomials with multiple variables) does not have this property. This difference—phrased in terms of algebra—has its translation in terms of geometry; the fact that $\mathbb{C}[x]$ is a PID corresponds to a statement that one can find an appropriate normal coordinate (or a defining equation f) for a divisor Y.

4.2 One-to-one Correspondence

Theorem 4.2.1. On a non-singular algebraic variety X, there is one-to-one correspondence among A) line bundles (ie., rank-1 vector bundles), B) locally free rank-1 sheaves and C) linear equivalence classes of Cartier divisors $\mathrm{Pic}(X)$.

A $\Rightarrow$ B: From a line bundle L on X, a sheaf $\mathcal{F}$ of $\mathcal{O}_X$ module is constructed as follows. For a Zariski open subset U of X, let $\mathcal{F}(U)$ be all the sections of L that remains holomorphic in $U \subset X$. This $\mathcal{F}(U)$ can be regarded as a module over a ring $\mathcal{O}_X(U)$; a section $s \in \mathcal{F}(U)$ multiplyed by a function f that remains regular on $U \subset X$—$f \cdot s$—is still a holomorphic section of L, and hence $f \cdot s \in \mathcal{F}(U)$. The restriction map $\rho_{U'U} : \mathcal{F}(U) \to \mathcal{F}(U)$ is obtained by just restricting a section $s \in \mathcal{F}(U)$ to a subset $U' \subset U$. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$ module is constructed that way.

[10]**Def**. In a module M over a ring R, $M \ni m \neq 0$ is a *torsion element*, if $r \cdot m \neq 0$ for any $r \in R$ that is not a zero divisor.
In a ring R, and element r is a *zero divisor*, if there exists $r' \in R$ so that $rr' = 0$. [e.g., when $R = \mathbb{C}[x, y]/(xy)$, both $x, y \in R$ are zero divisors.]

We still need to prove that this sheaf is locally free and rank-1. Let $\cup_{i\in I} V_i$ be an open covering of X that provides local trivialization of the line bundle L; then $\mathcal{F}|_{V_a} \cong \mathcal{O}_X|_{V_a}$ because $L|_{V_a}$ is a trivial line bundle. $\bullet$

B $\Rightarrow$ A: Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$ module whose rank is 1. Then there is an open covering $\cup_{i\in I} V_i$ of X so that $\mathcal{F}|_{V_i} \cong \mathcal{O}_X|_{V_i}$. There is an element $1 \in \mathcal{O}_X(V_a)$, the function whose value remains 1 everywhere in $V_i \subset X$, and let e_i be its corresponding element in $\mathcal{F}(V_i)$. Now, suppose that $V_i \cap V_j \neq \phi$. Then both $\rho_{(V_j\cap V_i)V_j}(e_j)$ and $\rho_{(V_j\cap V_i)V_i}(e_i)$ belong to $\mathcal{F}(V_j \cap V_i)$, but they are not necessarily the same. There must be a function $g_{ji} \in \mathcal{O}_X(V_j \cap V_i)$ so that

$$\rho_{(V_j\cap V_i)V_i}(e_i) = \rho_{(V_j\cap V_i)V_j}(e_j)\; g_{ji}, \tag{35}$$

because $\mathcal{F}(V_j \cap V_i) \cong \mathcal{O}_X(V_j \cap V_i)$. Now, a line bundle L over X is constructed by using $\cup_{i\in I} V_i$ as an open covering of X for local trivialization, and g_{ji} as the transition function; e_i is a frame in a trivialization patch V_i of the line bundle L. $\bullet$

A $\Rightarrow$ C: Let L be a line bundle on X, $\pi : L \to X$ be the projection to the base space, and $\cup_{i\in I} V_i$ the local trivialization patches covering X; by definition, there is an isomorphism $\pi^{-1}(V_i) \cong \mathbb{C} \times V_i$ and let ξ_i be the fibre coordinate in this trivialization patch. The transition function g_{ji} on $V_j \cap V_i$ sets the relation between the fibre coordinates in the overlapping trivialization patches, $\xi_j = g_{ji}\xi_i$. g_{ji} remains holomorphic and has neither a pole nor a zero in $V_j \cap V_i$.

Now we can use those data to specify a linear equivalence class of Cartier divisors. Let us choose $i_0 \in I$ and choose $f_{i_0} \in \mathbb{C}(V_{i_0})$ whatever one likes it to be. For other $j \in I$, we set $f_j = g_{ji_0} f_{i_0}$. (V_i, f_i) defines a Cartier divisor on X; due to the ambiguity/freedom in the choice of f_{i_0}, however, only a linear equivalence class of Cartier divisors is specified. $\bullet$

C $\Rightarrow$ A: Let $D = (V_i, f_i)$ be a Cartier divisor on X. Then a line bundle L on X is defined by using $\cup_i V_i$ as the trivialization patches of L, and $g_{ji} := f_i/f_j$ the transition function. Thanks to the definition of a Cartier divisor, f_i/f_j neither has a pole nor a zero in $V_j \cap V_i$. $\bullet$

C $\Rightarrow$ A $\Rightarrow$ B: Let $D = (V_i, f_i)$ be a Cartier divisor on X. A sheaf $\mathcal{F}$ over X is given by

$$\mathcal{F}(U) = \{\varphi \in \mathbb{C}(U) \mid (D + \mathrm{div}(\varphi))|_U \geq 0\} \tag{36}$$

for any open set $U \subset X$. This sheaf $\mathcal{F}$ is locally free rank-1, because $\mathcal{F}(U) \cong \{\varphi_i/f_i \mid \varphi_i \in \mathbb{C}[U]\} \cong \mathcal{O}_X(U)$ for any $U \subset V_i$, and $\cup_{i\in I} V_i$ is an open covering of X. $\bullet$

Single line summaary is

$$\xi_i = f_i\varphi, \qquad \xi_j = g_{ji}\xi_i, \qquad e_j g_{ji} = e_i, \qquad e_j\xi_j = e_i\xi_i, \qquad g_{ji} = (f_j/f_i), \tag{37}$$

where e_i, ξ_i and g_{ji} are the trivialization frames, fibre coordinates and transition functions of a line bundle L over X, and f_i rational functions used in defining a Cartier divisor.

4.2.2. The one-to-one correspondence in the theorem is so important and basic in algebraic geometry that an object in one of the three categories A–C is often regarded as the corresponding object in another category quite often in algebraic geometry literature, without referring to the theorem even a bit. The locally free rank-1 sheaf $\mathcal{F}$ corresponding to a Cartier divisor D is denoted by $\mathcal{O}_X(D)$; when the choice of X is obvious, $\mathcal{O}(D)$ may also be used. A locally free rank-1 sheaf $\mathcal{F}$, often in caligraphic style, may sometimes be regarded as the corresponding line bundle, without even changing the notation from $\mathcal{F}$ to F.

4.2.3. It is known[11] that there is a following exact sequence for a compact Kahler manifold X:

$$0 \to H^1(X;\mathbb{Z}) \to H^{0,1}(X;\mathbb{C}) \to H^1(X;\mathbb{C}^\times) \to H^2(X;\mathbb{Z}) \to H^{0,2}(X;\mathbb{C}). \tag{39}$$

$H^1(X;\mathbb{C}^\times)$ in the middle of this sequence is meant to be the $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$ valued Čech cohomology. For a line bundle L over X, its transition functions g_{ji} give an element in $H^1(X;\mathbb{C}^\times)$. Conversely, any element in $H^1(X;\mathbb{C}^\times)$ can be used as transition functions of a line bundle on X. Its image in $H^2(X;\mathbb{Z})$ is the first Chern class of L, $c_1(L)$. Line bundles L with a given $c_1(L)$ are parametrized by a complex torus $H^{0,1}(X;\mathbb{C})/H^1(X;\mathbb{Z})$.

The one-to-one correspondence between the linear equivalence classes of Cartier divisors C) and line bundles A) implies that $\mathrm{Pic}(X)$ should be identified with $H^1(X;\mathbb{C}^\times)$. In the case of $\dim_{\mathbb{C}} X = 1$ (when X is a curve), $H^{0,2}(X;\mathbb{C}) = \{0\}$ and the homomorphism $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z}) \cong \mathbb{Z}$ is surjective; this homomorphism just extracts the number of points, $\sum_i n_i =: \deg(D)$ from a divisor $D = \sum_i n_i \mathrm{pt}_i$, where pt_i's are points in the curve X. The kernel, divisors whose degree is zero, still forms a variety (complex torus) $H^{0,1}(X;\mathbb{C})/H^1(X;\mathbb{Z})$. We have seen in the case X is an elliptic curve that degree zero divisors modulo linear equivalence still comes with a family parametrized by E. So, this result is consistent with what we have learned before. The modulo linear equivalence classification of Cartier divisors, $\mathrm{Pic}(X)$, retains the information of Wilson lines encoded in $H^{0,1}(X;\mathbb{C})$, not just the field strength (first Chern class) in $H^2(X;\mathbb{Z})$, if we are to use the language suitable for line bundles. $H^{0,1}(X;\mathbb{C})/H^1(X;\mathbb{Z})$ is trivial when $X = \mathbb{P}^1$, which is once again consistent with what we learned before.

[11] This exact sequence of vector spaces over $\mathbb{C}$ follows as the long cohomology exact sequence associated with the following short exact sequence of sheaves,

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^\times \longrightarrow 0. \tag{38}$$

Since we have not introduced sheaf cohomology in this lecture note at this moment, we do not use this logic for explanation here.

4.3 Examples (also mappings to projective spaces and GAGA)

Example 4.3.1. Let $X = \mathbb{P}^1$. X has an open covering $U_{\xi_0 \neq 0} \cup U_{\xi_1 \neq 0}$, where $[\xi_0 : \xi_1]$ is the set of homogeneous coordinates on $X = \mathbb{P}^1$. We have seen that $\mathrm{Cl}(X) = \mathrm{Pic}(X) \cong \mathbb{Z}$ in this case. For a divisor class $[dH]$, where $d \in \mathbb{Z}$, we can choose a Cartier divisor $(U_{\xi_i \neq 0}, f_i)$ with $f_0 = 1$ and $f_1 = (\xi_0/\xi_1)^d$ as its representative. Corresponding to this divisor class is the line bundle where the fibre coordinates s_i in the $U_{\xi_i \neq 0}$ patch are identified under the relation $s_0 = g_{01} s_1$ in $U_{\xi_0 \neq 0} \cap U_{\xi_1 \neq 0}$ with $g_{01} = f_0/f_1 = (\xi_1/\xi_0)^d$. This line bundle is denoted by $\mathcal{O}_{\mathbb{P}^1}(dH)$, although this notation is primarily meant to be for the corresponding locally free rank-1 sheaf.

Holomorphic sections of the bundle $\mathcal{O}_{\mathbb{P}^1}(dH)$ correspond to rational functions $\mathbb{C}(X)$ that are polynomials of (ξ_1/ξ_0) in the $U_{\xi_0 \neq 0}$ patch, and are $(\xi_1/\xi_0)^d$ times those of (ξ_0/ξ_1) in the $U_{\xi_1 \neq 0}$ patch. Therefore the sections that are holomorphic everywhere on $\mathbb{P}^1$ correspond to polynomials of (ξ_1/ξ_0) of degree at most d; for the divisor classes $[dH]$ with $d < 0$, there is no section that is holomorphic anywhere in $\mathbb{P}^1$, or put differently, the complete linear system $|dH|$ is empty.

The cotangent bundle $T^*\mathbb{P}^1$ corresponds to $\mathcal{O}_{\mathbb{P}^1}(-2H)$. To see this, note that $d(\xi_1/\xi_0)$ and $d(\xi_0/\xi_1)$ are the frames of this line bundle in the $U_{\xi_0 \neq 0}$ and $U_{\xi_1 \neq 0}$ patches, respectively. The transition function is therefore $g_{01} = d(\xi_0/\xi_1)/d(\xi_1/\xi_0) = -(\xi_0/\xi_1)^2$. The (-1) sign in g_{01} can be absorbed by changing the frame in one of the two patches by (-1). Thus, this bundle $T^*\mathbb{P}^1$ corresponds to the $d = -2$ case. Similar computation reveals that $T\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2H)$.

Example 4.3.2. As an obvious generalization, consider $X = \mathbb{P}^n$. $X = \mathbb{P}^n$ is covered by $(n+1)$ open Affine patches, $U_{\xi_i \neq 0}$ with $i = 0, \cdots, n$, where $[\xi_0 : \xi_1 : \cdots : \xi_n]$ is a set of homogeneous coordinates of $\mathbb{P}^n$. $\mathrm{Cl}(X) = \mathrm{Pic}(X) \cong \mathbb{Z}$ in this case; for a divisor class $[dH]$ with $d \in \mathbb{Z}$, we can choose the Cartier divisor $(U_{\xi_i \neq 0}, f_i)$ with $f_i = (\xi_0/\xi_i)^d$ as a representative. In the corresponding line bundle $\mathcal{O}_{\mathbb{P}^n}(dH)$, the fibre coordinates s_i in the trivialization in the patches $U_{\xi_i \neq 0}$ are identified with $s_j = g_{ji} s_i$ with the transition functions $g_{ji} = (f_j/f_i) = (\xi_i/\xi_j)^d$ in $U_{\xi_j \neq 0} \cap U_{\xi_i \neq 0}$. Sections of this line bundle holomorphic everywhere in $\mathbb{P}^n$ are in one-to-one correspondence with polynomials in n-varaibles $(\xi_i/\xi_0)|_{i=1,\cdots,n}$ of degree at most d when $d \geq 0$; if $d < 0$, then there is no such section.

The cotangent bundle $T^*\mathbb{P}^n$ is a rank-n vector bundle on $X = \mathbb{P}^n$, and its determinant bundle $\det(T^*X) = \wedge^n T^*X$ is a line bundle. Explicit computation of the transition functions reveals that $\wedge^n T^*\mathbb{P}^n \cong \mathcal{O}_{\mathbb{P}^n}(-(n+1)H)$.

