QFT II homework VIII (Nov. 21)

- **At the head of your report, please write your name, student ID number and a list of problems that you worked on in a report (like "II-1, II-3, IV-2").**

1. **M1, E2 transitions etc.** [C]
   Electric dipole emission (E1 transition) is not the only possible mechanism of transitions between atomic energy eigenstates. Explore more about those higher order effects, following your intellectual curiosity.

2. **Positronium Decay** [B (or C)]
   Let us work out how to use Bethe–Salpeter wavefunction to compute the decay rate of a positronium (a bound state of a pair of e^-e^+) to two photons. Here, we need to note that each photon carries energy that is approximately m_e (in the rest frame of the initial bound state). The photon momenta, or derivatives acting on a photon field in the Lagrangian, is therefore not smaller than m_e. So, $\vec{\partial}/m_e$-expansion is not particularly useful in computing the matrix element for the decay rate. For this computation, it is better to use the frame of four-component spinor where

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \vec{\tau} \\ -\vec{\tau} & \end{pmatrix}. \tag{1}$$

The Bethe–Salpeter wavefunction $\chi(p)$ in (here, $p^\mu := (p_1 - p_2)^\mu/2 =: (\omega, \vec{p})$)

$$\langle \Omega | T\left\{ \Psi(p_1)\overline{\Psi}(p_2) \right\} | s_{\rm tot}, s_z; n, l, m\rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_{\rm CM}) \left[\chi_{nls;s_z m}(p)\right]_{4\times 4} \tag{2}$$

is a 4×4 matrix valued function, and is approximately given by

$$[\chi_{nls;s_z m}(\omega, \vec{p})]_{4\times 4} \simeq \begin{pmatrix} \mathbf{1}_{2\times 2} \\ \frac{\vec{p}\cdot\vec{\tau}}{2m_e} \end{pmatrix} [P(s_{\rm tot}, s_z)]_{2\times 2} \left(-\frac{\vec{p}\cdot\vec{\tau}}{2m_e}, \; -\mathbf{1}_{2\times 2} \right) \chi_{nlm}(\omega, \vec{p}), \tag{3}$$

$$[P(s_{\rm tot}, s_z)]_{2\times 2} = \begin{cases} \mathbf{1}_{2\times 2} & s_{\rm tot} = 0 \\ \tau^3 & s_{\rm tot} = 1, \; s_z = 0, \\ (\tau^1 \pm i\tau^2)/\sqrt{2} & s_{\rm tot} = 1, \; s_z = \pm 1 \end{cases} \tag{4}$$

$$\int \frac{d\omega}{2\pi} \chi_{nlm}(\omega, \vec{p}) \simeq \sqrt{4m_e}\, \psi^{\rm NRQM}_{nlm}(\vec{p}), \tag{5}$$

using the Fourier transform of the wavefunction of a state $|n, l, m\rangle$ (with the redueced mass $m_e/2$) in the non-relativistic quantum mechanics (that is, $\psi^{\rm NRQM}_{nlm}(\vec{p})$). In the rest of this problem, we set $\vec{p}_{\rm CM} = \vec{0}$.

   (a) Verify that the matrix element of positronium $\to \gamma + \gamma$ is given by

$$i\mathcal{M} \simeq \int \frac{d\omega}{2\pi} \int \frac{d^3\vec{p}}{(2\pi)^3} (-ieQ_e)^2 \, \epsilon^*_\mu(\vec{k}) \epsilon^*_\nu(-\vec{k}) \tag{6}$$

$$\mathrm{tr}_{4\times 4}\left[\left(\frac{\gamma^\nu i[\omega\gamma^0 - (\vec{p} - \vec{k})^i \gamma^i + m_e]\gamma^\mu}{\omega^2 - (\vec{p} - \vec{k})^2 - m_e^2} + \frac{\gamma^\mu i[\omega\gamma^0 - (\vec{p} + \vec{k})^i \gamma^i + m_e]\gamma^\nu}{\omega^2 - (\vec{p} + \vec{k})^2 - m_e^2} \right) [\chi_{4\times 4}] \right],$$

if this expression is not trivial for you.

(b) Because $|\vec{k}| = m_e + (\Delta E)/2 \approx \mathcal{O}(m_e)$, while $\vec{p}$ is typically $\mathcal{O}(m_e\alpha)$ and ω even less, it makes sense to drop all of ω and $\vec{p}$ (and retain only m_e and $\vec{k}$) from the vertex-and-propagator $(\cdots + \cdots)$ part in the expression above. You will then notice that $d\omega$ integral can be carried out, and χ turns into $\psi^{\rm NRQM}$. Now, carry out the rest of the computation to find the decay rate of the positronium $(n, l, m, s_{\rm tot}, s_z) = (1, 0, 0; 0, 0)$ state. [It is not as important to get the O(1) coefficient precisely as to get the right power of m_e and α.] You will also be able to confirm that the two outgoing photons have oppsite angular momentum.

