QFT II   homework I (Sep. 26)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**

1. **Final State Phase Space** [B]

   (a) **cross section 1**: Let us consider a process of a pair of relativisitc (approximately massless) particles collide at the center of mass energy $\sqrt{s}$, and scatter into a pair of non-identical particles. Suppose that this pair of final state particles have the same mass, m (imagine $e^+ + e^- \to \mu^+ + \mu^-$, for example). Verify that the cross section is

   $$d\sigma = \frac{d\varphi}{2\pi} \frac{d(\cos\theta)}{32\pi s} \beta |\mathcal{M}|^2, \tag{1}$$

   where (φ, θ) are the azimuthal angle and the scattering angle at the center of mass frame. [i.e., the initial state particles come along the z-axis, and the final state particles are moving out to $\vec{p} = |\vec{p}|(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$] β is the velocity $|\vec{p}|/E$ of the final state particles.
   (The primary lesson to take out from this problem is that the cross section has an overall dependence β/s due to the kinematics (final state phase space). To see how powerful this understanding is, visit the web page
   http://pdg.lbl.gov/2015/reviews/contents_sports.html
   and click on the last entry, "Plots of cross sections and related quantities", and look at Figure 50.5.)

   (b) **cross section 2**: Suppose now that the final state particles are (approximately) massless, for simplicity. Verify, then, that the expression above can also be written as

   $$d\sigma \simeq \frac{d\varphi}{2\pi} \frac{dt}{16\pi s^2} |\mathcal{M}|^2, \tag{2}$$

   where t is the Mandelstam variable of the 2-body $\longrightarrow$ 2-body scattering. (it is an option to skip this problem)

   (c) **decay rate 1** Let us think of a particle at rest with mass M decaying to a pair of non-identical (approximately) massless particles. Verify, then, that

   $$d\Gamma \simeq \frac{d^2\Omega}{4\pi} \frac{1}{16\pi M} |\mathcal{M}|^2. \tag{3}$$

   (d) **decay rate 2** Suppose that a particle at rest with mass M decays to a pair of a particle with mass $M' = M - \Delta m$ and an approximately massless particle. Verify, when $0 < \Delta m \ll M$, that

   $$d\Gamma \simeq \frac{d^2\Omega}{4\pi} \frac{\beta}{8\pi M} |\mathcal{M}|^2, \tag{4}$$

where β is the velocity of the massive particle in the final state. [A lesson: when a scattering/decay process is barely allowed kinematically, the measure over the final state phase space yields a positive power of $\beta \ll 1$, and hence the cross section/decay rate is suppressed.]

(e) **decay rate 3** Let us now think of a particle at rest with mass M decaying to three non-identical particles (e.g., $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$). The final state phase space is of 5-dimensions, after exploting the energy-momentum conservation. When the decaying particle is a scalar, or with a spin but without a polarization, however, there is SO(3) symmetry of space ($\mathbb{R}^3$) rotation acting on the final state phase, and the integral over the 5-dimensional phase space is reduced to one on a 2-dimensional space. We can take, for example, (E_1, E_2)—the energy of two particles (e.g., ν_μ and $\bar{\nu}_e$) as the coordinates of the 2-dimensional space, and the integral is in the form of

$$d\Gamma = \frac{1}{(2\pi)^3 8M} |\mathcal{M}|^2 dE_1 dE_2. \tag{5}$$

Now, assume that all the three particles in the final states are approximately massless, and work out the region in the (E_1, E_2) space that is kinematically possible; the area should be $M^2/8$. So, when the matrix element $\mathcal{M}(E_1, E_2)$ does not have a particular structure (such as singularity),the total decay rate is something like

$$\Gamma \sim \frac{M}{(8\pi)^3} \left\langle |\mathcal{M}|^2 \right\rangle. \tag{6}$$

[The absence of structure in $|\mathcal{M}|^2$ is an appropriate assumption for $\mu \to e\nu\bar{\nu}$, but not quite for $t \to W + b + g$.]

(f) (not meant as a homework problem) Suppose that the matrix element $\mathcal{M}$ for a 2-body decay in (3) is approximately M times the matrix element $\mathcal{M}$ for a 3-body decay in (6). The 3-body decay rate is smaller then the 2-body decay rate by a factor $1/32\pi^2$ then.

(g) (not meant as a homework problem) If you are interested in learning more on kinematics (such as Dalits plot), you might think of visiting the web page referred to earlier, and download a review article "Kinematics (rev.)" from that page. Derivation of (5) is also found there.

2. **Fermi Surface and Hole Excitation** [B]

Consider a Lagrangian (and corresponding Hamiltonian) of non-relativisitic electron.

$$\mathcal{L} = \psi^\dagger \left[i\partial_t + \frac{1}{2m} \vec{\partial} \cdot \vec{\partial} \right] \psi, \tag{7}$$

$$H = \int d^3x \, \psi^\dagger \left[-\frac{1}{2m} \vec{\partial} \cdot \vec{\partial} \right] \psi. \tag{8}$$

ψ is a 2-component spinor field (i.e., in the 2-dimensional representation of the space rotation SO(3) symmetry group). With the creation and annihilation operators of states with a given momentum, the field operators are written as

$$\psi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \sum_r \xi_r a_{\vec{p},r} e^{-iE_{\vec{p}}t + i\vec{p}\cdot\vec{x}}, \qquad \left\{ a_{\vec{p},r},\, a^\dagger_{\vec{q},s} \right\} = \delta_{r,s}(2\pi)^3 \delta^3(\vec{p}-\vec{q}). \tag{9}$$

Here, $E_{\vec{p}} = |\vec{p}|^2/(2m)$, and ξ_r is a dimensionless 2-component spinor; one can take $\xi_{r=\uparrow} = (1,0)^T$ and $\xi_{r=\downarrow} = (0,1)^T$, for example.

[OK to skip (a–c) if trivial for you]

(a) Verify that

$$H' \equiv H - \epsilon_F \int d^3x\, \psi^\dagger \psi = \int \frac{d^3p}{(2\pi)^3} \sum_r (E_{\vec{p}} - \epsilon_F)\, a^\dagger_{\vec{p},r} a_{\vec{p},r} + \text{const.} \tag{10}$$

(b) Verify that $\psi^\dagger\psi$ is the $\mu = 0$ component of the Noether current corresponding to the electron-number symmetry (phase rotation of ψ) in (7). [remark: Thus, $H' = H - \epsilon_F N_e$, where N_e is the electron number. ϵ_F is regarded as the chemical potential (Fermi energy).]

(c) Let us define $b_{\vec{p},r} \equiv a^\dagger_{-\vec{p},r}$ for $|\vec{p}|$ below the Fermi momentum p_F. Rewrite H' in terms of $a_{\vec{p},r}$ with $|\vec{p}| \geq p_F$ and $b_{\vec{p},r}$ with $|\vec{p}| < p_F$, and show that the state with all the levels below the Fermi surface filled is the ground state of this Hamiltonian H'. It will be easy to see that $b_{\vec{p}}$ and $b^\dagger_{\vec{p}}$ are annihilation and creation operators of a hole.

(d) What is the propagator like?

$$\langle 0| T \left\{ \psi_I(x) \psi_I^\dagger(y) \right\} |0\rangle \tag{11}$$

(e) (not a homework problem) Note that this system can be described by a Lagrangian

$$\mathcal{L}' = \psi^\dagger \left[i\partial_t + \frac{1}{2m} \vec{\partial}\cdot\vec{\partial} + \epsilon_F \right] \psi. \tag{12}$$