Definition 4.3.3. For a non-singular variety X with $\dim_{\mathbb{C}} X = n$, the determinant bundle $\det(T^*X) = \wedge^n T^*X$ of the holomorphic cotangent bundle T^*X is called *canonical bundle*. It is always rank-1. Let $\{x^{(a)}_{i=1,\cdots,n}\}$ be the local coordinates in a patch U_a, and let $\cup_{a \in A} U_a$ be an open covering of X. Then we can choose $\wedge_{i=1}^n dx_i^{(a)}$ as a frame in the patch U_a for the local trivialization of the canonical bundle. Sections of this bundle that look $s = s_a \wedge_{i=1}^n dx_i^{(a)}$ in the patch U_a are identified in a pair of overlapping patches when $s_b = g_{ba} s_a$, with $g_{ba} = |\partial x^{(a)}/\partial x^{(b)}|$.

The line bundle that corresponds to the canonical divisor K_X introduced in 3.4.2 is this canonical bundle $\wedge^n T^*X$ for any non-singular variety X; one can just verify that $g_{ba} = f_b/f_a$ to see this. Therefore, $\det(T^*X) = \wedge^n T^*X = \mathcal{O}_X(K_X)$. With an extra abuse of notation, K_X is often use for the canonical bundle, not just for the canonical divisor.

4.3.4. Let X be a non-singular variety, and Y be a non-singular hypersurface in X; $\mathrm{codim}_{\mathbb{C}} Y = 1$ because Y is a hypersurface. Then the normal bundle $N_{Y|X}$ is a rank-1 line bundle on Y. Due to the one-to-one correspondence, there must be a linear equivalence class of Cartier divisors on Y. It is known that $N_{Y|X} = \mathcal{O}_Y(Y|_Y)$; a divisor $Y|_Y$ on Y is obtained by replacing the divisor Y of X by another divisor Y' that is linear equivalent to Y but not the same as Y, and $Y|_Y := Y' \cdot Y$.

Intuitively, Y' is a continuous deformation of Y within X, and $Y' \cdot Y$ is the fixed locus in this deformation. Such an intuition is absent, however, when we cannot find a divisor $Y' \sim Y$ so that Y' is effective.

Example 4.3.5. Consider $X = \mathbb{P}^n$ and choose a non-singular hypersurface $Y \in |dH|$. Then $N_{Y|X} \cong \mathcal{O}_Y(dH|_Y)$. In particular, when $X = \mathbb{P}^2$ and Y a degree-d curve in $\mathbb{P}^2$, Y and its deformation intersect at d^2 points.

Example 4.3.6. Consider $X = \mathrm{Bl}_{\vec{0}}\mathbb{C}^n$ and its hypersurface $Y = E$, the exceptional divisor of the blow-up. We have seen that $E \cdot E \sim -H_E$, where H_E is the hyperplane divisor of $E \cong \mathbb{P}^{n-1}$. Thus, $N_{E|X} \cong \mathcal{O}_E(-H_E)$.

4.3.7. For a line bundle L over a non-singular variety X, $\Gamma(X, L)$ denotes all the sections of L that remain holomorphic everywhere in X. Those sections are called *global holomorphic section*s. They form a vector space over $\mathbb{C}$, because a sum of two global holomorphic sections is still holomorphic everywhere. When $\mathcal{L} = \mathcal{O}_X(D)$ is the locally free rank-1 sheaf corresponding to the line bundle L, then $\Gamma(X; L) = \mathcal{L}(X)$. Therefore,

$$\Gamma(X; L) = \Gamma(X; \mathcal{O}_X(D)) = \{\varphi \in \mathbb{C}(X) \mid D + \mathrm{div}(\varphi) \geq 0\}\,. \tag{40}$$

The dimension of this vector space is denoted by $\ell(D)$. The complete linear system $|D|$ is the $\mathbb{C}^\times$ quotient of this vector space (except $\vec{0} \in \mathbb{C}^{\ell(D)}$).

The family of hypersurfaces that belong to the divisor class $[D]$ represented by an effective divisor D in X can be captured as a whole as a hypersurface $\mathcal{Y}$ in $X \times |D|$. Let us take a basis $\{\varphi_{i=0,\cdots,\ell(D)-1}\}$ of the vector space $\Gamma(X; \mathcal{O}_X(D))$; we can always take $\varphi_0 = 1$; when the effective divisor D corresponds to the zero locus of a homogeneous function F_0, the divisor $D' = D + \mathrm{div}(\sum_i c_i \varphi_i)$ corresponds to the zero locus of $F_0(\sum_i c_i\varphi_i) =: \sum_i c_i F_i$. The family $\mathcal{Y} \subset X \times |D|$ is given as the zero locus of $\sum_i c_i F_i$, where $[c_0 : c_1 : \cdots : c_{\ell(D)-1}]$ is the homogeneous coordinates of $|D| \cong \mathbb{P}^{\ell(D)-1}$. $\mathcal{Y}$ is regarded as fibration over $|D|$; $\pi : \mathcal{Y} \hookrightarrow X \times |D| \to |D|$; when one point $[c_0 : \cdots : c_{\ell(D)-1}] \in |D|$ is chosen, then the fibre is $D' \subset X$.

Infinitesimal deformation of such an effective divisor D, or the tangent space of $|D|$ can also be captured in the language of geometry of the subvariety D rather than that of the entire X. It is known that there is an exact sequence

$$0 \to [\Gamma(X; \mathcal{O}_X) \cong \mathbb{C}] \to \Gamma(X; \mathcal{O}_X(D)) \to \Gamma(D; N_{D|X}) \to H^{0,1}(X; \mathbb{C}). \tag{41}$$

The first term $\cong \mathbb{C}$ corresponds to the subspace $\mathrm{Span}_{\mathbb{C}}\{\varphi_0\} \subset \mathrm{Span}_{\mathbb{C}}\{\varphi_{i=0,\cdots,\ell(D)-1}\}$ of the second term. The quotient, which is parametrized by $(c_{i=1,\cdots,\ell(D)-1}/c_0)$, is the space of sections of the normal bundle $N_{D|X}$ that are holomorphic over the entire hypersurface $D \subset X$. Infenitesimal deformations of the hypersurface $D \subset X$, the third term, therefore corresponds to the tangent space of $|D|$.

4.3.8. One can use the complete linear system $|D|$ of an effective divisor D in X to construct a regular map from X to some projective space $\mathbb{P}^{\ell(D)-1}$. Let $\Gamma(X; \mathcal{O}_X(D)) = \mathrm{Span}_{\mathbb{C}}\{\varphi_{i=0,\cdots,\ell(D)-1}\}$ once again. A map

$$\Phi_{|D|} : X \longrightarrow \mathbb{P}^{\ell(D)-1} := \mathbb{P}\left[\Gamma(X; \mathcal{O}_X(D))^* \backslash \{0\}\right] \tag{42}$$

is not to the projectivisation of the space $\Gamma(X; \mathcal{O}_X(D))$ (where $[c_0 : c_1 : \cdots : c_{\ell(D)-1}]$ is the homogeneous coordinates), but to the projectivisation of the dual vector space of $\Gamma(X; \mathcal{O}_X(D))$. A point $p \in X$ is sent by this map $\Phi_{|D|}$ to a point $[\varphi_0(p) : \varphi_1(p) : \cdots : \varphi_{\ell(D)-1}(p)] \in \mathbb{P}^{\ell(D)-1}$, or put differently, to $[F_0(p) : F_1(p) : \cdots : F_{\ell(D)-1}(p)] \in \mathbb{P}^{\ell(D)-1}$. This map is not well-defined, however, if there is an irreducible divisor D_0 that is contained in every member D' of the divisor class $[D]$.

When $\ell(D)$ is sufficiently large (which means that there are many (ample) sections of the line bundle $\mathcal{O}_X(D)$), $\Phi_{|D|} : X \to \mathbb{P}^{\ell(D)-1}$ can be used as an embedding of X into a projective space. $\Phi_{|D|} : X \to \mathbb{P}^{\ell(D)-1}$ becomes a projection when $\ell(D)-1 < \dim_{\mathbb{C}} X$. Even when $\ell(D)-1 = \dim_{\mathbb{C}} X$, multiple poins in X may be mapped to a single point in $\mathbb{P}^{\ell(D)-1}$.

Hyperplanes in this target space $\mathbb{P}^{\ell(D)-1}$ are in the form of $\sum_i c_i\varphi_i = 0$ (or $\sum_i c_iF_i = 0$) for some $(c_{i=0,\cdots,\ell(D)-1}) \neq \vec{0}$. The intersection of such a hypersurface and the image $\Phi_{|D|}(X) \subset \mathbb{P}^{\ell(D)-1}$ is the image of the divisor $D' \in |D|$ given by $\sum_i c_iF_i = 0$. This divisor D' of X is the same as the inverse image of $[c_0 : \cdots : c_{\ell(D)-1}] \in |D|$ under $\pi : \mathcal{Y} \to |D|$; hyperplanes in this $\mathbb{P}^{\ell(D)-1}$ ($\ell(D) - 1$ dimensional spaces in $\mathbb{C}^{\ell(D)}$ before projectivisation) are dual to points in the previous $\mathbb{P}^{\ell(D)-1}$ (1-dimensional space $\mathbb{C} \times (c_{i=0,\cdots,\ell(D)-1})$ before projectivisation).

Example 4.3.9. Let $X = \mathbb{P}^1$. $\Phi_{|dH|} : X \to \mathbb{P}^d$. When $d = 1$, $\Phi_{|H|}$ is a trivial isomorphism between $\mathbb{P}^1$'s. When $d = 2$, $\Phi_{|dH|} : \mathbb{P}^1 \ni [X : Y] \mapsto [X^2 : XY : Y^2] = [\varphi_0 : \varphi_1 : \varphi_2] \in \mathbb{P}^2$. The image

$\Phi_{|2H|}(\mathbb{P}^1)$ forms a degree-2 curve $\{\varphi_0\varphi_2 - \varphi_1^2 = 0\} \subset \mathbb{P}^2$. For general d, the image $\Phi_{|dH|}(\mathbb{P}^1)$ is a curve in $\mathbb{P}^d$ that intersects d times against a hyperplane in $\mathbb{P}^d$. This example is not particularly interesting.

Example 4.3.10. Let X be an elliptic curve E, and e be the origin of E. Then $\Gamma(E; \mathcal{O}_E) = \mathbb{C}$, $\Gamma(E; \mathcal{O}_E(e)) = \mathbb{C}$, $\Gamma(E; \mathcal{O}_E(2e)) = \mathrm{Span}_{\mathbb{C}}\{1, x\} \cong \mathbb{C}^2$ and $\Gamma(E; \mathcal{O}_E(3e)) = \mathrm{Span}_{\mathbb{C}}\{1, x, y\} \cong \mathbb{C}^3$; $x = \wp$ has a pole of order two at $e \in E$, and $y = \wp'$ of order three at $e \in E$. The map $\Phi_{|3e|} : E \to \mathbb{P}^2$ given by $p \mapsto [1 : x : y] \in \mathbb{P}^2$ sends E to a subspace of $\mathbb{P}^2$ defined by $y^2 = 4x^3 - g_2 x - g_3$. Even when E is given only analytically, $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\langle\tau\rangle)$, this maps allows us to understand E as a zero locus of a polynomial equation of a projective space, i.e., an algebraic variety. The map $\Phi_{|2e|} : E \to \mathbb{P}^1$, on the other hand, is a projection $(x, y) \mapsto x$, where two points are mapped to one point.

Theorem 4.3.11. Let M be a compact Kähler manifold of complex n dimensions such that $h^{2,0}(M) = 0$. Then there exists an algebraic geometry implementation X such that $X^{an} = M$.

The proof of this statement is divided into multple steps. The first step is to note that $h^{2,0}(M) = 0$ implies $H^1(M; \mathbb{C}^\times) \to H^2(M; \mathbb{Z})$ is surjective. This means that one can find a line bundle L_ω on M for any $\omega \in H^2(M; \mathbb{Z})$ so that $c_1(L_\omega) = \omega$.

Let $\omega_M \in H^{1,1}(M; \mathbb{R})$ be the Kähler form of the Kähler manifold M. Then one can find $\omega \in H^2(M; \mathbb{Z})$ very close to $\lambda\omega_M$ for some $\lambda \in \mathbb{R}_{>0}$; the rationale here is that one can find a lattice point arbitrary close to a given line (in the form of $\{\lambda\omega_M \mid \lambda \in \mathbb{R}\}$). Let ω_* be such ω. The first step of the argument above implies that there is a line bundle L_{ω_*} corresponding to $\omega_* \in H^2(M; \mathbb{Z})$.

As the third step, consider the vector space $\Gamma(M; L_{\omega_*})$ and the map $M \to \mathbb{P}[\Gamma(M; L_{\omega_*})^*]$ denoted by $\Phi_{|L_{\omega_*}|}$. One then needs to argue that this map can be made an embedding (any pair of points in M have distinct image points under $\Phi_{|L_{\omega_*}|}$). The rationale behind this is that ω_M is the Kähler form, so the line $\{\lambda\omega_M\}$ must be within the Kähler cone (ample cone), so the line bundle L_{ω_*} must also be ample. By choosing $\lambda\omega_M \simeq \omega_*$ very large (λ very large), L_{ω_*} is expected to have ample global sections so all the points in M are resolved by those sections (i.e., very ample).