(c) Peskin–Schroeder Problem 5.4 (at the end of chapter 5) contains more information. If you are interested, you might think of exploring more. (category [C] then)

3. **Partial wave decomposition at work** [B]
Let us get the feeling how the partial wave decomposition works in practice, using the results of perturbative computations of 2body to 2body scattering amplitudes.

(a) We begin with the easiest example. Let us think of a 2-body to 2-body scattering in the s-channel, where a pari of scalar particles $\Phi^-(p_1)$ and $\Phi^+(p_2)$ coupled to a photon annihilates in pair and produce another pair of scalar particles $\Phi'^-(p_3)$ and $\Phi'^+(p_4)$. For simplicity, we only deal with the case where the center of mass energy is much higher than their rest mass (so that the mass parameters are negligible). The scattering amplitude is

$$\mathcal{M} = (-e^2 Q_\Phi Q_{\Phi'})\frac{(p_1 - p_2)\cdot(p_3 - p_4)}{s} \simeq (-e^2 Q_\Phi Q_{\Phi'})\frac{u-t}{2s} \simeq (-e^2 Q_\Phi Q_{\Phi'})\frac{\cos\theta}{2}, \quad (7)$$

where θ is the scattering angle in the center of mass frame. Verify, by fitting the result above into the following expansion,

$$\frac{\mathcal{M}}{2(4\pi)^2} \simeq \mathcal{M}_{\rm red} = \sum_{\ell=0}^{\infty} Y_{\ell,m}(\hat{\mathbf{p}}_3)\,[\mathcal{M}_\ell(s)]\,(Y_{\ell,m}(\hat{\mathbf{p}}_1))^{\rm cc}, \quad (8)$$

$$Y_{\ell,m=0}(\hat{n}) = P_\ell(\cos\theta)\sqrt{\frac{2\ell+1}{4\pi}}. \quad (9)$$

that only the $\ell = 1$ partial wave is non-zero in this scattering, and that

$$\mathcal{M}_{\ell=1}(s) \simeq \frac{-\alpha(Q_\Phi Q_{\Phi'})}{12}. \quad (10)$$

[So, in this example, $\mathcal{M}_{\ell=1}$ turns out to be independent of the center of mass energy $\sqrt{s}$, at this tree level calculation. The S-matrix in this $\ell = 1$ partial wave is $S_{\ell=1} \simeq 1 + i(-\alpha QQ')/12 \simeq e^{-i\alpha QQ'/12}$, while $S_{\ell\neq 1} = 1$ in all other partial waves.]

(b) (If you are also interested in working on this...) Let us now consider a little more complicated case, where the initial state is not a pair of scalar $\Phi^- + \Phi^+$, but a pair of spin-1/2 fermions, $e^-(p_1) + e^+(p_2)$. We still consider the case where the final state is a pair of scalars $\Phi'^-(p_3) + \Phi'^+(p_4)$. We know that the scattering amplitude (at the center of mass frame, in the relativisitic limit) is given by

$$\mathcal{M} = (e^2 Q_e Q_{\Phi'}) \begin{cases} \sin\theta e^{i\phi} & s_{e^-} = s_{e^+} = \uparrow, \\ \sin\theta e^{-i\phi} & s_{e^-} = s_{e^+} = \downarrow, \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

where θ and ϕ indicate the direction of the momentum $\hat{\mathbf{p}}_3$ after the scattering (in the center of mass frame). Since the final state pair $\Phi'^- + \Phi'^+$ are scalars (spin 0), we see that only the $J = 1$ diagonal block of $\mathcal{M}_{\text{red}}$ is non-trivial [$Y_{\ell=1,m=\pm1} = \sqrt{3/(8\pi)} \sin\theta e^{\pm i\phi}$]. On the side of initial state particle pair $e^- + e^+$, however, there are multiple ways to contribute to the $J = 1$ channel; $(S_{\text{tot}}, L_{e^-e^+}) = (1, 2)$, $(S_{\text{tot}}, L_{e^-e^+}) = (1, 1)$, $(S_{\text{tot}}, L_{e^-e^+}) = (1, 0)$ and $(S_{\text{tot}}, L_{e^-e^+}) = (0, 1)$. So, we can read out $(\mathcal{M}_{\text{red}})_{J=1}$ in the form of a 1×4 matrix (apart from the obvious $\otimes \delta_{J'_z, J_z}$). How can you see that this 1×4 matrix is non-zero only for $(S_{\text{tot}}, L_{e^-e^+}) = (1, 0)$?