The final step is to claim that the image of M under the map $\Phi_{|L_{\omega_*}|}$ is characterized by the zero loci of polynomials in the projective space $\mathbb{P}(\Gamma(M; L_{\omega_*})^*)$. An important observation behind this claim is that all the meromorphic functions of the manifold Y^{an} of a complete[12] algebraic variety Y are rational functions on Y [Shafarevic Chap. VIII, §2.3 Thm.3 + §3.1 Thm.1]. It follows from this observation that for any submaifold M' (such as the image of M under $\Phi_{|L_{\omega_*}|}$ in

[12] On $Y = \mathbb{C}$, $\tan(z)$ and e^z are meromorphic functions of Y^{an}, but are not rational functions of Y. On $Y = \mathbb{C}P^1$, however, neither $\tan(z)$ nor e^z is meromorphic on Y^{an}, because they cannot be expressed as a ratio of holomorphic functions in any neighbourhood of the $z = \infty$ point of Y^{an}. Whether Y is complete ($\mathbb{C}P^1$) or not ($\mathbb{C}$) makes a difference.

$\mathbb{P}(\Gamma(X^{an}; L_{\omega_*})^*))$ in the compact manifold Y^{an} of a complete variety Y (such as $\mathbb{P}(\Gamma(X^{an}; L_{\omega_*})^*))$, there exists a subvariety X of Y so that the $X^{an} = M'$ (locally holomorphic defining equations of M' must be locally regular functions of Y). See [Shafarevic VIII, §3.1 Thm.2 + Thm.3] or [Hartshorn §B.2, Thm. 2.2] for more information.

A similar statement holds also for a holomorphic map between compact Kähler manifolds, because maps are given by specifying functions locally.

4.4 Adjunction formula and examples of Calabi–Yau varieties

4.4.1. Let Y be a non-singular subvariety of a non-singular variety X. Then it is known that

$$0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow N_{Y|X} \longrightarrow 0 \tag{43}$$

is an exact sequence. Here, TY and TX are tangent bundles of Y and X, respectively, and $TX|_Y$ is the restriction of the vector bundle TX over X to the subvariety Y. $N_{Y|X}$ is the normal bundle of Y in X. When $\dim_{\mathbb{C}} X = n$ and $\dim_{\mathbb{C}} Y = n - r$, TY, $TX|_Y$ and $N_{Y|X}$ are all vector bundles on Y with rank $(n - r)$, n and r, respectively.

4.4.2. When there is a short exact sequence of vector bundles $0 \to E \to F \to G \to 0$ over a manifold M, there is a relation among characteristic classes of those bundles:

$$c(F) = c(E)c(G), \tag{44}$$

where $c(E) = 1 + c_1(E) + c_2(E) + \cdots$; this relation applied to the short exact sequence (43) is called adjunction formula. The same relation holds true for some other characteristic classes (such as $\mathrm{td}(E)$, $p(E)$, $\hat{A}(E)$). The Chern class of the bundle F does not depend on whether it has a structure $F \cong E \oplus G$ (the short exact sequence splits) or not.

4.4.3. Let Y be a hypersurface of X; that is, $\mathrm{codim}_{\mathbb{C}} Y = 1$ in X. Then

$$c(N_{Y|X}) = 1 + Y|_Y, \tag{45}$$

where Y is regarded as a divisor here; since $N_{Y|X}$ is a rank-1 bundle, its second Chern class and beyond all vanish. The adjunction formula allows us to determine the Chern classes of the hypersurface Y through

$$c(TY) = \frac{c(TX)|_Y}{1 + Y|_Y} = 1 + [c_1(TX)|_Y - Y|_Y] + [c_2(TX) - c_1(TX) \cdot Y]\,|_Y + \cdots . \tag{46}$$

The condition $c_1(TY)$ (one of the conditions of Y being a Calabi–Yau manifold) is therefore equal to $c_1(TX)|_Y = Y|_Y$. Whenever we choose Y to be a generic member of the anti-canonical class $[-K_X]$, therefore, it satisfies the $c_1(TY) = 0$ condition.

Example 4.4.4. Let $X = \mathbb{P}^2$, when $[-K_X] = [3H]$. Thus a generic member $Y \in [3H]$, i.e., a cubic curve in $X = \mathbb{P}^2$, satisfies $c_1(TY) = 0$. Y is an elliptic curve in this case.
Let us now take $X = \mathbb{P}^3$, when $[-K_X] = [4H]$. Now, a generic member $Y \in [4H]$, a quartic surface in $X = \mathbb{P}^3$, satisfies $c_1(TY) = 0$. Y is now a quartic K3 surface.
Let us take $X = \mathbb{P}^4$, when $[-K_X] = [5H]$. A generic member $Y \in [5H]$ is a Calabi–Yau three-fold called a quintic Calabi–Yau.
When $X = \mathbb{P}^4$, we can think of $Y = Y_1 \cap Y_2$, where $Y_1 \in [2H]$ and $Y_2 \in [3H]$. Then $c_1(N_{Y|X}) = 2H + 3H$, and $c_1(TY) = 0$. Y in this case is a K3 surface, but it is not a quartic K3, but it is known as a degree-6 K3 surface.
In all those examples, the complex structure of those Calabi–Yau manifolds of various deimensions is (roughly spealing) parametrized by how we choose global holomorphic sections of the relevant line bundles. So, we can find out the dimension of the complex structure moduli space (roughly) by computing the relevant $\ell(D) = \dim_{\mathbb{C}}[\Gamma(X; \mathcal{O}_X(D))]$'s.

4.5 Coherent Sheaves and Projective Resolution

4.5.1. In the case of the exact sequence (43), all the three pieces there are vector bundles on a variety Y. Pick two bundles TY and $TX|_Y$ and the injective morphism $i : TY \hookrightarrow TX|_Y$, then the kernel and cokernel of this morphisms are both vector bundles, the trivial vector bundle and $N_{Y|X}$, respectively. Similarly, pick the two vector bundles TX and $N_{Y|X}$ and the projection morphism $\pi : TX|_Y \to N_{Y|X}$, then the kernel and cokernel are both vector bundles, TY and the trivial bundle over Y, respectively.

It is not true in general, however, that the kernel and cokernel of a morphism between a two vector bundles $f : E \to F$ on a variety X are both vector bundles on X. We have already seen such an example; let us think of the exact sequence (9) in the case Y is a codimension-1 subvariety of X. As we have remarked already in 4.1.3, $\mathcal{I}_Y$ is locally free (not just $\mathcal{O}_X$ is). The cokernel of the morphism $\mathcal{I}_Y \hookrightarrow \mathcal{O}_X$, which is the pushforward of $\mathcal{O}_Y$, is supported in the subvariety Y of X, and is not at all a locally free over X. Here, we think of locally free sheaves and vector bundles as essentially the same thing, and use a fact that an exact sequence of locally free sheaves can also be read as an exact sequence of the corresponding vector bundles (though we have not stated this fact explicitly so far in this lecture note).

By giving up to think within the category of vector bundles over a given variety X, and by allowing to include at least some kind of sheaves (that are not necessarily locally free) into a broader category containing vector bundles over X, we have a chance that appropriate objects can be found within the broader category suitable for the kernel and cokernel of a morphism between two objects in the category. The question is, then, what the minimum list of objects that should be included in “the broader category.” Certainly the broader category should contains all the sheaves over X that can be the cokernel of a morphism between two vector bundles on X. So, here is a

Definition 4.5.2. A sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ over a variety X is a *coherent sheaf* if there are two locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{O}_X$ modules, both of finite rank r_1 and r_2, and there is a morphism $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ so that

$$\mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{F} \longrightarrow 0 \tag{47}$$

is exact; put differently, $\mathcal{F} = \mathrm{Coker}\,(\phi : \mathcal{E}_1 \to \mathcal{E}_2)$.

The definition (characterization) of coherent sheaves is about their local properties, just like that of locally free shaves is (requirement that a local trivialization should exists).

Example 4.5.3. Suppose that $X = \mathbb{C}^2$, so the ring $R := \mathcal{O}_X(X)$ is $R \cong \mathbb{C}[x,y]$, where x, y are the Affine coordinates on $X = \mathbb{C}^2$. Now, think of R-modules $M_1 \cong \mathbb{C}[x,y]$ and $M_2 \cong \mathbb{C}[x,y]$; this is to think of two rank-1 vector bundles E_1 and E_2 with a trivialization patch X, where M_1 and M_2 correspond to sections of E_1 and E_2 that remain holomorphic over X. A homomorphism between the two vector bundles $\phi : E_1 \to E_2$ is specified by a polynomial $\phi(1)$ in

$$\phi : M_1 \ni 1 \longmapsto \phi(1) \in M_2 \cong \mathbb{C}[x,y]. \tag{48}$$

The cokernel is

$$M_3 := \mathrm{Coker}\,(\phi : M_1 \longrightarrow M_2) = \mathbb{C}[x,y]/(\phi(1)(x,y)). \tag{49}$$

So, the sheaf $\mathcal{F}$ of $\mathcal{O}_X$ module that fits into this exact sequence $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{F} \to 0$ is such that $\mathcal{F}(X) = M_3$. Its support is the $\phi(1) = 0$ locus in X. When the polynomial $\phi(1)$ is an irreducible non-singular curve (hypersurface) Y in $X = \mathbb{C}^2$, then $\mathcal{F}$ is $i_*(\mathcal{O}_Y)$ on $X \cong \mathbb{C}^2$ (that is, a push forward of a locally free sheaf on the hypersurface). When the polynomial $\phi(1)$ is not irreducible, and the zero locus consists of multiple irreducible non-singular pieces, $Y = \cup_i Y_i$, that intersect transversely with one another, then $\mathcal{F}$ is still $\mathcal{O}_Y$, which is not the same as $\oplus_i i_{Y_i *}(\mathcal{O}_{Y_i})$. When $\phi(1)$ is the square of an irreducible polynomial $\phi_0 \in \mathbb{C}[x,y]$, say, $\phi_0 = y$ and $\phi(1) = y^2$, then $M_3 = \mathbb{C}[x,y]/(y^2) \cong \mathbb{C}[x] \oplus \mathbb{C}[x] \cdot y$ as an $\mathcal{O}_X$ module. The sheaf $\mathcal{F}$ is a coherent sheaf, by definition, even in such a case. $\mathcal{F}$ is not in the form of the pushfoward of a locally free sheaf on its support locus (the zero locus of ϕ_0) anymore, however.

Theorem 4.5.4. For a given variety X, and a morphism $\phi : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ between two coherent sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$, both the kernel and cokernel are coherent sheaves on X. This means that the category of coherent sheaves over X is what we looked for as “the broader category containing vector bundles over X.”

The spirit behind seeking for such a category is from the way to see that the vector bundle F that fits into an exact sequence $0 \to E \to F \to G \to 0$ is much like[13] the direct sum of the vector bundles $E \oplus G$. In this very crude way of looking at bundles, then, G can be regarded as something like $F \ominus E$, and E as something like $F \ominus G$, replacing the symbol $+$ for sum by $-$ for subtraction, whatever $\ominus$ means. If one is able to find a sum of two objects, it is a natural desire also to be able to find a difference between them. Within the category of natural numbers $\mathbb{N}$ (positive integers), we cannot always do so; we need to enlarge the category from $\mathbb{N}$ to $\mathbb{Z}$ so we can always find the difference. The relation between the category of coherent sheaves on a variety X and that of vector bundles on X is like that between $\mathbb{Z}$ and $\mathbb{N}$.

We do not provide a proof of the theorem above in this lecture note; we will work on one more example later instead. But before getting there, here is a recap:

4.5.5. Motivations / benefits of introducing the notion / concept "sheaves."

- the language with which we can keep track of the ring-geometry correspondence even for compact varieties
  - the language with which we can define the notion of variety without referring to weighted projective spaces explicitly
- the language that is necessary in characterizing the minimum category containing vector bundles where both kernels and cokernels are contained
- Poincare's lemma for differential forms on a manifold is promoted to the definition of exact sequence of sheaves, which allows
- generalization and abstraction of cohomology theory (which we will see more in section 4.6)

Example 4.5.6. We have argued in 4.1.3 that the ideal sheaf $\mathcal{I}_Y$ for a subvariety Y of X is not locally free, if $\mathrm{codim}_{\mathbb{C}} Y > 1$ in X. Such an ideal sheaf is still a coherent sheaf; we can see it as follows. Because the definition of coherent sheaf refers only to local properties of a sheaf in question, we take $X \cong \mathbb{C}^n = \{(x_1, \cdots, x_n)\}$, and $Y \cong \mathbb{C}^{n-r} = \{(x_{r+1}, \cdots, x_n)\}$, and let y denote the coordinates $(x_{r+1}, \cdots, x_n)$ collectively; instead of the sheaf $\mathcal{I}_Y$, we study the $\mathcal{O}_X$-module $\mathcal{I}_Y(X)$. First, $\mathcal{I}_Y(X) \cong (x_1) + (x_2) + \cdots + (x_r) =: M_0 \subset \mathbb{C}[x_{1,\cdots,r}, y]$. Second, let us prepare r copies of $\mathbb{C}[x_{1,\cdots,r}, y] = \mathbb{C}[X]$, and take $M_1 := \oplus_{a=1}(\mathbb{C}[X])_{(a)}$; similarly, ${}_rC_2$ copies of $\mathbb{C}[X]$ are used in $M_2 := \oplus_{1 \leq b < c \leq r}(\mathbb{C}[X])_{bc}$. Now,

$$M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0 \tag{50}$$

[13] This is not to say that F is not always isomorphic to the bundle $E \oplus G$.

is surjective, where the homomorphism $M_1 \longrightarrow M_0$ is via

$$M_1 = \oplus_{a=1}^{r}(\mathbb{C}[X])_{(a)} \ni (f_a(x,y)) \mapsto \sum_{a=1}^{r} x_a f_a(x,y) \in M_0, \tag{51}$$

and

$$M_2 = \oplus_{1\leq b<c\leq r}(\mathbb{C}[X])_{(bc)} \ni (f_{bc}(x,y)) \mapsto (\sum_{b=1}^{r} x_b f_{ab}) \in \oplus_{a=1}^{r}(\mathbb{C}[X])_{(a)} = M_1, \tag{52}$$

where $f_{cb} := -f_{bc}$ when $b < c$. So, the ideal sheaf $\mathcal{I}_Y$ can be obtained as the cokernel of a morphism between rank-r and rank-${}_rC_2$ locally free sheaves, and hence a coherent sheaf.

Example 4.5.7. Let $\mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{I}_Y \to 0$ be the sheaf exact sequence we discussed in detail above. Combining this exact sequence with another exact sequence $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_{Y*}(\mathcal{O}_Y) \to 0$, we see that $\mathcal{E}_1 \to \mathcal{O}_X \to i_{Y*}(\mathcal{O}_Y) \to 0$ is also an exact sequence, and both $\mathcal{E}_1$ and $\mathcal{O}_X$ are locally free. Therefore, the sheaf $i_{Y*}(\mathcal{O}_Y)$ is also a coherent sheaf, even when $\mathrm{codim}_{\mathbb{C}} Y > 1$ in X.

For string theorists, the sheaves $i_{Y*}(\mathcal{O}_Y)$ on X with various complex codimensions look like D-branes.

4.5.8. Projective resolution: The sequence $\mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{I}_Y \to 0$ we saw above is certainly exact, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are locally free, but alas, $\mathcal{E}_2 \to \mathcal{E}_1$ is not injective. By allowing this sequence to become longer, however, we can think of a sheaf exact sequence

$$0 \to \mathcal{E}_r \to \mathcal{E}_{r-1} \cdots \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{I}_Y \to 0, \tag{53}$$

where $\mathcal{E}_i(X) = M_i := \oplus_{a_1<a_2\cdots<a_i}(\mathbb{C}[X])_{(a_1,\cdots,a_i)}$. All of $\mathcal{E}_{1,\cdots,r}$ are locally free sheaves on X, and the chain complex of sheaves above, including $\mathcal{I}_Y$, is exact. This is an example of what is called "projective resolution" of a sheaf ($\mathcal{I}_Y$ in this case).
Combining the exact sequence above and $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_{Y*}(\mathcal{O}_Y) \to 0$, one also finds that

$$0 \to \mathcal{E}_r \to \mathcal{E}_{r-1} \cdots \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{O}_X \to i_{Y*}(\mathcal{O}_Y) \to 0 \tag{54}$$

is also exact, and all the sheaves except $i_{Y*}(\mathcal{O}_Y)$ are locally free on X. This is an example of projective resolution of the sheaf $i_{Y*}(\mathcal{O}_Y)$.

We do not write down the definition of projective resolution in this note, but the definition must be found in virtually any textbooks on homology algebra. More important than the precise definition, though, is that there exists the exact sequences (53, 54), and the fact that mathematicians entertain themselves with an idea that the chain complex $0 \to \mathcal{I}_Y \to 0$ is pretty much the same as another chain complex $0 \to \mathcal{E}_r \to \mathcal{E}_{r-1} \cdots \mathcal{E}_1 \to 0$; the chain complex $0 \to i_{Y*}(\mathcal{O}_Y) \to 0$ is much the same as the chain complex $0 \to \mathcal{E}_r \to \cdots \mathcal{E}_1 \to \mathcal{O}_X \to 0$. In string theory language, the relation between $0 \to i_{Y*}(\mathcal{O}_Y) \to 0$ on a divisor Y (ie, $r = 1$) on X and the chain complex $0 \to \mathcal{E}_1 \to \mathcal{O}_X \to 0$ is to regard a D7-brane as the difference between a D9-brane and a $\overline{\text{D9}}$–$\overline{\text{D7}}$ composite.

4.6 Sheaf Cohomology

Sheaf cohomology groups are vector spaces over $\mathbb{C}$ assigned to a sheaf $\mathcal{F}$ on a variety X under a certain rule, and are denoted by $H^i(X; \mathcal{F})$. Their definition is given by using resolutions of $\mathcal{F}$ with certain properties. It is quite often not practical or realistic, however, to compute the sheaf cohomology groups by following the definition faithfully. A technique called Čech cohomology is better suited for direct computation of sheaf cohomology groups in practice, and furthermore, one would usually combine various useful formula (which we explain in section 5) to compute the dimension of the sheaf cohomology groups without even using Čech cohomology.

We should still start off by providing some kind of definition. Instead of writing down a definition from the beginning (as in mathematics style), however, we begin with looking at notions well-known to physicists that are actually examples of sheaf cohomology groups. Essence is extracted from the examples, and will be promoted to a definition in the following.

Example 4.6.1. Consider the $\mathbb{R}$-coefficient de Rham cohomology groups of a manifold M. Over the topological space M endowed with analytic topology, the locally constant sheaf $\underline{\mathbb{R}}$ associated with the Abelian group $\mathbb{R}$ has a sheaf exact sequence starting from itself:

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots \to \mathcal{A}^m \to 0, \tag{55}$$

where $\mathcal{A}^p$ is the sheaf of $\mathbb{R}$-valued smooth p-forms on M, and $\dim_{\mathbb{R}} M =: m$; note that the kernel of $d : \mathcal{A}^0 \to \mathcal{A}^1$ is the locally constant sheaf. The process of computing $\mathbb{R}$-coefficient de Rham cohomology groups is to i) think of the locally constant sheaf $\underline{\mathbb{R}}$ instead of the Abelian group $\mathbb{R}$, ii) replace the sheaf $\underline{\mathbb{R}}$ by the chain complex of sheaves $0 \to \mathcal{A}^0 \to \mathcal{A}^1 \cdots \to \mathcal{A}^m \to 0$ that constitutes the rest of the sheaf exact sequence, iii) replace this chain complex of sheaves by a chain complex of Abelian groups, taking global sections of those sheaves,

$$0 \to \Gamma(M; \mathcal{A}^0) \to \Gamma(M; \mathcal{A}^1) \to \cdots \to \Gamma(M; \mathcal{A}^m) \to 0, \tag{56}$$

and iv) compute the cohomology groups of this chain complex. Although the $\mathbb{R}$-coefficient de Rham cohomology groups are usually denoted by $H^p(M; \mathbb{R})$ ($p = 0, \cdots, m$), they are also regarded as sheaf cohomology groups $H^p(M; \underline{\mathbb{R}})$ under the definition below, which is a generalization of the procedure ii)–iv) above.

Example 4.6.2. Dolbaux cohomology groups. Sections of a line bundle is a generalization of the notion of functions, and sections of a vector bundle is a further generalization of the notion of sections of a line bundle. Thus, vector-bundle-valued cohomology groups is a generalization of the de Rham cohomology groups. Let M be a Kähler manifold with $\dim_{\mathbb{C}} M = n$ and E be a holomorphic vector bundle on M. Now, let us abuse the notation and think of E also as the sheaf of holomorphic sections of E on the topological space M with analytic topology. Then

$$0 \to E \to \mathcal{A}^0(E) \to \mathcal{A}^1(E) \to \cdots \to \mathcal{A}^n(E) \to 0 \tag{57}$$

is an exact sequence of sheaves on M; here, $\mathcal{A}^q(E)$ is the sheaf of smooth (but not necessarily holomorphic) bundle E-valued $(0,q)$-forms, and the differential $d : \mathcal{A}^q(E) \to \mathcal{A}^{q+1}(E)$ is given by the covariant anti-holomorphic derivative $\bar{\partial}_A$. The anti-holomorphic part of the connection can be gauged away, because the vector bundle E is assumed to be holomorphic (when $(dA + A \wedge A)^{(0,2)} = 0$). So, the kernel of $\bar{\partial} : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ is the sheaf of holomorphic sections of E, which is E. The procedure of computing the E-valued cohomology groups $H^q(M; E)$ is i) to think of E as the sheaf of holomorphic sections of E (rather than the vector bundle), ii) replace the sheaf E by the chain complex of sheaves $0 \to \mathcal{A}^0(E) \to \mathcal{A}^1(E) \to \cdots \to \mathcal{A}^n(E) \to 0$ that constitutes the rest of the exact sequence above, iii) replace this chain complex of sheaves by a chain complex of Abelian groups, taking the global sections of those sheaves,

$$0 \to \Gamma(M; \mathcal{A}^0(E)) \to \Gamma(M; \mathcal{A}^1(E)) \to \cdots \to \Gamma(M; \mathcal{A}^n(E)) \to 0, \tag{58}$$

and iv) compute the cohomology groups of this chain complex. For more about vector bundle valued cohomology groups, physicists can consult with the Green–Schwarz–Witten textbook, for example.

4.6.3. Here is a side remark before putting down a definition of sheaf cohomology groups by generalizing the two examples above. Suppose that X is a Kähler manifold with $\dim_{\mathbb{C}} X = n$. Then it is known that

$$H^{p,q}(X; \mathbb{C}) \cong H^q(X; \wedge^p T^* X). \tag{59}$$

The right hand side is a vector-bundle valued cohomology group (and hence a sheaf cohomology group) for $E = \wedge^p T^* X$, the p-th exterior (rank-p totally anti-symmetric tensor) product of the holomorphic cotangent bundle of X. the end of the side remark.

Now, the procedure ii)–iv) is precisely the same for the two examples above. So, this procedure is promoted to a definition.

Definition 4.6.4. Let X be a variety with $\dim_{\mathbb{C}} X = n$ and $\mathcal{F}$ a sheaf on X. Then its sheaf cohomology groups $H^i(X; \mathcal{F})$ are obtained by finding a sheaf exact sequence

$$0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^n \to 0 \tag{60}$$

with certain properties (**) to be mentioned shortly, ii) replace the sheaf $\mathcal{F}$ by the chain complex of sheaves $0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^n \to 0$ that constitutes the rest of the exact sequence above, iii) replace this chain complex of sheaves by a chain complex of Abelian groups, taking global sections of those sheaves,

$$0 \to \Gamma(X; \mathcal{E}^0) \to \Gamma(X; \mathcal{E}^1) \to \cdots \to \Gamma(X; \mathcal{E}^n) \to 0, \tag{61}$$

and iv) compute the cohomology groups of this chain complex of Abelian groups.

Without specifying the conditions (**) to be imposed, there will be many choices in the exact sequence (60) starting with the sheaf $\mathcal{F}$ of interest. If the computed dimensions of the vector spaces $H^i(X; \mathcal{F})$ depend on the choice of an exact sequence (60), then the description above is not specific enough to be able to define something. In fact, $0 \to \mathcal{F} \to \mathcal{F} \to 0$ is always an exact sequence; if this choice is allowed, then we always have $H^i(X; \mathcal{F}) = \{0\}$ for all of $i > 0$, but we know that this is not always the case in de Rham cohomology or vector-bundle valued cohomology groups. So, we need to introduce some criterion on the choices of the exact sequence (60) so that the results $\dim_{\mathbb{C}}[H^i(X; \mathcal{F})]$ do not depend on the choices within the criterion.

A criterion often adopted in textbooks is to require that all of the sheaves $\mathcal{E}^i$ are injective. In this note, we do not write down the definition of sheaves that are injective; interested readers can refer to math textbooks. An exact sequence starting with a given sheaf $\mathcal{F}$ followed by injective sheaves is called an *injective resolution of* $\mathcal{F}$; roughly speaking, an injective resolution is a dual notion of projective resolution, which we have discussed in 4.5.8.

Another criterion adopted in textbooks is to require that all of $\mathcal{E}^i$'s are flabby (flasque) sheaves. A sheaf $\mathcal{E}$ is said to be *flabby*, if $\rho_{UV} : \mathcal{E}(V) \to \mathcal{E}(U)$ is surjective for any pair of open sets $U \subset V$; any section of $\mathcal{E}$ over U can be extended to V containing U. An exact sequence starting with a given sheaf $\mathcal{F}$ followed by flabby sheaves is called a *flabby resolution of* $\mathcal{F}$.

It is known that an injective sheaf is always a flabby sheaf. It is also known that an injective resolution (and hence a flabby resolution) always exists. It is further known that the dimensions of the vector space $H^i(X; \mathcal{F})$ computed from flabby resolutions do not depend on which resolution is used, even when there are multiple flabby resolutions. So, the sheaf cohomology groups $H^i(X; \mathcal{F})$ are defined to be those obtained by requiring (**) that (60) is an injective (or a flabby) resolution.

It is known that, if X is a variety with $\dim_{\mathbb{C}} X = n$, then $H^i(X; \mathcal{F}) = \{0\}$ for $i > n$.

Although existence of injective/flabby resolutions is guaranteed, the sheaves involved in such a resolution are often not the kind of objects one can deal with concretely. So, it is often not the best strategy to try to find such a resolution explicitly to compute sheaf cohomology groups. Here is an alternative:

4.6.5. Čech cohomology: Suppose that $\cup_{i\in I} U_i$ is an open covering $\mathcal{U}$ of a topological space X. Then a chain complex for a sheaf $\mathcal{F}$

$$0 \to C^0 \to C^1 \to \cdots \to C^p \to \cdots \tag{62}$$

is obtained by setting C^p as the subset of

$$\prod_{\{i_0, i_1, \cdots, i_p\} \subset I} \mathcal{F}(\cap_{m=0}^{p} U_{i_m}) \tag{63}$$

totally anti-symmetric in the indices $\{i_0, i_1, \cdots, i_p\}$. An element $f \in C^p$ is therefore obtained by specifying $f_{i_0 i_1 \cdots i_p} \in \mathcal{F}(\cap_{m=0}^{p} U_{i_m})$ for any ordered $(p+1)$-element subset $\{i_0, i_1, \cdots, i_p\}$ of I so that $f_{i_{\sigma(0)} i_{\sigma(1)} \cdots i_{\sigma(p)}} = \mathrm{sgn}(\sigma) \cdot f_{i_0 i_1 \cdots i_p}$ for $\sigma \in \mathfrak{S}_{p+1}$. This is a generalization of the case of computing Čech cohomology with the value in an Abelian group G (such as $G = \mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$); the definition of C^p in the G-valued cases can be regarded as a special case of the definition introduced above, where the sheaf $\mathcal{F}$ is a locally constant sheaf $\underline{G}$ associated with the Abelian group G. The map $C^p \to C^{p+1}$ for a general sheaf $\mathcal{F}$ is defined just the same way as in the case of G-valued Čech cohomology group computations. Now, the cohomology groups of the chain complex above are denoted by $H^p(\mathcal{U}; \mathcal{F})$.

The Čech cohomology groups defined in this way depends on the choice of an open covering $\mathcal{U}$ of X. An extreme example is to just use one open seubset X itself to cover the whole X. Then $H^i(\mathcal{U}; \mathcal{F}) = \{0\}$ for $i > 0$, for any sheaf $\mathcal{F}$. That is not what we want. It is necessary to use an open covering $\mathcal{U}$ of X that consists of enough number of open subsets in order for the Čech cohomology groups to be the same as sheaf cohomology groups defined by using injective/flabby resolutions.

It is known that if an open covering $\mathcal{U}$ of X has enough open subsets so that $H^q(\bar{U}; \mathcal{F}) = 0$ for $1 \leq {}^\forall q$ for the intersection of any $p+1$ open subsets in the covering, $\bar{U} := \cap_{m=0}^{p} U_{i_m}$, then $H^p(X; \mathcal{F})$ can be computed by Čeck cohomology $H^p(\mathcal{U}; \mathcal{F})$. of that open covering.

4.6.6. For a sheaf $\mathcal{F}$ on a topological space X,

$$H^0(X; \mathcal{F}) \cong \Gamma(X; \mathcal{F}). \tag{64}$$

To see this, suppose that $\mathcal{U} = \{U_i \mid i \in I\}$ is an open covering of X satisfying the condition we referred to above. Then a general element f of C^0 is a set of $f_i \in \mathcal{F}(U_i)$'s specified for each $i \in I$. It is in the kernel of $d : C^0 \to C^1$ if and only if

$$\rho_{(U_i \cap U_j)U_j}(f_j) - \rho_{(U_i \cap U_j)U_i}(f_i) = 0. \tag{65}$$

This means that there must be $f_* \in \mathcal{F}(X)$ so that its restriction to U_i reproduces f_i (the property (i) in footnote 5). Moreover, the element f_* is unique (the property (ii) in footnote 5). Thus, for $f \in H^0(\mathcal{U}; \mathcal{F})$, we can asign $f_* \in \Gamma(X; \mathcal{F}) = \mathcal{F}(X)$.

5 Riemann–Roch Theorem

While sheaf cohomology groups are defined in quite an abstract way, as we have seen above, we quite often compute sheaf cohomology groups, not by being faithful to the definition, but by using powerful formulas that we explain in the following. The three major tools are i) Serre duality, ii) Hirzebruch–Riemann–Roch formula and iii) cohomology long exact sequence. Let us now take a look at them, one by one.

5.1 Serre Duality

5.1.1. Let X be a compact variety, $\dim_{\mathbb{C}} X = n$, and V be a holomorphic vector bundle on X. Then there is a natural bilinear form

$$H^k(X; V) \times H^{n-k}(X; K_X \otimes V^\times) \longrightarrow H^n(X; K_X) \cong \mathbb{C}. \tag{66}$$

Note that $H^n(X; K_X) = H^n(X; \wedge^n(T^*X))$ is the (n, n) Hodge component, and hence is isomorphic to $\mathbb{C}$. The bundle $V^\times$ is the bundle in the dual representation of V (where we take the transpose–inverse of the (matrix valued) transition function of V to obtain that of $V^\times$). Serre duality states that this bilinear form is non-degenerate. That is, when the bilinear form is presented as a coefficient matrix, the matrix has a non-zero determinant. This means that

$$H^k(X; V) \cong \left[H^{n-k}\left(X; K_X \otimes V^\times\right)\right]^*, \tag{67}$$

where the superscript $*$ means that we are taking a dual vector space. In particular, the dimensions of the two vector spaces over $\mathbb{C}$ are the same.

It is relatively easier to compute $H^0(X; \mathcal{F})$ for a sheaf $\mathcal{F}$ on X, because of their characterization as the space of all possible global sections. We just have to be faithful to this property. Serre duality also allows us to compute H^n, when $\mathcal{F}$ is a vector bundle (i.e., a locally free sheaf). When

X is a curve, then we can compute both H^0 and H^1, and that is enough. When X is a surface, we can compute both H^0 and H^2. Just one more information, hopefully linear in H^0, H^1 and H^2, would make it possible to determine H^1. So, ...

5.2 Hirzebruch–Riemann–Roch Formula

Definition 5.2.1. For a sheaf $\mathcal{F}$ on an algebraic variety X, *Euler characteristics* is

$$\chi(X;\mathcal{F}) := \sum_{k=0}^{n} (-1)^k h^k(X;\mathcal{F}). \tag{68}$$

The topological Euler number (or Euler characteristics) $\chi(X)$ should be regarded as that of the locally constant sheaf $\underline{\mathbb{R}}$, $\chi(X) = \chi(X;\underline{\mathbb{R}})$. For a Kahler manifold X, $\sum_{q=0}^{n}(-1)^q h^{p,q} = \chi(X;\wedge^p(T^*X))$. Now, here is the "one more information" we crave for.

5.2.2. Hirzebruch–Riemann–Roch formula: For a vector bundle V on X,

$$\chi(X;V) = \int_X \mathrm{ch}(V)\ \mathrm{td}(TX). \tag{69}$$

When V is a line bundle, $D = c_1(V)$, then $\mathrm{ch}(V) = e^D$. Also,

$$\mathrm{td}(TX) = 1 + \frac{1}{2}c_1(TX) + \frac{c_2(TX) + c_1(TX)^2}{12} + \frac{c_1(TX)c_2(TX)}{24} + \frac{c_2^2 + c_4 +}{72} + \cdots . \tag{70}$$

Example 5.2.3. Let $X = \mathbb{P}^2$, and $V = \mathcal{O}_{\mathbb{P}^2}(d)$. Then

$$\mathrm{td}(TX) = 1 + \frac{3}{2}H + \frac{3+3^2}{12}\mathrm{pt}, \tag{71}$$

retaining up to the $2n = 4$-form part, and dropping higher order parts. So,

$$\chi(X;\mathcal{O}(d)) = \int_X \left(1 + dH + \frac{d^2H^2}{2}\right)\left(1 + \frac{3}{2}H + H^2\right) = \frac{(d+1)(d+2)}{2}. \tag{72}$$

Now, we compute $H^0(X;\mathcal{O}(d))$ and $H^2(X;\mathcal{O}(d))$. When $d < 0$, $H^0(X;\mathcal{O}(d))$ is empty, because the divisor dH is not effective. When $-3 < d$, on the other hand, $K_X - dH \sim (-3-d)H$ is not effective, and hencee $H^2(X;\mathcal{O})$ is empty (where we used the Serre duality). Remembering that $H^0(\mathbb{P}^2;\mathcal{O}(d))$ corresponds to the set of all the homogeneous functions on $\mathbb{P}^2$ of degree d, where there can be ${}_{d+2}C_2$ monomials,

$$h^0(\mathbb{P}^2;\mathcal{O}(d)) = \begin{cases} d \geq 0, & \frac{(d+2)(d+1)}{2}, \\ d = -1,-2, & 0, \\ -3 \geq d, & 0, \end{cases} \qquad h^2(\mathbb{P}^2;\mathcal{O}(d)) = \begin{cases} d \geq 0, & 0, \\ d = -1,-2, & 0, \\ -3 \geq d, & \frac{(-1-d)(-2-d)}{2}. \end{cases} \tag{73}$$

One can further verify that the value of $h^0(\mathbb{P}^2;\mathcal{O}(d)) + h^2(\mathbb{P}^2;\mathcal{O}(d))$ is the same as $\chi(\mathbb{P}^2;\mathcal{O}(d))$ in (72), for any $d \in \mathbb{Z}$. So, we conclude that $h^1(\mathbb{P}^2;\mathcal{O}(d)) = 0$.

5.3 Cohomology Long Exact Sequence

Now, we have a fairly good tools to compute sheaf cohomology groups, when the sheaf in question is locally free (a vector bundle). If it is not, one way to go is to define the Chern character for sheaves that are not locally free, which is to invoke Grothendieck–Riemann–Roch formula. The other way to go is to use the cohomology long exact sequence. A first step is, to find a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{74}$$

over the same variety X, so a sheaf whose cohomology you hope to compute is contained in there.

5.3.1. It is known that

$$\begin{tikzcd} 0 \arrow[r] & H^0(X;\mathcal{E}) \arrow[r] & H^0(X;\mathcal{F}) \arrow[r] & H^0(X;\mathcal{G}) \arrow[dll, rounded corners] \\ & H^1(X;\mathcal{E}) \arrow[r] & H^1(X;\mathcal{F}) \arrow[r] & H^1(X;\mathcal{G}) \arrow[dll, rounded corners] \\ & H^2(X;\mathcal{E}) \arrow[r] & H^2(X;\mathcal{F}) \arrow[r] & H^2(X;\mathcal{G}) \arrow[r] & {} \end{tikzcd} \tag{75}$$

is an exact sequence. This exact sequence ends at finite terms, because $H^i(X;\mathcal{E})$, $H^i(X;\mathcal{F})$ and $H^i(X;\mathcal{G})$ all vanish for $i > n$, when $\dim_{\mathbb{C}} X = n$. So, even when $\mathcal{G}$ is not locally free, if it is possible to find an exact sequence with locally free $\mathcal{E}$ and $\mathcal{F}$, and then one can use this exact sequence to determine the dimensions of the vector spaces $H^i(X;\mathcal{G})$ from those of $H^i(X;\mathcal{E})$ and $H^i(X;\mathcal{F})$.

When the support of a sheaf $\mathcal{G}$ is a closed subvariety Y of X with $\mathrm{codim}_{\mathbb{C}} Y > 0$, therefore, computation of its sheaf cohomology often begins with finding a useful short exact sequence. In the case of $\mathcal{G} = i_{Y*}(\mathcal{O}_Y)$, the exact sequence (9) is always available. While $\mathcal{O}_X$ is locally free, $\mathcal{I}_Y$ is not necessarily locally free; this is not a problem because a projective resolution (53) is avialble for $\mathcal{I}_Y$. First, one computes sheaf cohomology groups of $\mathcal{E}_r$ and $\mathcal{E}_{r-1}$ by using $H^0(X;\mathcal{F}) = \Gamma(X;\mathcal{F})$, the Serre duality and the Riemann–Roch theorem, and then uses the long exact sequence to compute the sheaf cohomology groups of $\mathcal{E}_r \backslash \mathcal{E}_{r-1}$. Secondly, the sheaf cohomology groups of $\mathcal{E}_r \backslash \mathcal{E}_{r-1}$ and $\mathcal{E}_{r-2}$ are used in yet another long exact sequence to compute those of $\mathcal{E}_{r-1} \backslash \mathcal{E}_{r-2}$. One can go along this procedure to compute the sheaf cohomology groups of $\mathcal{E}_2 \backslash \mathcal{E}_1 = \mathcal{I}_Y$, and then $\mathcal{I}_Y \backslash \mathcal{O}_Y = i_{Y*}(\mathcal{O}_Y)$.

Example 5.3.2. Let $X = \mathbb{P}^2$ and $Y = C \in |dH|$ be a non-singular degree $d > 0$ curve in X. The sheaf cohomology groups of $i_{C*}(\mathcal{O}_C)$ can be computed by using the exact sequence

$0 \to \mathcal{O}_C(-C) \to \mathcal{O}_X \to i_{C*}(\mathcal{O}_C) \to 0$; note that $\mathcal{I}_C = \mathcal{O}_X(-C)$, because the subvariety C is a divisor of X in this case. Now,

$$h^0(X;\mathcal{O}_X) = 1, \qquad h^1(X;\mathcal{O}_X) = 0, \qquad h^2(X;\mathcal{O}_X) = 0, \qquad h^3(X;\mathcal{O}_X) = 0, \tag{76}$$

$$h^2(X;\mathcal{O}_X(-d)) = \begin{cases} \frac{(d-2)(d-1)}{2}, & d \geq 3, \\ 0, & 2 \geq d, \end{cases} \qquad h^1(X;\mathcal{O}_X(-d)) = \begin{cases} 0, & d \geq 0, \\ 0, & 2 \geq d, \end{cases} \tag{77}$$

$$h^0(X;\mathcal{O}_X(-d)) = 0, \qquad h^3(X;\mathcal{O}_X(-d)) = 0; \tag{78}$$

So, we can use the long exact sequence to find out that

$$h^0(X;i_{C*}(\mathcal{O}_C)) = 1, \qquad h^1(X;i_{C*}(\mathcal{O}_C)) = \begin{cases} \frac{(d-2)(d-1)}{2}, & d \geq 3, \\ 0, & 2 \geq d, \end{cases} \qquad h^{i\geq 2}(X;i_{C*}(\mathcal{O}_C)) = 0. \tag{79}$$

As a little more general class of sheaves $\mathcal{G}$ supported on a closed subvariety Y with $\mathrm{codim}_{\mathbb{C}} Y > 0$, one can think of sheaves of the form $\mathcal{G} = i_{Y*}(\mathcal{L})$, $\mathcal{L} \in \mathrm{Pic}(Y)$. For $\mathcal{L}$ that is obtained in the form of $\mathcal{L} = \mathcal{O}_Y(D|_Y)$ for some $\mathcal{O}_X(D) \in \mathrm{Pic}(X)$, it is useful to remember that

Theorem 5.3.3. $0 \to \mathcal{E} \otimes \mathcal{O}_X(D) \to \mathcal{F} \otimes \mathcal{O}_X(D) \to \mathcal{G} \otimes \mathcal{O}_X(D) \to 0$ is exact, whenever $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is exact. Moreover, the same is true for any exact sequence (not just a short exact sequence).

6 Curves and Surfaces

6.1 Curves

6.1.1. a few definitions: For a curve Σ, its *genus* is $g := h^{0,1} = h^{1,0}$. It follows that $2 - 2g = \chi_{\mathrm{top}}(\Sigma) = \int_\Sigma c_1(T\Sigma) = -\int_\Sigma K_\Sigma$.

For a divisor $D = \sum_i n_i \mathrm{pt}_i$ on a curve Σ, its *degree* $\deg(D) := \sum_i n_i$ counts the number of points in D, including the multiplicity. The space of degree-0 divisors, $\mathrm{Pic}^0(\Sigma)$, in

$$0 \longrightarrow \mathrm{Pic}^0(\Sigma) \longrightarrow \mathrm{Pic}(\Sigma) \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{80}$$

is called the *Jacobi variety of* Σ.

6.1.2. The Riemann–Roch formula on a curve Σ is

$$h^0(\Sigma;\mathcal{O}(D)) - h^1(\Sigma;\mathcal{O}(D)) = \int_\Sigma (1+D)\,(1 + c_1(T\Sigma)/2) = \deg(D) + (1-g). \tag{81}$$

Popular examples of this theorem includes

- The $D=0$ case: $\chi(\Sigma;\mathcal{O}_\Sigma)=0+1-g$ (should be equal to $h^{0,0}-h^{0,1}=1-g$).
- The $D=K_\Sigma$ case: $\chi(\Sigma;K_\Sigma)=\deg(K_\Sigma)+(1-g)=g-1$ (should be equal to $h^{1,0}-h^{1,1}=g-1$).
- $h^0(\Sigma;T\Sigma)$ is the number of globally defined holomorphic tangent vectors, and $h^1(\Sigma;T\Sigma)$ that of complex structure deformations. String theorists learn in textbooks that

$$\begin{aligned}&\#(\mathrm{b-ghosts})-\#(\mathrm{c-ghosts})\\ &=-\chi(\Sigma,T\Sigma)=\chi(\Sigma;K_\Sigma\otimes T^*\Sigma)=(1-g)+\deg(2K_\Sigma)=3g-3.\end{aligned} \tag{82}$$

6.1.3. ramification: Consider a regular map between two curves, $\phi:C_c\to C_b$, where $\phi(C_c)$ is not just a point in C_b. When $\phi^{-1}(p)$ consisits of d points for generic points $p\in C_b$, then the map is called *degree d map*. At isolated points in C_c and C_b, however, the map ϕ may behave not as local isomorphism, but as $y\mapsto x=y^n$ for some $n\geq 2$. This behavior is called *ramification*. The *ramification divisor* R on C_c is defined by $R=\sum_i(n_i-1)\mathrm{pt}_i$. There is a relation $K_c=\phi^*(K_b)+R$, and by counting their degree,[14]

$$(2g_c-2)=d(2g(C_b)-2)+\deg(R). \tag{83}$$

Exercise 6.1. Think of $\phi:C_c\to C_b$ where C_c is the compactification of a hyperelliptic curve $\{(x,y)\in\mathbb{C}^2\mid y^2=P^{(2g+2)}(x)\}$ and $C_b\simeq\mathbb{P}^1$ that of $\{x\in\mathbb{C}\}$. Confirm that the relation above holds for this example.

Exercise 6.2. Prove that there is no surjective (non-constant) map from $\mathbb{P}^1$ to a curve Σ of genus $g\geq 1$; note that $R\geq 0$.

6.1.4. resolution of double point singularity Note that, when A is blown up at one non-singular point to become $\tilde{A}$, $K(\tilde{A})=\nu^*(K(A))+(d-1)E$, where E is the exceptional divisor, and d the dimension of A. Now, let $X=D$ be a hypersurface of A, and is singular at the center point of the blow-up, where the defining equation of X is of degree m, at least. Then the proper transform $\overline{D}$ of D corresponds to $\nu^*(D)-mE$. Using the adjunction formula,

$$K(\overline{D})=K(\tilde{A})+\overline{D}=(K(A)+D)+(d-1-m)E=K(D)+(d-1-m)E|_{\overline{D}}. \tag{84}$$

For example, when a curvee Σ forms a double point singularity, and becomes $\overline{\Sigma}$ after the double point singularity is resolved, then $m=2$ and $d=2$. So, $K(\overline{\Sigma})=K(\Sigma)-2\mathrm{pt}s$, $g(\tilde{\Sigma})=g(\Sigma)-1$. [example, elliptic curve becomes $\mathbb{P}^1$]

[14] From the exact sequence $0\to\phi^*(T^*C_b)\to T^*C_c\to\oplus_i\mathbb{C}_{\mathrm{pt}_i}^{\oplus(n_i-1)}$, we have $\chi(C_c;T^*T_c)=\chi(C_c;\phi^*(T^*C_b))+\chi(C_c;\oplus_i\mathbb{C}_{\mathrm{pt}_i}^{\oplus(n_i-1)}$. The equation in the main text also follows from this.

6.2 Curves in a Surface

6.2.1. When a curve C is a divisor in a surface S, the adjunction formula

$$K_C = (K_S + C)|_C \tag{85}$$

determines K_C of the curve C. It then follows that

$$2g(C) - 2 = \deg(K_C) = C \cdot (K_S + C). \tag{86}$$

Examples: If $S = \mathbb{P}^2$, and $C \in |dH|$, then $2g - 2 = dH \cdot (d-3)H = d(d-3)$. $g = (d-1)(d-2)/2$. Hyperplanes and conics: $g = 0$, cubics: $g = 1$, quartic: $g = 3$. (cf Example 5.3.2: $h^0(S; i_{C*}(\mathcal{O}_C))) = 1$ and $h^1(S; i_{C_*}(\mathcal{O}_C)) = g$ are the same as $h^0(C; \mathcal{O}_C) = 1$ and $h^1(C; \mathcal{O}_C) = g$ here.)

If S is a K3 surface, where $K_S = 0$, $2g(C) - 2 = C \cdot C$. So, $C^2 \in 2\mathbb{Z}$. The lattice of algebraic curves $S_X \subset H^2(X; \mathbb{Z})$ is therefore an even lattice. A curve C with a genus g comes in a g-dimensional family (g-dimensional space of deformation) in a K3 surface S, because

$$\begin{aligned} h^0(S; \mathcal{O}_S(C)) \; &\geq \chi(S; \mathcal{O}_S(C)) = \left[\frac{c_2(TS) + c_1(TS)^2}{12} = 2 \right] \times 1 + c_1(TS) \cdot C + 1 \times \frac{C^2}{2} && (87) \\ &= 2 + 0 + \frac{2g(C) - 2}{2} = g(C) + 1; && (88) \end{aligned}$$

the dimension of $|C|$ is $h^0(S; \mathcal{O}_S(C)) - 1 \geq g(C)$.

6.3 Surfaces

del Pezzo surfaces $S = dP_k$ ($k \leq 8$) are obtained by blowing up $\mathbb{P}^2$ at arbitrary k points in $\mathbb{P}^2$ successively. Thus, $K_S = -3H + \sum_{i=1}^{k} E_i$, where E_i is the exceptional divisor of the i-th blow-up. So, for the range $0 \leq k \leq 8$, $c_1(TS)^2 = 9 + k(-1) = 9 - k$ remains positive. It is known that $h^{1,1}(S) = 1 + k$, and $h^{2,0}(S) = h^{1,0}(S) = 0$, so $\chi_{\text{top}}(S) = 3 + k$. Consistency check: $\chi(S; \mathcal{O}_S) = [c_2 + c_1^2]/12 = [(3 + k) + (9 - k)]/12 = 1 = h^{0,0}$.

Examples: $S = (d) \subset \mathbb{P}^3$: the adjunction formula is used:

$$\begin{aligned} c(S) = \frac{(1+H)^4}{1+dH} &= (1 + 4H + 6H^2 + 4H^3 + \cdots)(1 - dH + d^2H^2 - d^3H^3 + \cdots) \\ &= 1 + (4-d)H + (6 - 4d + d^2)H^2 + (4 - 6d + 4d^2 - d^3)H^3. \end{aligned} \tag{89}$$

So, $c_1(TS) = (4-d)H|_S$ and $c_2(TS) = (6 - 4d + d^2)H^2|_S$. Thus, in particular,

$$\chi_{\text{top}}(S) = \int_S c_2(TS) = \big(dH \cdot (6 - 4d + d^2)H^2\big)_{\mathbb{P}^3} = d(6 - 4d + d^2). \tag{90}$$

Interpretation: $S = \mathbb{P}^1$ if $d = 1$ (obvious), $S = \mathbb{P}^1 \times \mathbb{P}^1$ if $d = 2$, $S = dP_6$ if $d = 3$, $S = \mathrm{K3}$ if $d = 4$, etc. At least $\chi_{\rm top} = 3, 4, 9, 24$ for $d = 1, 2, 3, 4$ agrees with those interpretations. $\chi_{\rm top}(S) = 55$ for $d = 5$.

$$\chi(S; \mathcal{O}_S) = \frac{[c_2 + c_1^2]}{12} = \frac{dH}{12}\left((6 - 4d + d^2)H^2 + ((4 - d)H)^2\right) = \frac{d(2d^2 - 12d + 22)}{12}, \tag{91}$$

which remains $1, 1, 1$ for $d = 1, 2, 3$ ($\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, dP_6) and $2, 5$ for $d = 4, 5$. The degree-5 hypersurface should have $h^{2,0}(S) = 4$.

7 Introduction to Toric Geometry

Toric variety X is a complex n dimensional variety that has an automoprhism group $\mathrm{Aut}(X)$ containing $(\mathbb{C}^\times)^n$. The word toric came from this algebraic torus of the symmetry group; see XXXXXX for more.

Toric varieties are therefore nothing more than a class of algebraic varieties that happen to have a very special property. Geometry of a toric variety is, however, known to be described by using certain combinatorial data (due to the large symemtry group $\mathrm{Aut}(X)$), and it is even possible to construct a variety of this class by dealing with such combinatorial data. So, it is very easy to handle varieties of this class. For this usefulness (as well as for the fact that toric varieties are generalizations of projective spaces),[15] toric varieties have been used in many situations.

Not a small fraction of textbooks and review articles on toric varieties start declaring right from the beginning that this xxx is the combinatorial data to use, whitout explaining why one has come to think of using such combinatorial data. So, this section of this note explains the idea that led to the use of those combinatorial data. We do not intend to provide full account of all the techniques on toric geometry here, but the contents of this section will get a reader prepared to the extent that he/she is not going to be puzzled or bewildered when she/he reads papers that involve toric data.

7.1 Introduction to the combinatorical data

7.1.1. First, for $X = \mathbb{P}^2$, consider expressing complete linear system $|dH|$ of a divisor $[dH] \in \mathrm{Pic}(X)$ by using Newton polygon; we know that $\Gamma(X; \mathcal{O}_X(dH))$ consists of homogeneous functions

[15] $\mathbb{P}^2$ is an example of toric varieties because it has $\mathbb{C}^\times \times \mathbb{C}^\times$ as a part of its automorphism group. Let $[X_0 : X_1 : X_2]$ be the homogeneous coordinates of $X = \mathbb{P}^2$. Then for $(\lambda, \mu) \in \mathbb{C}^\times \times \mathbb{C}^\times$,

$$\phi_{(\lambda,\mu)} : X \ni [X_0 : X_1 : X_2] \longmapsto [X_0 : X_1 : X_2]' = [X_0 : \lambda X_1 : \mu X_2] \in X. \tag{92}$$

of degree d; if we use a set of inhomogeneous coordinates in an Affine patch $U_0 = \{(x_1, x_2) \in \mathbb{C}^2\}$, then possible monomials forms a pyramid (triangle) of lattice points of height d. We say that

$$\begin{aligned}|dH| =&\mathbb{P}\left[\mathrm{Span}_{\mathbb{C}}\left\{x_1^{m_1}x_2^{m_2} \mid m \in \Delta\right\} \backslash \{0\}\right] \Longleftrightarrow \\ &\{(m_1, m_2) \in M \mid 0 \leq m_{1,2},\ m_1 + m_2 \leq d\ \} =: \Delta \subset M := \mathbb{Z}^{\oplus 2}.\end{aligned} \tag{93}$$

This is the first example of an idea of expressing complete linear systems by its Newton polygon in a lattice $M = \mathbb{Z}^{\oplus n}$ for an n-dimensional toric variety X.

7.1.2. On the vector space of rational functions $\mathbb{C}(X)$ of a toric variety X, the group $(\mathbb{C}^\times)^n \subset \mathrm{Aut}(X)$ acts by pull-back. In the case of $X = \mathbb{P}^2$, eigenstates of the $(\mathbb{C}^\times)^{n=2}$ action are the monomials $\{x_1^{m_1}x_2^{m_2} = (X_1/X_0)^{m_1}(X_2/X_0)^{m_2} = (X_0/X_1)^{-m_1-m_2}(X_2/X_1)^{m_2} \mid (m_1, m_2) \in M\}$. The origin ($0 \in M$) of the lattice $M \cong \mathbb{Z}^{\oplus n=2}$ corresponds to the rational function 1 on $X = \mathbb{P}^2$. Two generators $\hat{e}_1$ and $\hat{e}_2$ may be chosen so that they correspond to the rational functions $x_1 = X_1/X_0$ and $x_2 = X_2/X_0$. Certainly those (x_1, x_2) are the regular coordinates of an Affine chart $U_0 \subset \mathbb{P}^2$; for other Affine charts, the regular coordinates are $(X_0/X_1, X_2/X_1) = (x_1^{-1}, x_2/x_1)$ in U_1 and $(X_0/X_2, X_1/X_2) = (x_2^{-1}, x_1/x_2)$ in U_2. Those regular functions in other Affine charts correspond to $-\hat{e}_1$, $\hat{e}_2 - \hat{e}_1$, $-\hat{e}_2$ and $\hat{e}_1 - \hat{e}_2$, respectively. There are $\mathrm{SL}(n = 2; \mathbb{Z})$ basis transformations on the lattice $M = \mathbb{Z}^{\oplus n=2}$ mapping $(\hat{e}_1, \hat{e}_2)$ for the chart U_0 to $(-\hat{e}_1 + \hat{e}_2, -\hat{e}_1)$ for the chart U_1, or to $(-\hat{e}_2, \hat{e}_1 - \hat{e}_2)$ for the chart U_2. So, we are not doing injustice to any one of the Affine charts.

7.1.3. When we converted the complete system $|dH|$ into a Newton polygon on $M \cong \mathbb{Z}^{\oplus n}$, we implicitly replaced a homogeneous function of degree d with a degree-d polynomial in the inhomogeneous coordinates (x_1, x_2) by dividing by $(X_0)^d$. We could have also said that we chose one divisor $D = dD_{X_0=0}$ and expressed the vector space $\Gamma(X; \mathcal{O}_X(D))$ as the set of rational functions of $X = \mathbb{P}^2$. We could have done the conversion by using any degree-d monomial of the homogeneous functions, like $(X_1)^d$, $X_0^{d-1}X_1$ etc. ($D = dD_{X_1=0}$, $D = (d-1)D_{X_0=0} + D_{X_1=0}$, etc.). Depending on how we did it, the Newton polygon shifts its location in $M \cong \mathbb{Z}^{\oplus n=2}$. So, only the shape of the Newton polygon carries the information of $|dH|$; the position of the polygon in M does not.

7.1.4. The Newton polygon of $|dD_{X_0=0}|$ in $M \cong \mathbb{Z}^{\oplus n=2}$ for $X = \mathbb{P}^2$ can be regarded as the intersection of three displaced cones; the three cones are

$$\hat{\sigma}_0 := \mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{\hat{e}_1, \hat{e}_2\}, \tag{94}$$

$$\hat{\sigma}_1 := \mathrm{Span}_{\mathbb{Z}\ geq0}\{-\hat{e}_1 + \hat{e}_2, -\hat{e}_1\}, \tag{95}$$

$$\hat{\sigma}_2 := \mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{-\hat{e}_2, \hat{e}_1 - \hat{e}_2\}. \tag{96}$$

and when they are displaced so that the vertices of the cones $\hat{\sigma}_0$, $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are displaced to $m_0^{(d)} := (0,0) \in M$, $m_1^{(d)} := (d,0) \in M$ and $m_2^{(d)} := (0,d) \in M$, respectively, then the intersection of the three displaced cones reproduces the Newton polygon for $|dH|$.

The three cones are in one-to-one correspondence with the Affine charts of $X = \mathbb{P}^2$. The coordinate ring of the Affine charts are

$$\mathbb{C}[U_0] = \mathbb{C}[x_1, x_2] \;= \mathbb{C}[\{x_m \mid m \in \hat{\sigma}_0\}]/(x_m x_{m'} - x_{m+m'}), \tag{97}$$

$$\mathbb{C}[U_1] = \mathbb{C}[x_2/x_1, 1/x_1] \;= \mathbb{C}[\{x_m \mid m \in \hat{\sigma}_1\}]/(x_m x_{m'} - x_{m+m'}), \tag{98}$$

$$\mathbb{C}[U_2] = \mathbb{C}[1/x_2, x_1/x_2] \;= \mathbb{C}[\{x_m \mid m \in \hat{\sigma}_2\}]/(x_m x_{m'} - x_{m+m'}). \tag{99}$$

For more general n-dimensional toric varieties X, Affine charts of X—$\{U_i\}$—are in one to one correspondence with n-dimensional cones in $M \cong \mathbb{Z}^{\oplus n}$ denoted by $\hat{\sigma}_i$. The coordinate ring on U_i is

$$\mathbb{C}[U_i] = \mathbb{C}\left[\{x_m \mid m \in \hat{\sigma}_i\}\right]/(x_m x_{m'} - x_{m+m'}). \tag{100}$$

Those funcions regular in a particular Affine patch are regarded as rational functions on the entire variety X. So, we can find algebraic relations among such functions that are regular in separate Affine charts, by comparing them as rational functions on X (as lattice points in M). So, the relative arrangement of the cones $\hat{\sigma}_i$ indicates how Affine patches are glued together, by the algebraic relations among regular/rational function encoded in the lattice of rational functions M.

7.1.5. There are some rules for the cones $\hat{\sigma}_i \subset M \subset \mathbb{Z}^{\oplus n}$ of a toric variety X to satisfy. It is not obvious how to find out such a rule by just looking at one example $X = \mathbb{P}^2$. So, here we state a known result. First, consider a lattice $N \cong \mathbb{Z}^{\oplus n}$ dual to M, and then define the dual cone σ_i for each one of the cones $\hat{\sigma}_i$.

$$\sigma := \left\{ n \in N \cong \mathbb{Z}^{\oplus n} \mid \langle n, m \rangle \geq 0 \text{ for } {}^\forall m \in \hat{\sigma} \right\}. \tag{101}$$

When X is a toric variety, the cones σ_i in N do not overlap with each other (apart from the faces of the cones). The variety X is compact, if and only if the cones $\{\sigma_i \otimes \mathbb{R}\}$ completely cover the entire $N \otimes \mathbb{R}$.

In the example $X = \mathbb{P}^2$, the dual cones are

$$\sigma_0 \;= \mathrm{Span}_{\mathbb{Z}_{\geq 0}} \{e_1, e_2\}, \tag{102}$$

$$\sigma_1 \;= \mathrm{Span}_{\mathbb{Z}_{\geq 0}} \{e_2, -(e_1 + e_2)\}, \tag{103}$$

$$\sigma_2 \;= \mathrm{Span}_{\mathbb{Z}_{\geq 0}} \{-(e_1 + e_2), e_1\}. \tag{104}$$

So, the conditions above are satisfied indeed. Here, e_i's are the lattice elements in N, and are the dual basis of $\hat{e}_i$ in M.

7.2 Examples

1-dimensional toric varieties
$X = \mathbb{P}^1$: introduce two cones in $N \cong \mathbb{Z}$. One is $\sigma_0 = \mathrm{Span}_{\mathbb{Z}_{\geq 0}} \{e_1\}$ and the other $\sigma_1 = \mathrm{Span}_{\mathbb{Z}_{\geq 0}} \{-e_1\}$. For $X = \mathbb{C}$, just retain one cone σ_0 in N.

2-dimensional toric varieties
We introduce a notation

$$\langle v_1, v_2, \cdots, \rangle := \mathrm{Span}_{\mathbb{Z}_{\geq 0}} \{v_1, v_2, \cdots, \}. \tag{105}$$

Now, $X = \mathbb{P}^2$ corresponds to introducing three cones in $N \cong \mathbb{Z}^{\oplus 2}$,

$$\sigma_0 = \langle e_1, e_2 \rangle, \qquad \sigma_1 = \langle e_2, -(e_1 + e_2) \rangle, \quad \text{and} \quad \sigma_2 = \langle -(e_1 + e_2), e_1 \rangle \tag{106}$$

in the new notation. As for $X = \mathbb{C}^2$, just retain σ_0 and get rid of two other cones $\sigma_{1,2}$.

Here is a family of surfaces, called Hirzebruch surfaces, which are labeled by an integer k. They are denoted by F_k. For $X = F_k$, we introduce four cones in $N \cong \mathbb{Z}^{\oplus 2}$. $\sigma_{++} := \langle e_1, e_2 \rangle$, $\sigma_{-+} = \langle e_2, -e_1 + ke_2 \rangle$, $\sigma_{--} := \langle -e_1 + ke_2, -e_2 \rangle$ and $\sigma_{+-} = \langle -e_2, e_1 \rangle$. The Hirzebruch surfaces F_k are all compact. Certainly the four cones completely cover $N \otimes \mathbb{R}$.

If we just retain the cones σ_{++} and σ_{-+}, then the corresponding toric variety X is known to be the total space of $\mathcal{O}_{\mathbb{P}^1}(k)$ (or $-k$?).

***n*-dimensional toric varieties**
$X = \mathbb{P}^n$: we introduce $n+1$ cones σ_0, $\sigma_{i=1,\cdots,n}$ in N, corresponding to $n+1$ Affine patches of $X = \mathbb{P}^n$. First, $\sigma_0 = \langle e_1, \cdots, e_n \rangle$. The n other cones are $\sigma_i = \langle e_0, \cdots, e_n(\text{w/o } e_i) \rangle$ for $i = 1, \cdots, n$, where $e_0 := -(e_1 + \cdots + e_n)$.

7.3 Divisors, Algebraic cycles, Linear Systems

It is much more convenient to use the cones σ's in N than $\hat{\sigma}$'s in M in stating all the conditions on the arrangments of the cones (on gluing of the Affine patches) and the condition for the compactness of toric varieties. There more benefits than that in using the data in the lattice N, in fact.

7.3.1. Let us use $X = \mathbb{P}^n$ as an example. The complete linear system $|dH|$ corresponds to a Newton polygon in the lattice M that is an n-dimensional pyramid with height d. We can take one of the $(n+1)$ vertices of the Newton polygon at $\vec{0} \in M$ (by dividing the homogeneous degree-d functions on $\mathbb{P}^n$ by $(X_0)^d$), when the n-other vertices are at $(d, 0, \cdots,)$, $(0, d, \cdots)$, $\cdots$ and $(0, \cdots, 0, d)$ in M. Those vertices correspond to the monomials $(X_0)^d$, $(X_1)^d$, $\cdots$ and $(X_n)^d$, respectively. Those vertices $m_i^{(d)}$ of the Newton polygon are the vertices of the diplaced cones $\hat{\sigma}$'s in M, and hence to the cones σ's in N. Noting that $(X_i)^d$ is the only monomial that remains non-vanishing at the point $X_i = 1$, $X_0 = X_1 = \cdots = X_n = 0$ in $\mathbb{P}^n$, we see that the n-dimensional cone σ_i corresponds to this special point (algebraic 0-dimensional cycle) in $\mathbb{P}^n$.

Next, think of one of the highest-dimensional faces (called *facets*) of the Newton polygon. The lattice points in a facet that is anti-podal to the vertex $m_i^{(d)}$ correspond to all the monomials where X_i is not contained. So, they are all the monomials that remain non-vanishing when the homogeneous functions of degree d are restricted to the divisor $X_i = 0$. The dual of this facet is a 1-dimensional ray along e_i, which is shared by n n-diensional cones, $\sigma_{0,\cdots,n}$ except σ_i. So, we are led to an idea that 1-dimensional rays in the data of N correspond to divisors in X.

So, this observation motivates us to regard the data in the lattice $N \cong \mathbb{Z}^{\oplus n}$ for a toric n-dimensional variety X, not as a collection of n-dimensional cones alone, but as a collection of n-dimensional cones, and other cones with lower dimensions that appear as intersection of those cones. The collection of those cones is called a *fan*. The toric fan for $X = \mathbb{P}^2$ is, for example,

$$\Sigma = \left\{ \langle e_1, e_2 \rangle, \langle e_2, e_0 \rangle, \langle e_0, e_1 \rangle, \langle e_1 \rangle, \langle e_2 \rangle, \langle e_0 \rangle, \vec{0} \right\}. \tag{107}$$

A subset $\Sigma(m)$ of a fan Σ is the collection of m-dimensional conles in Σ.

Here is a general story. For a toric variety X, elements of $\Sigma(1)$ have corresponding divisors in X; for $k = 1, \cdots, n$, those in $\Sigma(k)$ have corresponding codimension-k algebraic cycles in X. After removing all those subvarieties from X, there remains an algebraic torus $(\mathbb{C}^\times)^n$. So, an n-dimensional cone $\sigma \in \Sigma(n)$ should be regarded as a point; the collection of all the cones in a fan Σ that is geometrically contained in σ should be regarded as an Affine chart corresponding to $\hat{\sigma}$.

7.3.2. A complete linear system of a divisor D on a toric variety X is given by $(\hat{\sigma}_i, m_i^{(D)})$ where i runs over the Affine charts (cones in $\Sigma(n)$). So, a divisor class $[D]$ is specified by specifying a rational function $m_i^{(D)} \in M$ for each cone $\sigma_i \in \Sigma(n)$.

$$\Gamma(X; \mathcal{O}_X(D)) = \mathrm{Span}_{\mathbb{C}} \{ x_m \mid D + \mathrm{div}(x_m) \geq 0 \}, \tag{108}$$

$$= \mathrm{Span}_{\mathbb{C}} \left\{ x_m \mid {}^\forall i,\ \mathrm{div}(x_m) + f_i \geq 0 \right\}, \tag{109}$$

$$= \mathrm{Span}_{\mathbb{C}} \left\{ m \in M \mid {}^\forall i,\ m - m_i^{(D)} \in \hat{\sigma}_i \right\} \tag{110}$$

where, in the second line, we used the description of a Cartier divisor (U_i, f_i) for a divisor D. So, a Cartier divisor on X is given by $\{m_i \in M \mid i = 1, \cdots, |\Sigma(n)|\}$, whose interpretation is $(U_{\hat{\sigma}_i}, x_{-m_i})$.

Two Cartier divisors $\{m_i | i = 1, \cdots, |\Sigma(n)|\}$ and $\{m'_i \mid i = 1, \cdots, |\Sigma(n)|\}$ are linearly equivalent if and only if there is a rational function x_{m_0} with $m_0 \in M$, so that $m_i = m'_i + m_0$ for all $i = 1, \cdots, |\Sigma(n)|$ simultaneously. The Newton polygon in M of one of such divisors is obtained from that of the other by shifting by m_0.

When a Cartier divisor $(\hat{\sigma}_i, m_i)$ is given, its corresponding description as a Weil divisor is

$$D = \sum_{\tau_a \in \Sigma(1)} \langle v_a, -m_i \rangle D_a, \tag{111}$$

where D_a is the divisor of X corresponding to $\tau_a \in \Sigma(1)$, v_a the primitive vector (the first lattice point from $\vec{0} \in N$) along the 1-dimensional ray τ_a, and we should use $-m_i$ for an n-dimensional cone σ_i containing τ_a on its face; although there can be multiple n-dimensional cones σ_i that contain a given 1-dimensional cone τ_a, the value $\langle v_a, -m_i \rangle \in \mathbb{Z}$ remains the same, whichever $-m_i$ of that sort we use.

A a note on elliptic curves and elliptic functions

a text brook: section 1 of daen-kansuu-ron by Adachi and Komatsu, Springer–Verlag Tokyo ('91). This book is a Japanese translation of the Chap. 2 of "*Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen*" by Hurwitz and Courant.

A.1 Elliptic curve and the $\wp$-function

There are two ways to construct an elliptic curve E. One is to take a quotient of $\mathbb{C} = \{u \in \mathbb{C}\}$ by a rank-2 lattice $\Lambda = \mathbb{Z}\langle \omega_1 \rangle \oplus \mathbb{Z}\langle \omega_2 \rangle \subset \mathbb{C}$ generated by $u = \omega_1$ and $u = \omega_2$, with $\omega_2/\omega_1 =: \tau \in \mathbb{C}$; we assume that $\mathrm{Im}(\tau) > 0$.

$$E = E_\tau = \mathbb{C}/\Lambda. \tag{112}$$

This is called an alaytic representation of an elliptic curve with the complex structure τ; one can set $\omega_1 = 1$ because a pair of elliptic curves with the same τ, but with different ω_1 are mutually isomorphic (via $u \mapsto u \times \omega_1/\omega_1'$). The other is to consider a one-point compactification $\overline{E'}$ of an Affine variety

$$E' = \left\{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3\right\} \tag{113}$$

for some g_2, $g_3 \in \mathbb{C}$.

The relation between those two constructions is simply stated (in the following) by using Weierstrass' $\wp$ function. The $\wp$ function is a complex valued function from the complex u-plane $\mathbb{C}$ given by

$$\wp(u; \tau, \omega_1) = \frac{1}{u^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(u - \omega)^2} - \frac{1}{\omega^2} \right) = \frac{1}{\omega_1^2} \wp(u/\omega_1; \tau, 1); \tag{114}$$

it is periodic under translation $u \to u + \omega$ $(\omega \in \Lambda)$, and holomorphic everywhere in the u-plane except $u \in \Lambda \subset \mathbb{C}$; it has a pole of order two at $u \simeq \omega$ for each one of $\omega \in \Lambda$.

Definition A.1.1. An *elliptic function* is a rational function on the u-plane that is periodic under translations by Λ. (so, it can be regarded as a function on $\mathbb{C}/\Lambda$)

A.1.2. For a given rank-2 lattice $\Lambda \subset \mathbb{C}$, elliptic functions form a field. Moreover, it is known that an elliptic function φ can be expressed in the form of

$$\varphi(u) = \frac{f_1(\wp(u))}{f_2(\wp(u))} + \left(\frac{d\wp}{du}\right) \frac{f_3(\wp(u))}{f_4(\wp(u))}, \tag{115}$$

where $f_{1,2,3,4}$ are polynomials appropriately chosen for a given φ. When $\varphi(u)$ is even under $u \to -u$, just the first term is enough; if $\varphi(u)$ is odd, then just the second term is enough. •

A.1.3. Here, we summarize various properties of an elliptic function without a proof. Suppose that $\varphi(u)$ is an elliptic function with its periodicity $\Lambda \subset \mathbb{C}$.

(a) Let $\{p_i + \Lambda \mid i \in I_0\}$ be the list of zero's of $\varphi(u)$, and $\{q_j + \Lambda \mid j \in I_\infty\}$ that of poles of $\varphi(u)$; p_i [resp. q_j] should be counted m-times if $\varphi(u)$ has a zero [resp. pole] of order m. Then $|I_0| = |I_\infty|$, first of all, and

$$\sum_i p_i \equiv \sum_j q_j \qquad \mathrm{mod}\ \Lambda, \tag{116}$$

secondly.

(b) Let $\{a_j \in \mathbb{C} \mid j \in I_\infty\}$ be the list of the residues at the pole $u = q_i + \Lambda$. Then $\sum_{j \in I_\infty} a_j = 0 \in \mathbb{C}$.

(c) $|I_0| = |I_\infty|$ is either zero, or $2, 3, \cdots$, but it cannot be 1. (if $|I_\infty| = 1$, then $\sum_{j \in I_\infty} a_j$ would be zero, so there is no pole to begin with)

(d) for any choice of $\{p_{i=1,\cdots,n}\} \in E$ and $\{q_{j=1,\cdots,n} \in E\}$ that satisfy the condition (116), there exists an elliptic function whose zero's and poles are the chosen $\{p_i\}$ and $\{q_j\}$, respectively. Such an elliptic function is unique up to multiplication by a non-zero complex number.

A.1.4. Since $(d\wp/du)^2 = (\wp')^2$ is an elliptic function that is even under $u \to -u$, it should be expressed in terms of a rational function of $\wp$. It is known that

$$(\wp')^2 = 4(\wp)^3 - g_2\wp - g_3, \tag{117}$$

for some appropriately chosen value[16] of $g_2, g_3 \in \mathbb{C}$. So, the elliptic functions $\wp$ and $\wp'$ provides a map

$$(\wp, \wp') \; : \; E = \mathbb{C}/\Lambda \ni (u+\Lambda) \longmapsto (\wp(u), \wp'(u)) \in \left\{(x,y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3\right\} = E'. \tag{120}$$

The origin $u = 0 + \Lambda$ in E is mapped to $(x,y) = (\infty, \infty)$, so this is a map from the compact space E to the one-point-compactification $\overline{E'}$ of E'.

The inverse map is obtained by noting that

$$u = \int_0^u du' = \int \frac{d\wp(u)}{[d\wp/du]} = \int_{(\infty,\infty)}^{(x,y)} \frac{dx}{y} = \int_{(\infty,\infty)}^{(x,y)} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}. \tag{121}$$

The periods 1 and τ are obtained by integrating dx/y over topological cycles in the elliptic curve $\overline{E'}$.

A.2 Elliptic curve in algebraic geometry

In algebraic geometry, we regard an elliptic curve $E = \overline{E'}$ as a subvariety of $\mathbb{P}^2$ given by

$$ZY^2 = 4X^3 - g_2 Z^2 X - g_3 Z^3, \tag{122}$$

where $[X:Y:Z]$ are the homogeneous coordinates of $\mathbb{P}^2$. $E \sim 3H$. The point $[X:Y:Z] = [0:1:0] \in E \subset \mathbb{P}^2$ is a closed subvariety of E given by $\{Z=0\}|_E$; its complement—Zariski open, by definition—is the Affine variety $E' \subset \mathbb{C}^2 = \{(x,y) = (X/Z, Y/Z)\}$. The ring of regular functions in this Zariski open subset is $\mathcal{O}_E(E') = \mathbb{C}[x,y]/(4x^3 - g_2 x - g_3 - y^2)$.

$x = X/Z$ and $y = Y/Z$ are both rational functions on $\mathbb{P}^2$, and upon restriction to E, both are also regarded as rational functions on the elliptic curve E. Since the divisors $\{X=0\}$ and $\{Z=0\}$ of $\mathbb{P}^2$ intersect the divisor E at $[X:Y:Z] = [0:1:0] \in \mathbb{P}^2$ with multiplicity $+1$ and $+3$, respectively, $\mathrm{div}(x)$ and $\mathrm{div}(y)$ have a pole of order 2 and 3 at that point, respectiveley. This conclusion agrees with the fact in the analytic representation $\mathbb{C}/\Lambda$ of E that $\wp(u)$ and $\wp'(u)$ have a pole of order 2 and 3 at $u=0$, respectively.

The field of rational functions of the algebraic variety E, $\mathbb{C}(E)$, is the field of elliptic functions on $E = \mathbb{C}/\Lambda$.

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[16] Using weight-4 and weight-6 modular forms ($q := e^{2\pi i \tau}$),

$$g_2 = 60 \sum_{\omega \in \Lambda \backslash 0} \frac{1}{\omega^4} = \frac{120\zeta(4)}{\omega_1^4} E_4(\tau), \qquad E_4(\tau) = 1 + 240(q + 9q^2 + \cdots). \tag{118}$$

$$g_3 = 140 \sum_{\omega \in \Lambda \backslash 0} \frac{1}{\omega^6} = \frac{280\zeta(6)}{\omega_1^6} E_6(\tau), \qquad E_6(\tau) = 1 - 504(q + 33q^2 + \cdots). \tag{119}$$

A.2.1. The property (a) implies that

$$\deg\left(\mathrm{div}(\varphi)\right) = 0, \qquad \varphi_{\neq 0} \in \mathbb{C}(E) \tag{123}$$

The property (d) implies that $p_1 + p_2 \sim (p_1 \boxplus p_2) + e$, where p_i for $i = 1, 2$ are two points (as well as divisors) in E, and e the origin point (as well as a divisor) of E. Here, $+$ is the sum in $\mathrm{Div}(E)$, and $\boxplus$ the sum in $\mathbb{C}/\Lambda$.

Any degree 0 divisor of E, $D = \sum_{i=1}^{n} p_i - \sum_{j=1}^{n} q_j \in \mathrm{Div}(E)$ with $p_i \in E$ and $q_j \in E$, is linear equivalent to the divisor $(\boxplus_i p_i \boxminus_j q_j) - e$. The divisor class $[D] \in \mathrm{Cl}(E)$ is the trivial one, $[0]$, if and only if $\boxplus_i p_i \boxminus_j q_j = e = 0 \in \mathbb{C}/\Lambda$.

A.3 Addition Theorem

A.3.1. One can use A.1.2 and A.1.3 to derive the addition theorem of the $\wp$-function,

$$\wp(u_1 \boxplus u_2) + \wp(u_1) + \wp(u_2) = \frac{1}{4}\left(\frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)}\right)^2. \tag{124}$$

Exercise A.1. Let us verify that the meromorphic (rational) function ϕ

$$\phi(u) = \frac{\wp'(u) - A\wp(u) - B}{\wp(u) - C} = \frac{y - Ax - B}{x - C}, \tag{125}$$

$$A = \frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)}, \quad B = \frac{\wp(u_1)\wp'(u_2) - \wp(u_2)\wp'(u_1)}{\wp(u_1) - \wp(u_2)}, \quad C = \wp(u_1 \boxplus u_2) \tag{126}$$

gives rise to a principal divisor

$$\mathrm{div}(\phi) = D_{u_1} + D_{u_2} - D_e - D_{u_1 \boxplus u_2}. \tag{127}$$

- The first step is to realize that the meromorphic (rational) function $\wp'(u) - A\wp(u) - B = y - Ax - B$ has just one pole of order 3 at e, so it must also have three zeros. Two among them are u_1 and u_2; that was how the coefficients A and B are chosen. So, the remaining one more zero must be $\boxminus(u_1 \boxplus u_2)$ (the property (a)).

$$\mathrm{div}(y - Ax - B) = -3D_e + D_{u_1} + D_{u_2} + D_{\boxminus(u_1 \boxplus u_2)}, \tag{128}$$

- The second step is to realize that the meromorphic (rational) function $\wp(u) - C = x - C$ has just one pole of order 2 at e, and two zeros, one at $u_1 \boxplus u_2$ and the other at $\boxminus(u_1 \boxplus u_2)$. So,

$$\mathrm{div}(x - x_{1\boxplus 2}) = -2D_e + D_{u_1 \boxplus u_2} + D_{\boxminus(u_1 \boxplus u_2)}. \tag{129}$$

- By combining $\mathrm{div}(y - Ax - B)$ and $\mathrm{div}(x - C)$, one can derive $\mathrm{div}(\phi)$.

A.3.2. In the language of algebraic geometry, $Y - AX - BZ = 0$ is a hyperplane (line) in $\mathbb{P}^2$, and intersects with $\overline{E'}$ at three points (because $\overline{E'} \cdot H = 3H \cdot H = 3$). They are u_1, u_2, and $\boxminus(u_1 \boxplus u_2)$. The line $X - CZ = 0$ also intersects with $\overline{E'}$ at three points. They are $(u_1 \boxplus u_2)$, $\boxminus(u_1 \boxplus u_2)$, and $e = [0 : 1 : 0] = [X : Y : Z]$. So, the point $u_1 \boxplus u_2 \in \overline{E'}$ can be worked out from $u_1, u_2 \in \overline{E'}$ by drawing two lines in $\mathbb{P}^2$.

The addition theorem may be regarded as the relation between the coefficients and the roots of a cubic polynomial. $\wp(u_1)$, $\wp(u_2)$ and $\wp(\boxminus(u_1 \boxplus u_2)) = \wp(u_1 \boxplus u_2)$ are the three roots of

$$0 = 4x^3 - g_2 x - g_3 - y^2|_{y=Ax+B} = 4x^3 - A^2 x^2 + \cdots , \tag{130}$$

so $(x_1 + x_2 + x_{1\boxplus 2}) = A^2/4$.